An Optimal Control of Transverse Vibrations of an Euler-Bernoulli Beam-String Complex System

Ismail Kucuk and Melih Cinar

Abstract—**In this paper, an optimal control of transverse vibrations of an Euler-Bernoulli beam and string complex system is studied. The dynamic response of the system is measured by a performance index that consists of a modified energy functional of two coupled structures at the terminal time and the expenditure of the control forces as a penalty term. The minimization of the performance index over these forces is subject to the equation of motion governing the structural vibrations, the imposed initial condition as well as the boundary conditions. The optimal control of distributed parameter systems is transformed into the optimal control of a linear time-invariant lumped-parameter systems by employing Galerkin method in space. Fourier series approximation with unknown Fourier coefficients and frequencies is used to parameterize the control function. The applicability and effectiveness of the proposed method is demonstrated by a numerical example.**

Index Terms—**Optimal Control, beam, string.**

I. Introduction

SIMPLE continous elastic structures such as a string, beam etc. are foundational pieces of real mechanical systems. These simple structures are bonded through different elastic foundations such as Winkler, Pasternak or Vlasov, Flonenko - Borodich for modelling complex mechanical systems. The study of the dynamical response of the complex mechanical systems opens wide range of theoretical and practical applications in engineering. Controlling undesired vibrations in the mechanical systems is one of the core subjects attracting scientists and engineers. A concise briefing of vibratory analysis for the continuous systems is provided by [1]. The natural frequencies of different forms of the complex continuous systems have been studied by Oniszczuk (such as [2], [3] and [4].) Optimal control of the vibrations in complex continuous systems is studied in ([5], [6] are among others.)

In this paper, active control of transverse vibrations of an elastically connected beam-string complex system is studied. The complex system consists of parallel beam and string that are bonded through an elastic foundation. Such systems give rise to an important vibratory mechanical structures. Therefore, the objective is to rid of the excessive vibrations via applied actuators. The optimal control problem is formulated to minimize the performance index that is subjected to the equation of motion described by a distributed parameter system. The performance index of the system involves the physical energy of the system and the expenditure of the control forces is added as a penalty term to limit the usage of the forces. The distributed parameter system (DPS) is transformed into the lumped parameter system (LPS) in a finite dimensional space by employing Galerkin method in space. Optimal control forces derived via variational techniques that yield degenerate integral equations. A numerical example is provided to show the results. This study is elaborated in [7].

Ismail Kucuk, Department of Mathematical Engineering, Yildiz Technical University, 34220 TURKEY, e-mail: ikucuk@yildiz.edu.tr

Melih Cinar, Department of Mathematical Engineering, Yildiz Technical University, 34220 TURKEY, e-mail: mcinar@yildiz.edu.tr

A. Formulation of the Problem

The mathematical model of the free transverse vibrations of an elastically connected beam-string complex system is described as a set of two coupled homogeneous partial differential equations [4],

$$m_1\ddot{w}_1 + Kw_1^{(4)} + k(w_1 - w_2) = 0, \tag{1a}$$

$$m_2\ddot{w}_2 - Sw_2'' + k(w_2 - w_1) = 0, \tag{1b}$$

where $(x,t) \in [0,l] \times [0,t_f]$; $w_1(x,t)$ and $w_2(x,t)$ are the displacement of Beam and String, respectively; t_f is the final time in the control process; $K = E_1 J_1$ is the flexural rigidity of the beam, E_1 is Young's modulus and J_1 is the moment of inertia of the beam cross-section; S is the string tension; k is the thickness modulus of a Winkler elastic layer; m_i is the mass per unit length and

$$w_1^{(4)} = \frac{\partial^4 w_1}{\partial x^4}, w_2'' = \frac{\partial^2 w_2(x,t)}{\partial x^2}, \quad \ddot{w}_i = \frac{\partial^2 w_i(x,t)}{\partial t^2}, i = 1,2. \tag{2}$$

The boundary and initial conditions for Beam and String are introduced, respectively, as

