Method of Homogenization for the Study of the Propagation of Electromagnetic Waves in a Composite Part 1: Modeling, Scaling, Existence and Uniqueness Results

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***Abstract*—The purpose of this article is to study behavior of the electromagnetic field in 3D in and near composite material. For this, time-harmonic Maxwell equations, for a conducting two-phase composite and the air above, are considered. This paper is considered a first part in two parts study. In this part we give the setting of the problem, and we propose a rescaling of time-harmonic Maxwell system. then with a view to the homogenization, we demonstrate the uniqueness and the existence of a solution as well as an estimate.**

***Index Terms*—Harmonic Maxwell Equations; Electromagnetic Pulses, Electromagnetism; Homogenization; Asymptotic Analysis; Asymptotic Expansion; Two-scale Convergence; Effective Behavior; Frequencies; Composite Material.**

I. Introduction

We are interested in the time-harmonic Maxwell equations in and near a composite material with boundary conditions modeling electromagnetic field radiated by an electromagnetic pulse (EMP). An electromagnetic pulse is a short burst of electromagnetic energy. It may be generated by a natural occurrence such like a lightning strike. We study the electromagnetic pulse caused by this lightning strike and what happens over a period of time of a millisecond during the peak of the first return stroke.

EMP interference is generally damaging to electronic equipment. A lightning strike can damage physical objects such as aircraft structures, either through heating effects or disruptive effects of the very large magnetic field generated by the current. Structures and systems require some form of protection against lightning. Every commercial aircraft is struck by lightning at least once a year in average. With the increasing of use of composite materials, up to 53% for the latest Airbus A350, and 50% for the Boeing B787, aircrafts offer increased vulnerability facing lightning. Earlier generation aircrafts, whose fuselages were predominantly composed of aluminum, behave like a Faraday cage and offer maximum protection for the internal equipment. For these reasons, aircraft manufacturers are very sensitive to lightning protection and pay special attention to aircraft certification through testing and analysis.

We evaluate the electromagnetic field within and near a periodic structure when the period of this microstructure is small compared to the wavelength of the electromagnetic wave. Our model is composed by air above the composite fuselage and we study the behavior of the electromagnetic wave in the domain filled by the composite material, representing the skin aircraft, and the air. We build the 3D model, under simplifying assumptions, using linear time-harmonic Maxwell equations and constitutive relations for electric and magnetic fields. Composite materials consist of conducting carbon fibers, distributed as periodic inclusions in a matrix (epoxy resin). We impose a magnetic permeability μ_0 uniform and an electrical permittivity $\epsilon = \epsilon_0\epsilon^\star$, where $\epsilon^\star$ is the relative permittivity depending of the medium.

We account for some characteristic values, we focus on the boundary conditions as we consider them as the source. Then, we use on the upper frontier, the magnetic field induced by the peak of the current of the first return stroke

$$\overline{H_d} = \frac{I}{2\pi r}, \tag{1}$$

with current intensity $I = 200$ kA and r the radius of the lightning strike, this is the worst aggression that can suffer an aircraft, and we deduce a characteristic electric field $\overline{E} = 20$ kV/m. In our model we consider that we have very conductive - but not perfect conductors - carbon fibers and an epoxy resin whose conduction depends on its doping rate. The conductivity of the air is non-linear. Air is a strong insulator [1] with conductivity of the order of $10^{-14}\ S.m^{-1}$ but beyond some electric solicitation, the air loses its insulating nature and locally becomes suddenly conductive. The ionization phenomenon is the only cause that can make the air conductor of electricity. The ionized channel becomes very conductive.

On of the parameter we account for in our model: $\delta = \frac{1}{\sqrt{\overline{\omega}\ \overline{\sigma}\mu_0}}$, where $\overline{\sigma}$ is the characteristic conductivity and $\overline{\omega}$ the order of the magnitude of the pulsation shares much with the definition of theoretical thickness skin $\delta = \sqrt{\frac{2}{\overline{\omega}\overline{\sigma}\mu_0}}$. The thickness skin is the depth at which the surface current moves to a factor of e^{-1}. Indeed, at high frequency, the skin effect phenomenon appears because the current tends to concentrate at the periphery of the conductor. On the other side, at low frequencies, in our case, the penetration depth is much greater than the thickness of the plate which means that a part of the electric field penetrates the composite plate.

A. Notations and setting of the problem

We consider set $\widetilde{\Omega} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in R^3,\ \widetilde{y} \in (-\overline{L}, d)\}$ for $\overline{L}$ and d two positive constants, with two open subsets $\widetilde{\Omega}_a$ and $\widetilde{P}$. The air fills $\widetilde{\Omega}_a$ and we consider that the composite

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material, with two materials periodically distributed, stands in domain $\widetilde{P}$.

