He-Laplace Method for the Solutions of the Navier-Stokes Model

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***Abstract*— In this paper, He-Laplace method: a blend of Laplace transformation and Homotopy perturbation method via He's polynomials is applied for the solutions of the Navier-Stokes model. The solutions are in series form with easily computable components. This blended method appears to be very flexible, effective, efficient and reliable because it provides the exact solution of the solved problem with less computational work, while still maintaining high level of accuracy. Identification of Lagrange multipliers is not required. Hence, the proposed method is recommended for handling linear and nonlinear models of higher orders.**

***Index Terms*— Analytical solutions; Laplace transform; HPM; Navier-Stokes model.**

I. Introduction

In physical sciences: mathematics, engineering, computational fluid dynamics and other areas of pure and applied sciences; Navier-Stokes equations (NSEs) serve as vital models in the description of motion of viscous fluid substances. NSEs relate pressure and external forces acting on fluid to the response of the fluid flow [1]. In general form, the Navier-Stokes and continuity equations are given by:

$$\begin{cases} \dfrac{\partial w}{\partial t}+\left(\underline{w}\bullet\nabla\right)\underline{w}=-\rho^{-1}\nabla P+v\nabla^{2}\underline{w}, \\ \nabla\bullet\underline{w}=0, \end{cases} \tag{1.1}$$

where w is the flow velocity, $\underline{w}$ is the velocity, v is the kinematics viscosity, P is the pressure, t is the time, ρ is the density, and ∇ is a del operator. For a one dimensional motion of a viscous fluid in a tube; the equations of motion governing the flow field in the tube are Navier-Stokes equations in cylindrical coordinates [1, 2]. These are denoted by:

$$\frac{\partial w}{\partial t}-P=v\left(\frac{\partial^{2}w}{\partial\eta^{2}}+\frac{1}{\eta}\frac{\partial w}{\partial\eta}\right),\quad w(\eta,0)=g(\eta), \tag{1.2}$$

where $P=-\dfrac{\partial P}{\rho\partial z}$.

Providing solutions (numerical or exact) to linear and nonlinear differential equations has led to the development and adoption of direct and semi-analytical methods [3-8].

The Homotopy Perturbation Method (HPM) stands out for its simplicity in overcoming the difficulties involved in calculating the nonlinear terms [9]. This has wider applications when dealing with models in applied sciences [10-16]. Ghorbani *et al.* [17, 18] modified the HPM by introducing the He's polynomials where nonlinear terms were split into series of polynomials. The He's polynomials are in agreement with Adomian's polynomials, yet it is remarked that the He's polynomials can be computed easily.

Recently, many researchers have been combining (hybridizing) solution methods for simplicity, fast convergence rate and so on. These include Laplace Adomian Decomposition Method (LADM), Laplace HPM, Laplace DTM and so on [19, 20]. For the solutions of the NSEs, some of the semi-analytical methods have been applied [1-3, 21-23].

In this work, our aim is to provide analytical solutions to the NSEs using the He-Laplace method which combines the basic features of the Laplace transform and those of He's polynomials method.

II. The Overview of the Method [17, 18, 25]

A. *The He's Method*

Let Ξ be an integral or a differential operator on the equation of the form:

$$\Xi\left(\Im\right)=0. \tag{2.1}$$

Let $H\left(\Im,p\right)$ be a convex homotopy defined by:

$$H\left(\Im,p\right)=p\Xi\left(\Im\right)+\left(1-p\right)G\left(\Im\right), \tag{2.2}$$

where $G\left(\Im\right)$ is a functional operator with $\Im_0$ is a known solution. Thus, we have:

$$H\left(\Im,0\right)=G\left(\Im\right) \text{ and } H\left(\Im,1\right)=\Xi\left(\Im\right), \tag{2.3}$$

whenever $H\left(\Im,p\right)=0$ is satisfied, and $p\in\left(0,1\right]$ is an embedded parameter. In HPM, p is used as an expanding parameter to obtain:

Manuscript received March 13, 2017. Revised April 15, 2017.
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$$\Im = \sum_{j=0}^{\infty} p^j \Im_j = \Im_0 + p\Im_1 + p^2\Im_2 + \cdots. \quad (2.4)$$

From (2.4) the solution is obtained as $p \to 1$. The convergence of (2.4) as $p \to 1$ has been considered in [26]. The method considers $N(\Im)$ (the nonlinear term) as:

$$N(\Im) = \sum_{j=0}^{\infty} p^j H_j\,, \quad (2.5)$$

where $H_k\,'s$ are the so-called He's polynomials, which can be computed using:

$$H(\Im) = \frac{1}{i!}\frac{\partial^i}{\partial p^i}\left(N\left(\sum_{j=0}^{i} p^j \Im_j \right)\right)_{p=0}, \; n \geq 0, \quad (2.6)$$

where $H(\Im) = H_i(\Im_0, \Im_1, \Im_2, \Im_3, \cdots, \Im_i)$.