BC's

$$\begin{aligned} &w_1(0,t) = w_1''(0,t) = w_1(l,t) = w_1''(l,t) = 0,\\ &w_2(0,t) = w_2(l,t) = 0, \end{aligned} \tag{3}$$

IC's

$$w_i(x,0) = w_i^0(x), \dot{w}_i(x,0) = v_i^0(x), i = 1,2.$$

In order to avoid any unwanted resonance we put forward

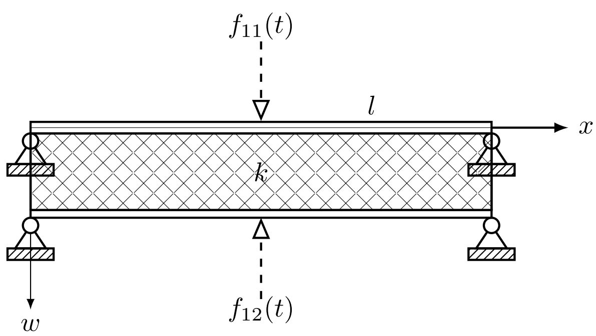

Fig. 1. An elastically connected beam-string complex system including actuators.

ISBN: 978-988-14047-4-9 (revised on 10 April 2018) WCE 2017
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

a new system where actuators are included to domain of beam and string shown Fig. 1. Then, the set of coupled non-homogeneous partial differential equations in (1a) and (1b) is transformed into the following form:

$$m_1\ddot{w}_1 + Kw_1^{(4)} + k(w_1 - w_2) = f_{11}(t)\delta(x - x_B), \quad (4)$$
$$m_2\ddot{w}_2 - Sw_2'' + k(w_2 - w_1) = f_{12}(t)\delta(x - x_S), \quad (5)$$

where x_B and x_S are the locations of the actuators in the domain of beam and string, respectively; $f_{1j}(t) \in L^2([0, t_f]), j = 1, 2$ are the amplitude (or influence) of distributed actuators. In order to measure the performance of the system where actuators are included, performance index function is introduced as

$$\mathcal{E}(f_{11}(t), f_{12}(t)) = \frac{1}{2}\int_0^l \Big(\mu_1 w_1^2(x,t_f) + \mu_2\dot{w}_1^2(x,t_f) + \mu_3 w_2^2(x,t_f) + \mu_4\dot{w}_2^2(x,t_f)\Big)dx. \quad (6)$$

In Eq. (6), the elements of the weighting factors μ_i are greater than zero for $i = 1, 2, 3, 4$ such that $\mu_1 + \mu_2 + \mu_3 + \mu_4 \neq 0$.

Control Problem Determine an optimal $f_{1i}^{\star}(t) \in L^2([0, t_f])$ for $i = 1, 2$ such that
$\mathcal{J}(f_{11}^{\star}(t), f_{12}^{\star}(t)) \leq \mathcal{J}(f_{11}(t), f_{12}(t))$,
where

$$\mathcal{J}(f_{11}(t), f_{12}(t)) = \mathcal{E}(f_{11}(t), f_{12}(t)) + \frac{1}{2}\int_0^{t_f}(\mu_5 f_{11}^2(t) + \mu_6 f_{12}^2(t))dt, \quad (7)$$

in which $\mathcal{E}(f_{11}(t), f_{12}(t))$ is given in Eq. (6), and μ_5 and μ_6 are the weight factors that determine the influence of the actuators. The first term in right-hand side of Eq. (7), $\mathcal{E}(f_{11}(t), f_{12}(t))$, stands for the contribution of the modified energies of the beam and string, and the other term represent a contribution of the energy that accumulates over the control duration where the final time t_f is fixed.