We assume that the thickness $\overline{L}$ of the composite material is much smaller than its horizontal size. We denote by e the lateral size of the basic cell $\widetilde{Y}^e$ of the periodic microstructure of the material. The cell is composed of a carbon fiber in the resin. We define now more precisely the material, introducing:

$\widetilde{P} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in R^3 / -\overline{L} < \widetilde{y} < 0\}$, which is the domain containing the material. Now we describe precisely the basic cell. For this we first introduce the following cylinder with square base:

$\widetilde{Z}^e = [-\frac{e}{2}, \frac{e}{2}] \times [-e, 0] \times R$, We consider α such that $0 < \alpha < 1$, and $\widetilde{R}^e = \alpha\frac{e}{2}$. We set

$\widetilde{D}^e = \{(\widetilde{x}, \widetilde{y}) \in R^2 / (\widetilde{x}^2 + (\widetilde{y} + \frac{e}{2})^2) < (\widetilde{R}^e)^2\}$. We define the cylinder containing the fiber as:

$\widetilde{C}^e = \widetilde{D}^e \times R$. Then the part of the basic cell containing the matrix is $\widetilde{Y}^e_R = \widetilde{Z}^e \setminus \widetilde{C}^e$, and by definition, the basic cell $\widetilde{Y}^e$ is the couple $(\widetilde{Y}^e_R, \widetilde{C}^e)$.

The composite material results from a periodic extension of the basic cell. More precisely the part of the material that contains the carbon fibers is $\widetilde{\Omega}_c = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{C}^e, i \in Z, j \in Z^-\}$, where the intersection with $\widetilde{P}$ limits the periodic extension to the area where stands the material. Set $\{(ie, je, 0) + \widetilde{C}^e, i \in Z, j \in Z^-\}$ is a short notation for $\{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in R^3, \exists i \in Z, \exists j \in Z^-, \exists (x_b, y_b, z_b) \in \widetilde{C}^e; \widetilde{x} = x_b + ie, \widetilde{y} = y_b + je, \widetilde{z} = z_b\}$. In the same way the part of the material that contains the resin is $\widetilde{\Omega}_r = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{Y}^e_R\}$, or equivalently $\widetilde{\Omega}_r = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{Z}^e \setminus \widetilde{C}^e\} = (R \times (-\overline{L}, 0) \times R) \setminus \widetilde{\Omega}_c$.

So the geometrical model of our composite material is couple $(\widetilde{\Omega}_c, \widetilde{\Omega}_r)$. Now, it remains to set the domain that contains the air: $\widetilde{\Omega}_a = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) / 0 \leq \widetilde{y} < d\}$. We consider that d is of the same order as $\overline{L}$ and we introduce the upper frontier $\widetilde{\Gamma}_d = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) / \widetilde{y} = d\}$ of domain $\widetilde{\Omega}$. On this frontier we will consider that the electric field and magnetic field are given. We also introduce the lower frontier $\widetilde{\Gamma}_L = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) / \widetilde{y} = -\overline{L}\}$ with those definitions we have $\widetilde{\Omega}_a \cap \widetilde{P} = \emptyset$, $\widetilde{\Omega}_c \cap \widetilde{\Omega}_r = \emptyset$, $\widetilde{P} = \Omega_r \cup \Omega_c$, $\widetilde{\Omega} = \Omega_a \cup \widetilde{P} = \Omega_a \cup \Omega_r \cup \Omega_c$, and for any $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \partial\widetilde{\Omega} = \widetilde{\Gamma}_d \cup \widetilde{\Gamma}_L$ and, we write $\widetilde{n}$, the unit vector, orthogonal to $\partial\widetilde{\Omega}$ and pointing outside $\widetilde{\Omega}$. We have : $\widetilde{n} = e_2 \;\; on \;\; \widetilde{\Gamma}_d$

$\widetilde{n} = -e_2 \;\; on \;\; \widetilde{\Gamma}_L$.

In the following we need to describe what happens at the interfaces between resin and carbon fibers, and resin and air. So we define $\Gamma_{ra} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \;/\; \widetilde{y} = 0\}$ and Γ_{cr} the interface between the resin and the carbon fiber.

B. Time-harmonic Maxwell equations

We consider the harmonic version of the Maxwell equations which describe the electromagnetic radiation, they are written:

$$\left\{\begin{array}{ll} \nabla\times\widetilde{H} - i\widetilde{\omega}\epsilon_0\epsilon^\star\widetilde{E} = \sigma\widetilde{E}, & \text{Maxwell - Ampere equation} \\ \nabla\times\widetilde{E} + i\widetilde{\omega}\mu_0\widetilde{H} = 0, & \text{Maxwell - Faraday equation} \\ \nabla\cdot(\epsilon_0\epsilon^\star\widetilde{E}) = \widetilde{\rho}, & \\ \nabla\cdot(\mu_0\widetilde{H}) = 0, & \end{array}\right. \tag{2}$$

where $\widetilde{E}(t, \widetilde{x}, \widetilde{y}, \widetilde{z}) = \Re e(\widetilde{E}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \exp^{i\widetilde{\omega}t})$ and $\widetilde{H}(t, \widetilde{x}, \widetilde{y}, \widetilde{z}) = \Re e(\widetilde{H}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \exp^{i\widetilde{\omega}t})$, $\forall t \in R+$, $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \widetilde{\Omega}$, μ_0 and ϵ_0 are the permeability and permittivity of free space. $\epsilon^\star$ is the relative permittivity of the domains defined by

$$\epsilon^\star_{|\widetilde{\Omega_a}} = 1, \epsilon^\star_{|\widetilde{\Omega_r}} = \epsilon_r, \epsilon^\star_{|\widetilde{\Omega_c}} = \epsilon_c, \tag{3}$$

