Analytical Solution of (1+n) Dimensional Nonlinear Burgers' Equation using Variational Iteration Method

Vikramjeet Singh, Monika Rani, and H. S. Bhatti

Abstract- (1+n) dimensional Burgers' equation which can be represented as a simple form of Navier-stocks equation is a nonlinear partial differential equation. This equation is widely used to mull over a number of physically important phenomena, including shock waves, turbulent fluid motion, acoustic transmission and traffic flow with important applications in physics. In this manuscript, the analytical solution of (1+n) dimensional nonlinear Burgers' equation by He's Variational Iteration Method (VIM) has been presented. We have constructed a correctional function by a general Lagrange Multiplier which can be identified via variational theory to find the closed form solution of generalized (1+n) dimensional Burgers' equation without any unphysical restrictive assumptions.

Index Terms— (1+n) dimensional Burgers' equation, Variational Iteration Method, ADM, Nonlinear Partial Differential Equation, Lagrange Multiplier

I. INTRODUCTION

With the advancement in non linear science, the investigators are showing their interest in methodical techniques to resolve a broad diversity of linear – nonlinear, ordinary - partial differential equations with initial and boundary value problems. Several methods and analytical models have been suggested and proposed for solving higher order non linear equations, although these methods and models have their own definitions, statements, hypothesizes, insufficiencies, boundaries and restrictions [1]. Burgers' equation has been suggested by J. M. Burgers in an application of unstable fluid motion model through a number of publications. This equation comes into sight as an ordinary type i.e. it represents the qualitative performance of a wide variety of equations and occurs in the discovery of blueprint patterns, in the background of modulations of spatially-periodic waves [2]. One and two dimensional Burgers' equations are quite famous in wave theory, gas dynamics, plasma physics and their applications etc.

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Due to its broad variety of relevance, the researchers are aiming to represents this equation in the generalized form of higher dimensions [3].

Ji Huan He developed variation iteration method (VIM) in 1999 to solve all type of linear - non linear, ordinary - partial differential equations with initial and boundary value problems. Initially, VIM has been suggested by Innokuti, Sekine and Mura, but the actual prospective has been investigated by He [4]. This method is based on variational theory, in which a correctional function is implemented by using general Lagrange multiplier in a way that the overall precision of the solution has been improved with each iteration [5, 6].

Unlike perturbation method and Adomian decomposition method (ADM), it does not involve small and auxiliary parameter and Adomian polynomials etc. Due to faster rate of convergence of solution, this method is more influential than ADM, Homotopy Perturbation Method and their modified versions. These features of VIM make it broadly applicable for solving higher order non-linear problems [7-11].

II. BRIEF DISCRIPTION OF VIM

Consider a general non linear differential equation

$$Ły + ɳy = q(x) \qquad (1)$$

Where q(x) represents a non homogeneous term, Ł and ɳ are linear and non linear operators respectively. Now, consider a correction functional as

$$Ym+1\ (x) = ym\ (x) + \int_0^x \lambda\, Ł\ (ym\ (\eta) + ɳ\ \widetilde{ym}\ (\eta) - q\ (\eta))\ d\ \eta \qquad (2)$$

where λ is a Lagrange multiplier, m represents the mth approximation, $\widetilde{ym}$ can be shown as a restricted variation i.e. $\delta\ \widetilde{ym} = 0$. The successive approximation ym+1, m ≥ 0 of the solution y can be readily obtained by the determined Lagrange multiplier and any selective zeroth approximation y0, consequently, the solution is given by

$$y = \lim_{m\to\infty} y_m \qquad (3)$$

III. ANALYTICAL SOLUTION OF (1+N) DIMENSIONAL NONLINEAR BURGER'S EQUATION

We consider (1+n) dimensional nonlinear Burgers' equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} + ----- + \frac{\partial^2 v}{\partial x_n^2} + v\frac{\partial v}{\partial x_1} \quad (4)$$

With initial condition

$$v(x_1, x_2, x_3, ----, x_n, 0) = x_1 + 2x_2 + 3x_3 + --- + nx_n \quad (5)$$

Equation (4) can be rewritten as

$$(L+N)v = h(x_1, x_2, x_3, ----, x_n, t) \quad (6)$$

Where L symbolizes linear term, N represents non linear term and h is the source term. Define correctional functional for equation (4) as:

$$v_{k+1}(x_1, x_2, x_3, --, x_n, t) = v_k(x_1, x_2, x_3, ----, x_n, t) + \int_0^t \lambda(\eta)\left[\frac{\partial}{\partial\eta} v_k(x_1, x_2, x_3, ----, x_n, \eta) - \tilde{v_k}(x_1, x_2, x_3, ----, x_n, \eta)\frac{\partial}{\partial x_1} v_k(x_1, x_2, x_3, ----, x_n, \eta) - h(x_1, x_2, x_3, ----, x_n, \eta)\right] d\eta,$$

$$k \geq 0 \quad (7)$$

Where $h(x_1, x_2, x_3, ----, x_n, t) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ----- + \frac{\partial^2}{\partial x_n^2}\right) v(x_1, x_2, x_3, ----, x_n, t)$

Taking variation with respect to independent variable v_k with

$$\delta N(\tilde{v_k}) = 0$$

$$\delta v_{k+1}(x_1, x_2, x_3, ----, x_n, t) = \delta v_k(x_1, x_2, x_3, ----, x_n, t) + \delta\int_0^t \lambda(\eta)\left[\frac{\partial}{\partial\eta} v_k(x_1, x_2, x_3, ----, x_n, \eta) - \tilde{v_k}(x_1, x_2, x_3, ----, x_n, \eta)\frac{\partial}{\partial x_1} v_k(x_1, x_2, x_3, ----, x_n, \eta) - h(x_1, x_2, x_3, ----, x_n, \eta)\right] d\eta \quad (8)$$