II. Solution of the vibration problem

The system of coupled non-homogeneous partial differential equations in (4) and (5) with boundary conditions can be solved by using the eigenfunction expansion technique. Let $w_1(x,t)$ and $w_2(x,t)$ be the solutions of the equations of (4) and (5), respectively, such that

$$w_1(x,t) = \lim_{n\longrightarrow\infty}\sum_{n=1}^{N}\mathit{\Gamma}_n(x)T_{n1}(t), \quad (8)$$

$$w_2(x,t) = \lim_{n\longrightarrow\infty}\sum_{n=1}^{N}\mathit{\Gamma}_n(x)T_{n2}(t), \quad (9)$$

where $T_{n1}(t)$ and $T_{n2}(t)$ are unknown functions of time, and N is the number of modes used in the calculations. In practice, it is customary to expand the displacement functions $w_i(x,t), i = 1, 2$ with high accuracy through the truncated form of Eqs. (8) and (9), i.e., N is taken a finite number.

The orthonormal eigenfunctions of operator $\pounds w = \partial^2 w/\partial x^2$ has the following form:

$$\{\mathit{\Gamma}_n(x)\}_{n=1}^{\infty} = \left\{\sqrt{\frac{2}{l}}\sin(a_n x)\right\}_{n=1}^{\infty}, \quad (10)$$

where $a_n = l^{-1}n\pi$.

Substituting the solutions in Eqs. (8) and (9) into Eqs. (4) and (5), where number N is a finite number, we have

$$\sum_{n=1}^{N}\mathit{\Gamma}_n(x)\{m_1\ddot{T}_{n1}(t) + (Ka_n^4 + k)T_{n1}(t) - kT_{n2}(t)\} = f_{11}(t)\delta(x - x_B), \quad (11)$$

and

$$\sum_{n=1}^{N}\mathit{\Gamma}_n(x)\{m_2\ddot{T}_{n2}(t) + (Sa_n^2 + k)T_{n2}(t) - kT_{n1}(t)\} = f_{12}(t)\delta(x - x_S). \quad (12)$$

Using the orthonormal property of eigenfunctions $\mathit{\Gamma}_n(x)$, last two equations are transformed into new system of second-order differential equations as follows:

$$\ddot{T}_{n1}(t) + \varphi_{n1}T_{n1}(t) - \varphi_{01}T_{n2}(t) = F_{n1}(t), \quad (13)$$
$$\ddot{T}_{n2}(t) + \varphi_{n2}T_{n2}(t) - \varphi_{02}T_{n1}(t) = F_{n2}(t), \quad (14)$$

where
$\varphi_{n1} = \dfrac{K(a_n)^4 + k}{m_1}, \varphi_{n2} = \dfrac{S(a_n)^2 + k}{m_2}$, $\varphi_{0i} = \frac{k}{m_i}, i = 1, 2$,

$$F_{n1}(t) = m_1^{-1}f_{11}(t)\mathit{\Gamma}_n(x_B), \quad (15)$$
$$F_{n2}(t) = m_2^{-1}f_{12}(t)\mathit{\Gamma}_n(x_S). \quad (16)$$

Now, in order to solve the new coupled system of second-order differential equations, we introduce new variables $y_{n1}(t) = T_{n1}(t), y_{n2}(t) = \dot{T}_{n1}(t), y_{n3}(t) = T_{n2}(t)$, $y_{n4}(t) = \dot{T}_{n2}(t)$. Substituting the new variables into Eqs. (13) and (14), we immediately get the following form of the first-order system of differential equations in time:

$$\begin{aligned}
\dot{y}_{n1}(t) &= y_{n2}(t),\\
\dot{y}_{n2}(t) &= -\varphi_{n1}y_{n1}(t) + \varphi_{01}y_{n3}(t) + F_{n1}(t),\\
\dot{y}_{n3}(t) &= y_{n4}(t),\\
\dot{y}_{n4}(t) &= \varphi_{02}y_{n1}(t) - \varphi_{n2}y_{n3}(t) + F_{n2}(t).
\end{aligned} \quad (17)$$