where ϵ_r and ϵ_c are positives constants. And σ is the electric conductivity. Its value depends on the location: $\sigma_{|\widetilde{\Omega_a}} = \sigma_a, \sigma_{|\widetilde{\Omega_r}} = \sigma_r, \sigma_{|\widetilde{\Omega_c}} = \sigma_c$, where $\widetilde{\Omega_a}$, $\widetilde{\Omega_r}$ and $\widetilde{\Omega_c}$ were defined in the first paragraph. The magnetic field $\widetilde{H}$ can be directly computed from the electric field $\widetilde{E}$

$$\widetilde{H} = -\frac{1}{i\omega\mu_0}\nabla\times\widetilde{E}. \tag{4}$$

Inserting $\nabla\times\widetilde{H}$ in Maxwell - Faraday equation we get the following equation for the electric field:

$$\nabla\times\nabla\times\widetilde{E} + (-\widetilde{\omega}^2\mu_0\epsilon_0\epsilon^\star + i\widetilde{\omega}\mu_0\sigma)\widetilde{E} = 0 \text{ in } \widetilde{\Omega}. \tag{5}$$

Taking the divergence of the equation Maxwell - Ampere equation yields the natural gauge condition:

$$\nabla\cdot[(i\widetilde{\omega}\epsilon_0\epsilon^\star + \sigma)\widetilde{E}] = 0 \text{in} \widetilde{\Omega}. \tag{6}$$

Notice that $i\widetilde{\omega}\epsilon_0 + \sigma$ is equal to $i\widetilde{\omega}\epsilon_0 + \sigma_a$ in $\widetilde{\Omega_a}$, to $i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r$ in $\widetilde{\Omega_r}$ and to $i\widetilde{\omega}\epsilon_0\epsilon_c + \sigma_c$ in $\widetilde{\Omega_c}$, those quantities being all nonzero. Then (6) is equivalent to:

$$\nabla\cdot\widetilde{E}_{|\Omega_a} = 0 \text{ in } \widetilde{\Omega_a}, \;\; \nabla\cdot\widetilde{E}_{|\Omega_r} = 0 \text{ in } \widetilde{\Omega_r}, \;\; \nabla\cdot\widetilde{E}_{|\Omega_c} = 0 \text{ in } \widetilde{\Omega_c}. \tag{7}$$

with the transmission conditions

$$\left\{\begin{array}{l} (i\widetilde{\omega}\epsilon_0 + \sigma_a)\widetilde{E}_{|\widetilde{\Omega_a}}.\widetilde{n} = (i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r)\widetilde{E}_{|\widetilde{\Omega_r}}.\widetilde{n} \;\text{ on }\; \widetilde{\Gamma_{ra}}, \\ (i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r)\widetilde{E}_{|\widetilde{\Omega_r}}.\widetilde{n} = (i\widetilde{\omega}\epsilon_0\epsilon_c + \sigma_c)\widetilde{E}_{|\widetilde{\Omega_c}}.\widetilde{n} \;\text{ on }\; \widetilde{\Gamma_{cr}}. \end{array}\right. \tag{8}$$

Summarizing, we finally obtain the PDE model:

$$\nabla\times\nabla\times\widetilde{E} + (-\widetilde{\omega}^2\mu_0\epsilon_0\epsilon^\star + i\widetilde{\omega}\mu_0\sigma)\widetilde{E} = 0 \text{ in } \widetilde{\Omega}. \tag{9}$$

We have to set boundary conditions on $\widetilde{\Gamma_d}$ and $\widetilde{\Gamma_L}$. On $\widetilde{\Gamma_d}$ we will write conditions that translate that $\widetilde{E}$ and $\widetilde{H}$ are given by the source located in $\widetilde{y} = d$. The way we chose consists in setting:

$$\widetilde{E} \times \widetilde{n} = \widetilde{E}_d \times \widetilde{n}; \quad \widetilde{H} \times \widetilde{n} = \widetilde{H}_d \times \widetilde{n} \quad \text{on} \widetilde{\Gamma_d}, \tag{10}$$

where $\widetilde{E}^\star_d$, $\widetilde{H}^\star_d$ are functions defined on $\widetilde{\Gamma}_d$ for any $t \in R$. On $\widetilde{\Gamma}_L$, we chose something simple, i.e :

$$\nabla\times\widetilde{E} \times \widetilde{n} = 0 \quad \text{on} \widetilde{\Gamma_L}, \tag{11}$$

that translate that $\widetilde{E}$ does not vary in the $\widetilde{y}$-direction near $\widetilde{\Gamma}_L$. According to the tangential trace of the Maxwell-Faraday equation (2) we obviously obtain that using boundary condition (10), is equivalent to using:

$$\nabla\times\widetilde{E} \times e_2 = -i\widetilde{\omega}\mu_0\widetilde{H}_d(\widetilde{x}, \widetilde{z}) \times e_2 \quad \text{on} \widetilde{\Gamma_d}. \tag{12}$$

And on $\widetilde{\Gamma_L}$ we have the following boundary condition:

$$\nabla\times\widetilde{E} \times e_2 = 0 \quad \text{on} \widetilde{\Gamma_L}. \tag{13}$$