B. The He-Laplace Method

Let $\hbar(y', y, x) = f(x)$ expressed as:

$$y' + p_1 y + p_2 g(y) = g(x), \; y(0) = \beta \quad (2.7)$$

be a first order initial value problem (IVP), where $p_1(x)$ and $p_2(x)$ are coefficient of y and $g(y)$ respectively, $g(y)$ a nonlinear function and $g(x)$ a source term. Suppose we define the Laplace transform (resp. inverse Laplace transform) as $\tilde{L}\{(\cdot)\}$ $(resp.\ \tilde{L}^{-1}\{(\cdot)\})$. So the Laplace transform of (2.7) is as follows:

$$\tilde{L}\{y'\} + \tilde{L}\{p_1 y\} + \tilde{L}\{p_2 g(y)\} = \tilde{L}\{g(x)\}. \quad (2.8)$$

Applying linearity property of Laplace transform on (2.8) yields:

$$\tilde{L}\{y'\} + p_1\tilde{L}\{y\} + p_2\tilde{L}\{g(y)\} = \tilde{L}\{g(x)\}. \quad (2.9)$$

Therefore, by differential property of Laplace transform, (2.9) is expressed as follows:

$$\begin{cases} s\tilde{L}\{y\} - y(0) = \tilde{L}\{g(x)\} \\ \qquad\qquad -\left(p_1\tilde{L}\{y\} + p_2\tilde{L}\{g(y)\}\right). \end{cases} \quad (2.10)$$

$\Rightarrow$

$$\tilde{L}\{y\} = \frac{y(0)}{(s+p_1)} + \frac{1}{(s+p_1)}\left[\tilde{L}\{g(x)\} - p_2\tilde{L}\{g(y)\}\right]. \quad (2.11)$$

Thus, by inverse Laplace transform, (2.11) becomes:

$$y(x) = H(x) + \tilde{L}^{-1}\left(\frac{1}{(s+p_1)}\left[\tilde{L}\{g(x)\} - p_2\tilde{L}\{g(y)\}\right]\right), \quad (2.12)$$

$$\tilde{L}^{-1}\left(\frac{\beta}{(s+p_1)}\right) = H(x). \quad (2.13)$$

Suppose the solution $y(x)$ assumes an infinite series, then by convex homotopy, (2.12) can be expressed as:

$$\sum_{i=0}^{\infty} p^i y_i = z(x) + \tilde{L}^{-1}\left(\frac{1}{(s+p_1)}\left[\tilde{L}\{g(x)\} - p_2 p\tilde{L}\left\{\sum_{i=0}^{\infty} p^i H_i(y)\right\}\right]\right), \quad (2.14)$$

where $g(y) = \sum_{i=0}^{\infty} p^i H_i(y)$ for some He's polynomials H_i, and p an expanding parameter as defined earlier.

III. The He-Laplace Method Applied

In this subsection, the He-Laplace approach is applied to the Navier-Stokes model as follows:

*A. **Application:** Consider the following Navier-Stokes model:*

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial \eta^2} + \dfrac{1}{\eta}\dfrac{\partial w}{\partial \eta}, \\ w(\eta, 0) = \eta. \end{cases} \quad (3.1)$$

Procedure w.r.t Application:

We take the Laplace transform (LT) of (3.1) as follows:

$$\tilde{L}\left\{ \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} \right\}. \quad (3.2)$$

$$\Rightarrow \quad \tilde{L}\{w\} = \frac{w(0)}{s} + \frac{1}{s}\tilde{L}\left\{ \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} \right\}. \quad (3.3)$$

By applying the inverse Laplace transform, $\tilde{L}^{-1}\{(\cdot)\}$ of $\tilde{L}\{(\cdot)\}$ on both sides of (3.3), we have:

$$w = w(\eta, t) = \tilde{L}^{-1}\left\{\frac{\eta}{s}\right\} + \tilde{L}^{-1}\left\{ \frac{1}{s}\tilde{L}\left\{ \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} \right\}\right\}$$

$$= \eta + \tilde{L}^{-1}\left\{ \frac{1}{s}\tilde{L}\left\{ \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} \right\}\right\}. \quad (3.4)$$

By Convex Homotopy Approach (CHA), (3.4) becomes:

$$\sum_{i=0}^{\infty} p^n w_i = \eta + \tilde{L}^{-1}\left\{ \frac{1}{s}\tilde{L}\left\{ \sum_{i}^{\infty} p^{i+1}\left(\frac{\partial^2 w_i}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w_i}{\partial \eta} \right)\right\}\right\}. \quad (3.5)$$

