On a Modified Iterative Method for the Solutions of Advection Model

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Abstract— **Variational Iterative Method (VIM) has been reported in literature as a powerful semi-analytical method for solving linear and nonlinear differential equations; however, it has also been shown to have some weaknesses such as calculation of unneeded terms, and time-consumption regarding repeated calculations for series solution. In this work, a modified VIM is applied for approximate-analytical solution of homogeneous advection model. The result attest to the robustness and efficiency of the proposed method (MVIM).**

Index Terms— **Exact solutions; modified VIM; HPM; advection model.**

I. INTRODUCTION

In pure and applied sciences, Nonlinear Partial Differential Equations (PDEs) stand out for modelling real life problems. An example of such nonlinear model is the advection model of the form:

$$\begin{aligned} &u_t(x,t)+uu_x = f(x,t), \\ &u(x,0)=h(x). \end{aligned} \tag{1.1}$$

The importance of providing solutions (numerical or exact) to linear and nonlinear differential equations cannot be overemphasized. This has led to various methods of solution [1-10].

Ji-Huan He [11] proposed the popular variational iteration method (VIM) for a nonlinear differential equation. VIM has been widely applied [12-16]. However, some of the weaknesses of the VIM are remarked. This include: repeated computations and computations of unneeded terms, and so on. Hence, the VIM modification [17].

In this work, the modified VIM is applied to advection model for approximate-analytical solutions.

II. VIM AND THE MODIFIED VIM [17]

Considering the general nonlinear PDE of the form:

$$\begin{cases} Lu(x,t)+Ru(x,t)+Nu(x,t)=f(x,t) \\ u(x,t)=h(x) \end{cases} \tag{2.1}$$

where $L=\dfrac{\partial}{\partial t}$, R is a linear operator whose partial derivatives are w.r.t. x, $Nu(x,t)$ is a nonlinear term associated to (2.1) and $f(x,t)$ is a source term (which may be homogeneous or inhomogeneous), thus by the classical VIM, the solution of (2.1) is expressed as:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(Lu_n+\overline{Ru_n}+\overline{Nu_n}-f(x,t)\right)ds \tag{2.2}$$

where λ is a Lagrange multiplier [11, 12] to be identified optimally via variational theory, and the terms: $\overline{Ru_n}$ and $\overline{Nu_n}$ are being considered as restricted variations such that $\delta\overline{Ru_n}=0$ and $\delta\overline{Nu_n}=0$. Hence, by calculating the variations w.r.t. u_n using the stationary conditions:

$$\begin{cases} \lambda'(s)=0, \\ 1+\lambda(s)=0. \end{cases} \tag{2.3}$$

The Lagrange multiplier is identified as $\lambda=-1$. Therefore, (2.2) becomes:

$$\begin{cases} u_{n+1}(x,t)=u_n(x,t) \\ \qquad -\int_0^t\left(Lu_n+R(u_n)+N(u_n)-f(x,t)\right)ds. \end{cases} \tag{2.4}$$

Remark: Using (2.4) for the solution of special kind of nonlinear differential equations involve the calculation of unrequired terms, repeated calculations, and time-consumption, hence, the need for meaningful modification of the VIM. The modified VIM as proposed by [17] gives the iterative formula as follows:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(R(u_n-u_{n-1})+(G_n-G_{n-1})\right)ds, \tag{2.5}$$

where

Manuscript received March 20, 2017. Revised March 31, 2017.

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$$\begin{cases} u_{-1} = 0, \\ u_0 = h(x), \\ u_1 = u_0 - \int_0^t \left(R(u_n - u_{n-1}) + (G_n - G_{n-1}) - f \right) ds, \\ G_n = N u_n + \mathrm{O}\left(t^{n+1}\right). \end{cases} \tag{2.6}$$

III. Illustrative Application

In this subsection, the modified VIM is applied to homogeneous advection model as follows.

Application: Consider the following homogeneous advection model:

$$\begin{cases} u_t + u u_x = 0, \\ u(x,0) = -x. \end{cases} \tag{3.1}$$

Procedure:

By applying the modified VIM, it is therefore obvious that $Lu = u_t$, $Ru = 0$ and $Nu = uu_x$. Therefore, we have:

$$u_{n+1} = u_n - \int_0^t (G_n - G_{n-1}) ds \ , \tag{3.2}$$

$$\begin{cases} u_{-1} = 0, \\ u_0 = -x, \\ u_1 = u_0 - \int_0^t (G_0 - G_{-1}) ds, \\ G_n = u_n (u_n)_x \ . \end{cases} \tag{3.3}$$

Hence, for $n = 0$, we have:

$$G_0 = u_0 (u_0)_x = x,$$

$$u_1 = -x - \int_0^t (x) ds = -x - xt \ . \tag{3.4}$$

When $n = 1$, we have:

$$G_1 = u_1 (u_1)_x = (-x - xt)(-1 - t),$$

showing that $G_1 - G_0 = 2xt + xt^2$, as such:

$$\begin{aligned} u_2 &= u_1 - \int_0^t (G_1 - G_0) ds \\ &= -x - xt - \int_0^t \left(2xt + xt^2 \right) ds \\ &= -x - xt - xt^2 - \frac{xt^3}{3} \\ &\vdots \qquad \vdots \end{aligned} \tag{3.5}$$

From (3.5), the solution is therefore deduced as:

$$\begin{aligned} u(x,t) &= -x - xt - xt^2 - xt^3 + \cdots \\ &= -x\left(1 + t^2 + t^3 + \cdots\right) \simeq \frac{x}{t-1}. \end{aligned} \tag{3.6}$$

Eq. (3.6) is thus, the exact solution of the solved problem.

IV. Concluding Remarks

This paper demonstrated the robustness and efficiency of the modified VIM. For illustration, the MVIM is applied for approximate-analytical solution of homogeneous advection model. Applying this method does not require perturbation, or linearization.

Acknowledgement

The authors are sincerely thankful to Covenant University, and the anonymous referee(s) for financial support, and constructive suggestions respectively.

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