C. Scaling

In this subsection we propose a rescaling of system ((9)-(13)), we will consider a set of characteristic sizes related to our problem. Physical factors are then rewritten using those values leading to a new set of dimensionless and unitless variables and fields in which the system is rewritten. The considered characteristic sizes are : $\overline{\omega}$ the characteristic pulsation, $\overline{\sigma}$ the characteristic electric conductivity, $\overline{E}$ the characteristic electric magnitude, $\overline{H}$ the characteristic magnetic magnitude. We also use the already introduced thickness $\overline{L}$ of the plate $\widetilde{P}$. We then introduce the dimensionless variables: $\mathbf{x} = (x, y, z)$ with $x = \frac{\widetilde{x}}{\overline{L}}$, $y = \frac{\widetilde{y}}{\overline{L}}$, $z = \frac{\widetilde{z}}{\overline{L}}$ and fields E, H and σ that are such that

$$\left\{ \begin{array}{l} E(\omega, \mathbf{x}) = \frac{1}{\overline{E}} \widetilde{E}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \\ H(\omega, \mathbf{x}) = \frac{1}{\overline{H}} \widetilde{H}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \\ \sigma(\mathbf{x}) = \frac{1}{\overline{\sigma}} \widetilde{\sigma}(\overline{L}x, \overline{L}y, \overline{L}z), \end{array} \right. \tag{14}$$

Taking (I-B) into account, σ also reads:

$$\left\{ \begin{array}{lll} \sigma(\mathbf{x}) = \frac{\sigma_a}{\overline{\sigma}} & if & 0 \leq \overline{L}y \leq d, \\ \sigma(\mathbf{x}) = \frac{\sigma_r}{\overline{\sigma}} & if & (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_r, \\ \sigma(\mathbf{x}) = \frac{\sigma_c}{\overline{\sigma}} & if & (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_c. \end{array} \right. \tag{15}$$

Doing this gives the status of units to the characteristic sizes. Since, for instance:

$$\frac{\partial E}{\partial x}(\omega, \mathbf{x}) = \frac{\overline{L}}{\overline{E}} \frac{\partial \widetilde{E}}{\partial \widetilde{x}}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \tag{16}$$

using those dimensionless variables and fields and taking (15)-(15) into account, equation (9) gives:

$$\begin{array}{l} \overline{E}\ \nabla\times\nabla\times E(\omega, \mathbf{x}) - \big(\frac{\overline{L}^2\overline{\omega}^2}{c^2}\epsilon^\star\omega^2 \\ +i\overline{\sigma}\ \overline{\omega}\ \omega\overline{L}^2\mu_0\sigma(\mathbf{x}, \omega)\big)\overline{E}E(\omega, x, y, z) = 0, \end{array} \tag{16}$$

for any $(\omega, \mathbf{x})$ such that $(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}$. Now we exhibit

$$\overline{\lambda} = \frac{2\pi c}{\overline{\omega}}, \tag{17}$$

which is the characteristic wave length and

$$\overline{\delta} = \frac{1}{\sqrt{\overline{\omega}\ \overline{\sigma}\mu_0}}, \tag{18}$$

which is the characteristic skin thickness. Using those quantities equation (I-C) reads, for any $(\omega, \mathbf{x}) \in \Omega$:

$$\left\{ \begin{array}{l} \nabla\times\nabla\times E(\omega, \mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_a}{\overline{\sigma}}\omega)E(\omega, \mathbf{x}) = 0 \\ \text{when} \quad 0 \leq \overline{L}y \leq d, \\ \nabla\times\nabla\times E(\omega, \mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\epsilon_r\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_r}{\overline{\sigma}}\omega)E(\omega, \mathbf{x}) = 0 \\ \text{when} \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_r, \\ \nabla\times\nabla\times E(\omega, \mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\epsilon_c\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_c}{\overline{\sigma}}\omega)E(\omega, \mathbf{x}) = 0 \\ \text{when} \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_c. \end{array} \right. \tag{19}$$

In the following expressions, $\frac{\overline{L}}{\overline{\lambda}}$ and $\frac{\overline{L}}{\overline{\delta}}$ appearing in the equations above will be rewritten in terms of a small parameter ε.

The boundary conditions are written

$$\left\{ \begin{array}{l} \nabla\times E(\omega, \mathbf{x}) \times e_2 = -i\omega\overline{\omega}\mu_0\frac{\overline{L}}{\overline{E}}\widetilde{H_d}(\overline{L}x, \overline{L}z) \times e_2 \\ when \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Gamma_d}, \\ \nabla\times E(\omega, \mathbf{x}) \times e_2 = 0 \quad when \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Gamma_L}. \end{array} \right. \tag{20}$$

The characteristic thickness of the plate $\overline{L}$ is about 10^{-3}m and the size of the basic cell e is about 10^{-5}m. Since e is much smaller than the thickness of the plate $\overline{L}$, it is pertinent to assume the ratio $\frac{e}{\overline{L}}$ equals a small parameter ε:

$$\frac{e}{\overline{L}} \sim 10^{-2} = \varepsilon. \tag{21}$$

The lightning is seen as a low frequency phenomenon. Indeed, energy associated with radiation tracers and return stroke are mainly burn by low and very low frequencies (from 1kHz to 300kHz). Components of the frequency spectrum are however observed beyond 1GHz see [2]. In our study we will consider $\overline{\omega} = 10^6$rad/s), for medium frequency we set $\overline{\omega} = 10^{10}$ rad/s and for high frequency phenomena $\overline{\omega} = 10^{12}$ rad/s. Then, concerning the characteristic electric conductivity it seems to be reasonable to take for $\overline{\sigma}$ the value of the effective electric conductivity of the composite material. Yet this choice implies to compute a coarse estimate of this effective conductivity at this level.