Thus, comparing the coefficients of the p powers in (3.5) gives:

$$p^{(0)} : w_0 = \eta\,,$$

$$p^{(1)} : w_1 = \tilde{L}^{-1}\left\{ \frac{1}{s}\tilde{L}\left\{ \frac{\partial^2 w_0}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w_0}{\partial \eta} \right\}\right\},$$

$$p^{(2)} : w_2 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 w_1}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial w_1}{\partial\eta}\right\}\right\},$$

$$p^{(3)} : w_3 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 w_2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial w_2}{\partial\eta}\right\}\right\},$$

$$\vdots$$

$$p^{(k)} : w_k = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 w_{n-1}}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial w_{n-1}}{\partial\eta}\right\}\right\}, n \ge 1.$$

So, the values of $w_1, w_2, w_3, \cdots$, via $w_0 = \eta$ are as follows:

$$\begin{cases} w_0 = \eta, w_1 = \dfrac{t}{\eta}, w_2 = \dfrac{1}{2}\dfrac{t^2}{\eta^3}, w_3 = \dfrac{3}{2}\dfrac{t^3}{\eta^5}, w_4 = \dfrac{75}{8}\dfrac{t^4}{\eta^7}, \\ w_5 = \dfrac{735}{8}\dfrac{t^5}{\eta^9}, w_6 = \dfrac{19845}{16}\dfrac{t^6}{\eta^{11}}, w_7 = \dfrac{343035}{16}\dfrac{t^7}{\eta^{13}}, \cdots \end{cases}$$

$\therefore$

$$w(x,t) = \eta + \frac{t}{\eta} + \frac{1}{2}\frac{t^2}{\eta^3} + \frac{3}{2}\frac{t^3}{\eta^5} + \frac{75}{8}\frac{t^4}{\eta^7} + \frac{735}{8}\frac{t^5}{\eta^9} + \frac{19845}{16}\frac{t^6}{\eta^{11}} + \frac{343035}{16}\frac{t^7}{\eta^{13}} + \cdots$$

$$= \eta + \sum_{\iota=1}^{\infty}\frac{1^1\times 3^2\times 5^2\times\cdots\times(2\iota-3)^2}{\eta^{2\iota-1}}\frac{t^\iota}{\iota!}. \quad (3.6)$$

Note: For graphical purpose of the approximate solution, we use $\eta \in [0, 0.5]$ and $t \in [0, 0.2]$. Figure 1 and Figure 2 below represent the 3D plots of the solution for terms up to power seven and power five (in terms of the time variable t) respectively.

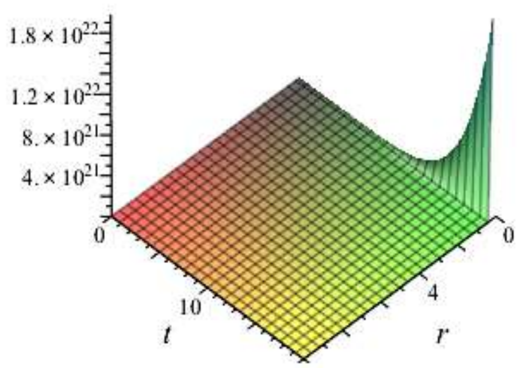

Fig 1: Solution via He-Laplace Method up to term t^7.

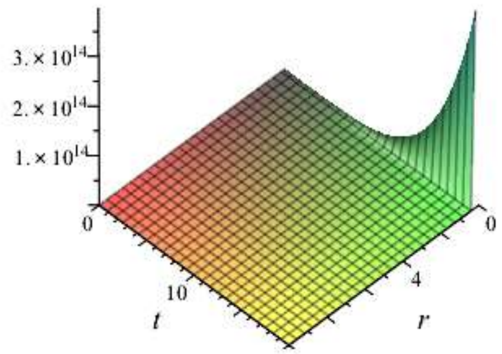

Fig 2: Solution via He-Laplace Method up to term t^5.

IV. Concluding Remarks

In this paper, He-Laplace method has been implemented for the solutions of the Navier-Stokes model. The solutions were in series form with easily computable components. This proposed method appeared to be very flexible, effective, efficient and reliable because it provides the exact solution of the solved problem with less computational work, while still maintaining high level of accuracy. Identification of Lagrange multipliers is not required. Hence, the proposed method is recommended for handling linear and nonlinear models of higher orders. Numerical computations, and graphics done in this work, are through *Maple* 18.

ACKNOWLEDGEMENT

The authors wish to thank Covenant University for financial support and provision of good working environment. Also, sincere thanks to the anonymous referee(s) for constructive and helpful suggestions.

AUTHOR CONTRIBUTIONS

All the authors contributed meaningfully and positively to this work, read and approved the final manuscript for publication.

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CONFLICT OF INTERESTS

The authors declare no conflict of interest as regards this paper.

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