This gives the stationary condition,

$$[1 + \lambda(\eta)]_{\eta=t} = 0 \quad (9)$$

$$[\lambda'(\eta)]_{\eta=t} = 0 \quad (10)$$

Equation (9) and (10) are called as Lagrange Euler equation and natural boundary condition respectively. Using these equations, the Lagrange multiplier can be calculated as $\lambda = -1$

Then equation (7) read as

$$v_{k+1}(x_1, x_2, x_3, ----, x_n, t) = v_k(x_1, x_2, x_3, ----, x_n, t) - \int_0^t \left[\frac{\partial}{\partial\eta} v_k(x_1, x_2, x_3, ----, x_n, \eta) - v_k(x_1, x_2, x_3, ----, x_n, \eta)\frac{\partial}{\partial x_1} v_k(x_1, x_2, x_3, ----, x_n, \eta) - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ----- + \frac{\partial^2}{\partial x_n^2}\right) v_k(x_1, x_2, x_3, ----, x_n, \eta)\right] d\eta \quad (11)$$

Using initial condition (5),

$$v_0(x_1, x_2, x_3, ----, x_n, t) = x_1 + 2x_2 + 3x_3 + ---- + nx_n$$

And the successive approximations can be calculated as

$$v_1(x_1, x_2, x_3, ----, x_n, t) = (x_1 + 2x_2 + 3x_3 + ---- + nx_n)(1+t) \quad (12)$$

$$v_2(x_1, x_2, x_3, ----, x_n, t) = (x_1 + 2x_2 + 3x_3 + ---- + nx_n)(1 + t + t^2) + \text{small terms} \quad (13)$$

$$v_3(x_1, x_2, x_3, ----, x_n, t) = (x_1 + 2x_2 + 3x_3 + ---- + nx_n)(1 + t + t^2 + t^3) + \text{small terms} + \ldots \quad (14)$$

Now, exact solution is

$$v(x_1, x_2, x_3, ----, x_n, t) = \lim_{k\to\infty} v_k(x_1, x_2, x_3, ----, x_n, t) = \frac{(x_1 + 2x_2 + 3x_3 + ---- + nx_n)}{1-t}$$

provided $1 \geq |t|$ (15)

IV. CONCULSION

In this paper, we have represented the Burgers' nonlinear differential equation and its solution using standard variational iteration method. An example of (1+n) dimensional non linear Burgers' equation is shown to confirm the applicability of the method, its fast convergence and accuracy of the solution. It is also suggested that the closed form solutions of higher order non linear initial and boundary value problems can be obtained using VIM method in less computations. The modified VIM method can also be applied to solve such problems in future.

REFERENCES

[1] S. T. Mohyud-Din, M. A. Noor, "Homotopy Perturbation Method for Solving Partial Differential Equations", Zeitschrift für Naturforschung A., vol. 64, no. 3-4, pp. 157-170, 2014.

[2] Burgers, J. M., "A mathematical model illustrating the theory of turbulence", Advances in Applied Mechanics, pp. 171, 1948.

[3] V. K. Srivastava, M. K. Awasthi, "(1 +n) - Dimensional Burgers' equation and its analytical solution: A comparative study of HPM, ADM and DTM", Engineering Physics and Mathematics, Ain Shams Engineering Journal, vol. 5, pp. 533–541, 2014.

[4] J. H. He, "Homotopy perturbation method: a new non linear analytical technique", Applied Mathematics and Computation, vol. 135, pp. 73-79, 2003.

[5] S. T. Mohyud-Din, M. A. Noor, "Solving Schrödinger Equations by Modified Variational Iteration Method", World Applied Sciences Journal, vol. 5, no. 3, pp. 352-357, 2008.

[6] S. Zhang, "Exp-function method for constructing explicit and exact solutions of a lattice equation", Applied Mathematics and Computation, vol. 199, no. 1, pp. 242-249, 2008.

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[7] O. Kıymaz, A. Cetinkaya, “Variational Iteration Method for a Class of Nonlinear Differential Equations”, Int. J. Contemp. Math. Sciences, vol. 5, no. 37, pp. 1819 – 1826, 2010.

[8] N. H. Sweilam, M. M. Khader, “Variational iteration method for one dimensional nonlinear thermo elasticity”, Chaos Solitons & Fractals, vol. 32, no. 1, pp. 145-149, 2007.

[9] A. S. Mohamed , R. A. Mahmoud, “ Picard, Adomian and predictor–corrector methods for an initial value problem of arbitrary (fractional) orders differential equation”, Journal of the Egyptian Mathematical Society, vol. 24, pp. 165–170, 2016.

[10] S. Paul, S. P. Mondal, Paritosh Bhattacharya, “Numerical solution of Lotka Volterra prey redator model by using Runge–Kutta–Fehlberg method and Laplace Adomian decomposition method”, Alexandria Engineering Journal, vol. 55, pp. 613–617, 2016.

[11] M. Paripour, E. Hajilou, A. Hajilou, H. Heidari, “Application of Adomian decomposition method to solve hybrid fuzzy differential equations”, Journal of Taibah University for Science, vol. 9, pp. 95–103, 2015.