For this we take into account that the composite material is composed of carbon fibers and epoxy resin. In our model, the resin can be doped, which increases strongly its conductivity. We also account for the fact there is not only one effective electric conductivity but a first one in the fiber direction : the effective longitudinal electric conductivity , and a second effective electric conductivity, in the direction transverse to the fibers. In this context, we consider the basic model which is based on the electrical analogy and the law of mixtures. It corresponds to the Wiener limits: the harmonic average and the arithmetic average. The effective values are the extreme limits of the conductivity of the composite introduced by Wiener in 1912 see S. Berthier p 76 [3].
The effective longitudinal electric conductivity corresponding of the upper Wiener limit is expressed by the equation:

$$\overline{\sigma} = \sigma_{\text{long}} = f_c\ \sigma_c + (1 - f_c)\ \sigma_r, \tag{22}$$

where $f_c = \pi\frac{\alpha^2}{4}$ is the volume fraction of the carbon fiber.

The effective transverse electric conductivity corresponding of the lower Wiener limit is expressed by

$$\overline{\sigma} = \sigma_{\text{trans}} = \frac{1}{\frac{f_c}{\sigma_c} + \frac{(1-f_c)}{\sigma_r}}. \tag{23}$$

For the computation, we take values close to reality. We consider composite materials with similar proportions of carbon and resin, this means that α is close to $\frac{1}{2}$. When the resin is not doped $\sigma_r \sim 10^{-10}S.m^{-1}$ is much smaller than $\sigma_c \sim 40000S.m^{-1}$. Then, $\overline{\sigma} = \sigma_{\text{long}}$ is close to $\pi\frac{\alpha^2}{4}\ \sigma_c \sim \sigma_c$ and $\overline{\sigma} = \sigma_{\text{trans}}$ is close to $\frac{\sigma_r}{(1-\pi\frac{\alpha^2}{4})} \sim \sigma_r$.

Now, we express the electric conductivity of the air in terms of $\overline{\sigma}$.We consider a situation with a ionized channel, so σ_a being $\sigma_{lightning} = 4242S.m^{-1}$ for an ionized lightning channel see [4]. In our model we perform the study for $\omega = 10^6\ rad.s^{-1}$, which corresponds to the air ionized, a resin doped and the effective longitudinal electric conductivity of the carbon fibers.

Now, we will discuss on the values of $\overline{E}$ and $\overline{\rho}$. It seems that the density of electrons in a ionized channel is about 10^{10} part.m^{-3}. Hence we take $\overline{\rho} = 10^{10}$. When the air is not ionized, the charge density is much smaller, and we choose: $\overline{\rho} = 1$.

For the boundary conditions, in our context, we consider the peak of the current of the first return stroke. Then the magnetic field magnitude $\overline{H}$ is $\overline{H_d}$ given by (1).

Then the dimensionless boundary conditions (12) writes:

$$\nabla\times E(\mathbf{x},\omega)\times e_2 = -i\omega\overline{\omega}\mu_0\frac{\overline{L}}{\overline{E}}\overline{H_d}H_d(x,z)\times e_2, \qquad (24)$$

where $\overline{H_d}H_d(x,z) = \widetilde{H_d}(\overline{L}x,\overline{L}z)$ and where $\overline{\omega}\mu_0\frac{\overline{L}}{\overline{E}}\overline{H_d}$ being of order 1, with the characteristic electric field $\overline{E} = 20$ kV/m.

From the physical spatial coordinates $(\widetilde{x},\widetilde{y},\widetilde{z}) \in \widetilde{\Omega}$ we define $\mathbf{y} = (\xi,\nu,\zeta)$ with $\xi = \frac{\widetilde{x}}{e}, \nu = \frac{\widetilde{y}}{e}, \zeta = \frac{\widetilde{z}}{e}$ or equivalently $\xi = \frac{x}{\varepsilon}, \nu = \frac{y}{\varepsilon}, \zeta = \frac{z}{\varepsilon}$. And we now introduce Y, the basic cell. It is built from: $Z = [-\frac{1}{2},\frac{1}{2}]\times[-1,0]\times R$ and the set $C = D\times R$ with the disc D defined by:

$$D = \{(\xi,\nu)\in R^2 \ /\xi^2 + (\nu+\frac{1}{2})^2 < R^2\}, \qquad (25)$$

and $R = \frac{\alpha}{2}$. The set Ω_c is then defined as:

$$\Omega_c = \{(i,j,0) + C, i\in Z, j\in Z^-\}. \qquad (26)$$

We denote Y_r as $Y_r = Z\backslash C$ and then the set

$$\Omega_r = \{(i,j,0) + Y_r, i\in Z, j\in Z^-\}. \qquad (27)$$

Then unit cell Y is defined as $Y = (Y_r, C)$. Finally, we define the domain Ω_a:

$$\Omega_a = \{\mathbf{y} = (\xi,\nu,\zeta) \ / \ \nu > 0\}. \qquad (28)$$

Using this, we will give a new expression of the sets in which the variables range in equations (19). We see the following:

$$(\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{\Omega}_r \Leftrightarrow \begin{cases} (\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{P}, \\ (\frac{\overline{L}}{e}x,\frac{\overline{L}}{e}y,\frac{\overline{L}}{e}z)\in\Omega_r, \end{cases} \qquad (29)$$

i.e.