Eq. (17) can be written as the following matrix form:

$$\frac{d\mathbf{Y}_n(t)}{dt} = \mathbf{A}\mathbf{Y}_n(t) + \mathbf{F}(t), \quad (18)$$

where

$$\mathbf{Y}_n(t) = \begin{pmatrix} y_{n1}\\ y_{n2}\\ y_{n3}\\ y_{n4}\end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -\varphi_{n1} & 0 & \varphi_{01} & 0\\ 0 & 0 & 0 & 1\\ \varphi_{02} & 0 & -\varphi_{n2} & 0\end{pmatrix}, \mathbf{F}(t) = \begin{pmatrix} 0\\ F_{n1}(t)\\ 0\\ F_{n2}(t)\end{pmatrix}. \quad (19)$$

The initial conditions of first-order system of differantial equations in (17) are rewritten as

$$y_{n1}(0) = T_{n1}(0) = \int_0^l \mathit{\Gamma}_n(x)w_1^0(x)dx, \quad (20)$$

ISBN: 978-988-14047-4-9 (revised on 10 April 2018)
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$y_{n2}(0) = \dot{T}_{n1}(0) = \int_0^l \Gamma_n(x) v_1^0(x) dx, \tag{21}$$

$$y_{n3}(0) = T_{n2}(0) = \int_0^l \Gamma_n(x) w_2^0(x) dx, \tag{22}$$

$$y_{n4}(0) = \dot{T}_{n2}(0) = \int_0^l \Gamma_n(x) v_2^0(x) dx. \tag{23}$$

To decouple differential equations in (18), we introduce a new variable $\mathbf{M}_n(t)$ such that

$$\mathbf{Y}_n(t) = \mathbf{Z}\mathbf{M}_n(t), \tag{24}$$

in here, the matrix $\mathbf{Z}$ is 4×4 modal matrix whose columns are the eigenvectors of matrix $\mathbf{A}$.
Substituting Eq. (24) into Eq. (18), we get

$$\dot{\mathbf{M}}_n(t) = (\mathbf{Z}^{-1}\mathbf{A}\mathbf{Z})\mathbf{M}_n(t) + \mathbf{Z}^{-1}\mathbf{F}(t) = \mathbf{D}\mathbf{M}_n(t) + \mathbf{G}(t), \tag{25}$$

where $\mathbf{D} = \mathbf{Z}^{-1}\mathbf{A}\mathbf{Z}$, $\mathbf{G}(t) = \mathbf{Z}^{-1}\mathbf{F}(t)$ defined as

$$\begin{aligned}\mathbf{G}(t) &= \beta \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} F_{n1}(t) + \begin{pmatrix} \Delta_1 \\ \Delta_1 \\ \Delta_2 \\ \Delta_2 \end{pmatrix} F_{n2}(t) \\ &= \mathbf{G}_1 F_{n1}(t) + \mathbf{G}_2 F_{n2}(t),\end{aligned} \tag{26}$$

where $\beta, \Delta_1, \Delta_2$ are some constants.
Eq. (18) is a system of four differantial equations for $M_{ni}(t), i = 1, 2, 3, 4$. In scalar form, we observe the following equations:

$$\frac{dM_{ni}(t)}{dt} = \lambda_i M_{ni}(t) + G_i(t), \tag{27}$$

with the proper form of the initial conditions given in the Eqs. (20)-(23). In here, λ_i's are the eigenvalues of $\mathbf{A}$. The solutions of FOSDE of Eq. (17) are of the folllowing form:

$$M_{ni}(t) = \int_0^t e^{\lambda_i(t-\tau)} G_i(\tau) d\tau + c_i e^{\lambda_i t}, \tag{28}$$

in here, c_i's are constants to be determined by the initial conditions in Eqs. (20)-(23). Finally, we can write the solutions for $T_{n1}(t)$ and $T_{n2}(t)$, and their derivatives as follows:

$$T_{n1}(t) = \sum_{j=1}^{4} z_{1j} M_{nj}(t), \tag{29}$$

$$\dot{T}_{n1}(t) = \sum_{j=1}^{4} z_{2j} M_{nj}(t), \tag{30}$$

$$T_{n2}(t) = \sum_{j=1}^{4} z_{3j} M_{nj}(t), \tag{31}$$

$$\dot{T}_{n2}(t) = \sum_{j=1}^{4} z_{4j} M_{nj}(t), \tag{32}$$

where $\mathbf{Z} = [z_{ij}]_{4\times4}$ whose columns are the eigenvectors of $\mathbf{A}$. Thus, the deflections in beam and string are respectively obtained as