$$(\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{\Omega}_r \Leftrightarrow \begin{cases} (\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{P}, \\ (\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon})\in\Omega_r. \end{cases} \qquad (30)$$

In the same way:

$$(\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{\Omega}_c \Leftrightarrow \begin{cases} (\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{P}, \\ (\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon})\in\Omega_c, \end{cases} \qquad (31)$$

and:

$$0\le\overline{L}y\le d \Leftrightarrow \begin{cases} y\in R^2 \\ \overline{L}y\le d, \end{cases} \qquad (32)$$

or

$$(\overline{L}x,\overline{L}y,\overline{L}z)\in\widetilde{\Omega}_a \Leftrightarrow \begin{cases} \overline{L}y\le d \\ (\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon})\in\Omega_a. \end{cases} \qquad (33)$$

We define:

$$\Sigma^\varepsilon(\mathbf{y}) = \Sigma^\varepsilon(\xi,\nu,\zeta) = \begin{cases} \Sigma_a^\varepsilon & \text{in } \Omega_a, \\ \Sigma_r^\varepsilon & \text{in } \Omega_r, \\ \Sigma_c^\varepsilon & \text{in } \Omega_c, \end{cases} \qquad (34)$$

where $\Sigma_a^\varepsilon = \frac{\sigma_a}{\overline{\sigma}}\frac{\overline{L}^2}{\delta^2}, \Sigma_r^\varepsilon = \frac{\sigma_r}{\overline{\sigma}}\frac{\overline{L}^2}{\delta^2}$ and $\Sigma_c^\varepsilon = \frac{\sigma_c}{\overline{\sigma}}\frac{\overline{L}^2}{\delta^2}$ have their expressions in term of ε. The detail of this expressions are in the article [5]. The model that we present is the case $\omega = 10^6\ rad.s^{-1}$, $\eta = 5$, $\Sigma_a^\varepsilon = \varepsilon$, $\Sigma_r^\varepsilon = \varepsilon^4$ and $\Sigma_c^\varepsilon = 1$.

Defining also mapping

$$\begin{array}{lll} \psi_\varepsilon: & R^3 & \to R^3 \\ & (x,y,z) & \mapsto (\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon}), \end{array} \qquad (35)$$

we can set Ω_a^ε as $\psi_\varepsilon^{-1}(\Omega_a)\cap(R\times[0,\frac{d}{\overline{L}}]\times R)$, Ω_r^ε as $\psi_\varepsilon^{-1}(\Omega_r)\cap\widetilde{P}$ and Ω_c^ε as $\psi_\varepsilon^{-1}(\Omega_c)\cap\widetilde{P}$. We also define the boundaries $\Gamma_d = \{\mathbf{x}\in R^3, \ \ y = \frac{d}{\overline{L}}\}$ and $\Gamma_L = \{\mathbf{x}\in R^3, \ \ y = -\overline{L}\}$ and interfaces $\Gamma_{ra} = \{\mathbf{x}\in R^3, \ \ y = 0\}$ and $\Gamma_{cr}^\varepsilon = \partial\Omega_c$. Hence equation (19) reads:

$$\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon^\star + i\,\omega\,\sigma^\varepsilon(x,y,z))E^\varepsilon = 0 \text{ in } \Omega, \quad (36)$$

where $\Omega = \Omega_a^\varepsilon\cup\Omega_r^\varepsilon\cup\Omega_c^\varepsilon = \{\mathbf{x}\in R^3, -1 < y < \frac{d}{\overline{L}}\}$ does not depend on ε. Only its partition in Ω_a^ε, Ω_r^ε and Ω_c^ε is ε-dependent where

$$\sigma^\varepsilon(x,y,z) = \Sigma^\varepsilon(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon}), \qquad (37)$$

with Σ^ε given by (34) and

$$\varepsilon^\eta = \frac{4\pi^2\overline{L}^2}{\lambda^2}, \qquad (38)$$

we replace E by E^ε, to clearly state that it depends on ε.

Equation (36) is provided with the following boundary conditions:

$$\nabla\times E^\varepsilon\times e_2 = -i\omega H_d(x,z)\times e_2 \ \ on \ \ R\times\Gamma_d, \qquad (39)$$

coming from (24). And, coming from (20),

$$\nabla\times E^\varepsilon\times e_2 = 0 \ \text{ on } \ \Gamma_L. \qquad (40)$$

From (36) we can deduce the condition on the divergence of E^ε which can be written in two ways. As previously in (6), (7) and (8) we obtain:

$$\nabla\cdot[(-\omega^2\varepsilon^\eta\epsilon^\star + i\omega\sigma^\varepsilon)E^\varepsilon] = 0 \ \text{ in } \ \Omega, \qquad (41)$$

which will be preferentially used with (36) and its second one is

$$\nabla\cdot E^\varepsilon_{|\Omega_a^\varepsilon} = 0 \text{ in } \Omega_a^\varepsilon, \ \nabla\cdot E^\varepsilon_{|\Omega_r^\varepsilon} = 0 \text{ in } \Omega_r^\varepsilon, \ \nabla\cdot E^\varepsilon_{|\Omega_c^\varepsilon} = 0 \text{ in } \Omega_c^\varepsilon, \qquad (42)$$

with the transmission conditions on the interfaces Γ_{ra} and Γ_{cr}^ε

$$\begin{cases} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)\ E^\varepsilon_{|\Omega_a^\varepsilon}\cdot n_{|\Omega_a^\varepsilon} \\ = (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)\ E^\varepsilon_{|\Omega_r^\varepsilon}\cdot n_{|\Omega_r^\varepsilon} \ \text{ on } \ \Gamma_{ra}, \\ (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)\ E^\varepsilon_{|\Omega_r^\varepsilon}\cdot n_{|\Omega_r^\varepsilon} \\ = (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon)\ E^\varepsilon_{|\Omega_c^\varepsilon}\cdot n_{|\Omega_c^\varepsilon} \ \text{ on } \ \Gamma_{cr}^\varepsilon. \end{cases} \qquad (43)$$

Before treating mathematically the question we are interested in, we make a last simplification. Since it seems clear that physical relevant phenomena occur in the upper part of the plate. The boundary condition on the lower boundary of the plate has very little influence on the physics of what happens in the upper part, we consider that the lower boundary of Ω is located in $y = -\infty$ in place of $y = -1$, making the second boundary condition useless. Besides, as $\overline{L}$ and d are of the same order it seems reasonable to set $\Gamma_d = \{\mathbf{x}\in R^3, \ \ y = 1\}$ and consequently

$$\begin{cases} \Omega = \{\mathbf{x}\in R^3, y < 1\} = \Omega_a^\varepsilon\cup\Omega_r^\varepsilon\cup\Omega_c^\varepsilon, \ \text{ with,} \\ \Omega_a^\varepsilon = \psi_\varepsilon^{-1}(\Omega_a), \\ \Omega_r^\varepsilon = \psi_\varepsilon^{-1}(\Omega_r), \\ \Omega_c^\varepsilon = \psi_\varepsilon^{-1}(\Omega_c), \end{cases} \qquad (44)$$

with ψ_ε defined in (35). We have that the border of Ω is Γ_d. In the following section we will establish existence and uniqueness results.

II. Mathematical analysis of the models

A. Preliminaries

We are going to make precise the variational formulation. First of all, we need to introduce the following functional spaces. We have the standard function spaces $\mathbf{L}^2(\Omega^\varepsilon) = [L^2(\Omega^\varepsilon)]^3$

$$\begin{array}{l}\mathbf{H}(\text{curl},\Omega) = \{u \in \mathbf{L}^2(\Omega): \quad \nabla\times u \in \mathbf{L}^2(\Omega)\},\\ \mathbf{H}(\text{div},\Omega) = \{u \in \mathbf{L}^2(\Omega): \quad \nabla\cdot u \in L^2(\Omega)\},\end{array} \tag{45}$$

with the usual norms:

$$\begin{array}{l}\|u\|^2_{\mathbf{H}(\text{curl},\Omega)} = \|u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\times u\|^2_{\mathbf{L}^2(\Omega)},\\ \|u\|^2_{\mathbf{H}(\text{div},\Omega)} = \|u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\cdot u\|^2_{L^2(\Omega)}.\end{array} \tag{46}$$

They are well known Hilbert spaces.

We use the trace spaces $H^{-\frac{1}{2}}(\text{curl},\Gamma_d)$ and $H^{-\frac{1}{2}}(\text{div},\Gamma_d)$ defined by

$$\begin{array}{l}H^{-\frac{1}{2}}(\text{curl},\Gamma_d) = \{u \in H^{-\frac{1}{2}}(\Gamma_d, R^3),\\ \quad (n\cdot u)_{|\Gamma_d} = 0,\ \text{curl}_{\Gamma_d} u \in H^{-\frac{1}{2}}(\Gamma_d, R^3)\},\end{array} \tag{47}$$

$$\begin{array}{l}H^{-\frac{1}{2}}(\text{div},\Gamma_d) = \{u \in H^{-\frac{1}{2}}(\Gamma_d, R^3),\\ \quad (n\cdot u)_{|\Gamma_d} = 0,\ \text{div}_{\Gamma_d} u \in H^{-\frac{1}{2}}(\Gamma_d, R^3)\}\end{array} \tag{48}$$

where the surface divergence $\text{div}_{\Gamma_d} u$ and the surface rotation $\text{curl}_{\Gamma_d} u$ are defined by $\forall\ V \in C^1(\Gamma_d)$

$$\begin{array}{l}(\text{div}_{\Gamma_d} u, V)_{L^2(\Gamma_d)} = -(u, \nabla_{\Gamma_d} V)_{L^2(\Gamma_d, R^3)},\\ \text{curl}_{\Gamma_d} u = n\cdot(\nabla\times u_{|\Gamma_d})\end{array} \tag{49}$$

and the surface gradient $\nabla_{\Gamma_d} V$ is defined by the orthogonal projection of ∇ on Γ_d, n denotes the outward unit vector normal to Γ_d. Finally we recall the trace theorems, see J.C Nédélec [6] for the demonstration, stating that the traces mappings $\gamma_T : \mathbf{H}(\text{curl},\Omega) \longrightarrow H^{-\frac{1}{2}}(\text{curl},\Gamma_d)$, that assigns any $u \in \mathbf{H}(\text{curl},\Omega)$ its tangential components $n\times(u\times n)$ is continuous and surjective, that is:

$$\|\gamma_T(u)\|_{H^{-\frac{1}{2}}(\text{curl},\Gamma_d)} \le C_{\gamma_T}\|u\|_{\mathbf{H}(\text{curl},\Omega)},\ \forall u \in \mathbf{H}(\text{curl},\Omega) \tag{50}$$