$$w_1(x,t) = \sum_{n=1}^{N} \Gamma_n(x) \sum_{j=1}^{4} z_{1j} M_{nj}(t), \tag{33}$$

$$w_2(x,t) = \sum_{n=1}^{N} \Gamma_n(x) \sum_{j=1}^{4} z_{3j} M_{nj}(t), \tag{34}$$

where $M_{ni}(t)$ is defined in (28).

III. Necessary conditions

In this section, to determine necessary conditions of optimality for the applied actuators, calculus of variation will be used. After taking first variation of the Eq. (7) with respect to f_{1i} for $i = 1, 2$, it will be equate to zero. In this calculations; flextural rigidity K, the tension of the string S, final time t_f, the location of the actuators in the system and weight factors will be fixed.
Substituting Eqs. (33) and (34) into Eq. (7) and using the orthonormal properties of $\Gamma_n(x)$ leads to the following:

$$\begin{aligned}\mathcal{J}(f_{11}(t), f_{12}(t)) &= \frac{1}{2} \sum_{n=1}^{N} \sum_{i=1}^{4} \mu_i \big(\sum_{j=1}^{4} z_{ij} M_{ni}(t_f) \big)^2 + \\ &\frac{1}{2} \int_0^{t_f} (\mu_5 f_{11}^2(t) + \mu_6 f_{12}^2(t)) dt.\end{aligned} \tag{35}$$

If we take variation of Eq. (35) with respect to $f_{11}(t)$, it leads to

$$\begin{aligned}&\boldsymbol{\delta}_{f_{11}} \mathcal{J}(f_{11}(t), f_{12}(t)) = \int_0^{t_f} \Bigg\{ \sum_{n=1}^{N} \sum_{i=1}^{4} \mu_i \big(\sum_{j=1}^{4} z_{ij} M_{ni}(t_f) \big) \times \\ &\sum_{j=1}^{4} z_{ij} e^{\lambda_j(t_f-\tau)} \overline{G}_{1j} \Gamma_n(x_B) + \mu_5 f_{11}(\tau) \Bigg\} \Delta f_{11}(\tau) d\tau = 0,\end{aligned} \tag{36}$$

or

$$\begin{aligned}&\sum_{n=1}^{N} \sum_{i=1}^{4} \mu_i \Bigg[\sum_{j=1}^{4} z_{ij} \Big\{ \int_0^{t_f} e^{\lambda_j(t_f-\tau)} \Big(\overline{G}_{1j} f_{11}(\tau) \Gamma_n(x_B) \\ &+ \overline{G}_{2j} f_{12}(\tau) \Gamma_n(x_S) \Big) d\tau + c_j e^{\lambda_j t_f} \Big\} \Bigg] \times \\ &\Big[\sum_{j=1}^{4} z_{ij} e^{\lambda_j(t_f-\tau)} \overline{G}_{1j} \Gamma_n(x_B) \Big] + \mu_5 f_{11}(\tau) = 0,\end{aligned} \tag{37}$$

where $\overline{G}_{1j}$ and $\overline{G}_{2j}$ are the terms from the column coefficient matrices of $F_{n1}(t)$ and $F_{n2}(t)$ given in (26).
Similarly, taking first variation of Eq. (35) with respect to $f_{12}(t)$, it can be derived that

$$\begin{aligned}&\sum_{n=1}^{N} \sum_{i=1}^{4} \mu_i \Bigg[\sum_{j=1}^{4} z_{ij} \Big\{ \int_0^{t_f} e^{\lambda_j(t_f-\tau)} \Big(\overline{G}_{1j} f_{11}(\tau) \Gamma_n(x_B) \\ &+ \overline{G}_{2j} f_{12}(\tau) \Gamma_n(x_S) \Big) d\tau + c_j e^{\lambda_j t_f} \Big\} \Bigg] \times \\ &\Big[\sum_{j=1}^{4} z_{ij} e^{\lambda_j(t_f-\tau)} \overline{G}_{2j} \Gamma_n(x_S) \Big] + \mu_6 f_{12}(\tau) = 0.\end{aligned} \tag{38}$$

We can write Eqs. (37) and (38) in a more compact form so that coupled nonhomogeneous Fredholm integral equations with degenerate kernel can be observed as

$$\begin{aligned}&\sum_{n=1}^{N} \Bigg[\sum_{i=1}^{4} \int_0^{t_f} \Big(\mathbf{U}_{ni}(\tau) f_{11}(r) + \overline{\mathbf{U}}_{ni}(\tau) f_{12}(r) \Big) e^{\lambda_i(t_f-r)} dr \\ &+ \mathbf{K}_n(\tau) \Bigg] + \mu_5 f_{11}(\tau) = 0,\end{aligned} \tag{39}$$

$$\sum_{n=1}^{N}\left[\sum_{i=1}^{4}\int_{0}^{t_f}\Big(\mathbf{V}_{ni}(\tau)f_{11}(r)+\overline{\mathbf{V}}_{ni}(\tau)f_{12}(r)\Big)e^{\lambda_i(t_f-r)}dr \right.$$
$$\left. +\,\mathbf{K}_n(\tau)\right]+\mu_6 f_{12}(\tau)=0, \tag{40}$$

where
$\mathbf{U}(\tau) = diag(\mathbf{G}_1)\mathbf{Z}^\top diag(\mu_1,\mu_2,\mu_3,\mu_4)\big(\mathbb{U}(\tau)\big)_{4\times 1}\Gamma_n(x_B)$ in which $(\mathbb{U}(\tau))_{4\times1} = \big(\sum_{j=1}^{4} z_{ij}\overline{G}_{1j}e^{\lambda_j(t_f-\tau)}\big)_{4\times1}$ and $\mathbf{K}_n(\tau) = (\mathbb{U}(\tau)diag(\mu_1,\mu_2,\mu_3,\mu_4))^\top\mathbf{M}(e^{\boldsymbol{\lambda}t_f}C)$ in which $\boldsymbol{\lambda}$ is the column matrix of eigenvalues of $\mathbf{A}$, and C is the column matrix of constants that are due to the new initial conditions obtained for the new system in (28). The other terms can be obtained similarly.
If we define r_{i1} and $\bar{r}_{i1}$ by

$$r_{i1} = \int_{0}^{t_f} e^{\lambda_i(t_f-r)}f_{11}(r)dr, \tag{41}$$

$$\bar{r}_{i1} = \int_{0}^{t_f} e^{\lambda_i(t_f-r)}f_{12}(r)dr, \tag{42}$$

then, the integral equations in Eqs. (39) and (40) can be transformed into the system of linear equations in terms of r_{i1} and $\bar{r}_{i1}$. And then, the system of equations can be written in the following compact form of linear equations:

$$\begin{aligned}&(\mathcal{R}+\mathbf{I}\mathcal{H})\mathcal{S}+\overline{\mathcal{R}\mathcal{S}}+\mathcal{V}=0,\\ &\mathcal{K}\mathcal{S}+(\overline{\mathcal{K}}+\mathbf{I}\mathcal{N})\overline{\mathcal{S}}+\overline{\mathcal{V}}=0.\end{aligned} \tag{43}$$

After computing $\mathcal{S}$ and $\overline{\mathcal{S}}$, the optimal actuators $f_{11}(\tau)$ and $f_{12}(\tau)$ can be calculated as follows

$$f_{11}(\tau) = -\frac{1}{\mu_5}\sum_{n=1}^{N}\Big\{\mathbf{K}_n(\tau)+\sum_{i=1}^{4}\big(\mathbf{U}_{ni}(\tau)r_{i1}+\overline{\mathbf{U}}_{ni}(\tau)\bar{r}_{i1}\big)\Big\}, \tag{44}$$

$$f_{12}(\tau) = -\frac{1}{\mu_6}\sum_{n=1}^{N}\Big\{\mathbf{K}_n(\tau)+\sum_{i=1}^{4}\big(\mathbf{V}_{ni}(\tau)r_{i1}+\overline{\mathbf{V}}_{ni}(\tau)\bar{r}_{i1}\big)\Big\}. \tag{45}$$

IV. Numerical example

In this chapter, to illustrate theoretical considerations presented in the previous part of the paper, the behavior of uncontrolled and controlled beam-string system is presented. In numerical simulations, the deflection and velocity of beam are observed at points $x_B = 0.46$ and $x_S = 0.63$ in the domain of beam and string, respectively , and $m_1 = 0.02$, $m_2 = 0.001$, $K = 10^{-2}$, $S = 30$, $l = 1$, $\mu_1 = 0.02, \mu_2 = 0.003, \mu_3 = 0.01, \mu_4 = 0.004$, and $t_f = 0.8$ are used. For simplicity of the analysis, it is assumed that the beam-string system is subjected to the initial conditions (3) of the form

$$\begin{aligned}&w_i(x,0) = \Gamma_1(x),\\ &\dot{w}_i(x,0) = 0, i = 1,2,\end{aligned} \tag{46}$$

where $\Gamma_1(x)$ is the fundamental mode of the system.

The deflection and velocity of the beam and string in both uncontrolled and controlled cases are illustrated in Figs. 2 and 3, respectively. As the numerical results show that the deflection and velocity of the system is drastically suppressed and hence the physical energy used in the system is reduced. The location of the actuators is determined by the designer but determining the location of the actuators is another problem to explore for future investigations.

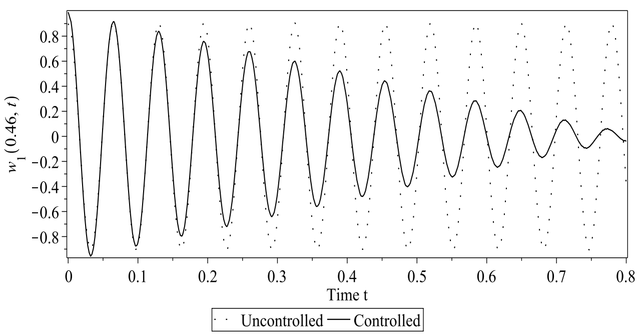

Fig. 2. The displacement of beam for various cases.

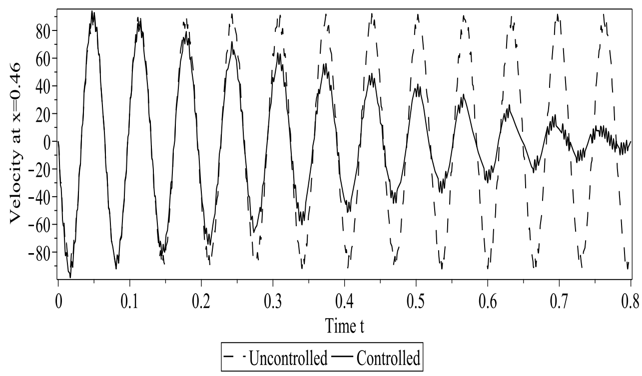

Fig. 3. The velocity of beam for various cases.

V. Conclusion

The active control of the free transverse vibrations of elastically bonded beam-string system is achieved by applying actuators. The performance index of the system is defined by its physical energy and control expenditure over the time as a penalty term. Galerkin method in space is used to transform the DPS into LPS in a finite-dimensional space. Hence, the performance index functional is expressed in temporal terms. To obtain the optimal control forces, variation of the new performance index is taken that yields integral equations. Thus, optimal control forces are obtained explicitly. Numerical simulation of the methodology developed to solve the problem validates the theoretical results.

Acknowledgment

The author Melih Cinar would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for its support under 2211-A Fellowship Program.

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Date of Modification: April 10, 2018. (Only the acknowledgment section was added.)