$\gamma_t : \mathbf{H}(\text{curl},\Omega) \longrightarrow H^{-\frac{1}{2}}(\text{div},\Gamma_d)$, that assigns any $u \in \mathbf{H}(\text{curl},\Omega)$ its tangential components $u\times n$, is continuous and surjective:

$$\|\gamma_t(u)\|_{H^{-\frac{1}{2}}(\text{div},\Gamma_d)} \le C_{\gamma_t}\|u\|_{\mathbf{H}(\text{curl},\Omega)},\ \forall u \in \mathbf{H}(\text{curl},\Omega). \tag{51}$$

Moreover, $H^{-\frac{1}{2}}(\text{div},\Gamma_d)$ is the dual of $H^{-\frac{1}{2}}(\text{curl},\Gamma_d)$ and one has the Green's formula $\forall(u,V)\ \in \mathbf{H}(\text{curl},\Omega)$:

$$\textstyle\int_\Omega(\nabla\times u\cdot V - u\cdot\nabla\times V)dx = \langle u\times n, V_T\rangle_{\Gamma_d}. \tag{52}$$

We define the next space:

$$\begin{array}{rl}\mathbf{X}(\Omega) & = \{u \in \mathbf{H}(\text{curl},\Omega) \mid \nabla\cdot u_{|\Omega_a^\varepsilon} \in L^2(\Omega_a^\varepsilon),\\ & \nabla\cdot u_{|\Omega_r^\varepsilon} \in L^2(\Omega_r^\varepsilon),\ \nabla\cdot u_{|\Omega_c^\varepsilon} \in L^2(\Omega_c^\varepsilon)\}.\end{array} \tag{53}$$

Our variational space is:

$$\begin{array}{rl}\mathbf{X}^\varepsilon(\Omega) & = \{u \in \mathbf{X}(\Omega) \mid (-\omega^2\varepsilon^\eta + i\omega\sigma^\varepsilon_{|\Omega_a^\varepsilon})u_{|\Omega_a^\varepsilon}\cdot e_2\\ & = (-\omega^2\varepsilon^\eta\varepsilon_r + i\omega\sigma^\varepsilon_{|\Omega_r^\varepsilon})u_{|\Omega_r^\varepsilon}\cdot e_2,\\ & (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\sigma^\varepsilon_{|\Omega_r^\varepsilon})u_{|\Omega_r^\varepsilon}\cdot n^\varepsilon_{|\Omega_r^\varepsilon}\\ & = (-\omega^2\varepsilon^\eta\varepsilon_c + i\omega\sigma^\varepsilon_{|\Omega_c^\varepsilon})u_{|\Omega_c^\varepsilon}\cdot n^\varepsilon_{|\Omega_c^\varepsilon}.\end{array} \tag{54}$$

Finally

$$\begin{array}{rl}\mathbf{X}^\varepsilon(\Omega) & = \{u \in \mathbf{X}(\Omega) \mid (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)u_{|\Omega_a^\varepsilon}\cdot e_2 =\\ & (-\omega^2\varepsilon^\eta\varepsilon_r + i\omega\Sigma_r^\varepsilon)u_{|\Omega_r^\varepsilon}\cdot e_2,\\ & (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)u_{|\Omega_r^\varepsilon}\cdot n^\varepsilon_{|\Omega_r^\varepsilon}\\ & = (-\omega^2\varepsilon^\eta\varepsilon_c + i\omega\Sigma_c^\varepsilon)u_{|\Omega_c^\varepsilon}\cdot n^\varepsilon_{|\Omega_c^\varepsilon}\}.\end{array} \tag{55}$$

This space is equipped with the norm

$$\begin{array}{l}\|u\|^2_{\mathbf{X}^\varepsilon(\Omega)} = \|u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\cdot u_{|\Omega_a^\varepsilon}\|^2_{L^2(\Omega_a^\varepsilon)} + \|\nabla\cdot u_{|\Omega_r^\varepsilon}\|^2_{L^2(\Omega_r^\varepsilon)}\\ + \|\nabla\cdot u_{|\Omega_c^\varepsilon}\|^2_{L^2(\Omega_c^\varepsilon)} + \|\nabla\times u\|^2_{\mathbf{L}^2(\Omega)}.\end{array} \tag{56}$$

B. Weak formulation

Now, we introduce the variational formulation of our problem (36), (39) and (40) for the electric field. Integrating (36) over Ω and using the Green's formula and (39) we obtain

$$\left\{\begin{array}{l}\int_\Omega \nabla\times E^\varepsilon\cdot\nabla\times\overline{V}\ d\mathbf{x} + \int_{\Omega_a^\varepsilon}(-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)E^\varepsilon\cdot\overline{V}\ d\mathbf{x}\\ + \int_{\Omega_c^\varepsilon}(-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon)E^\varepsilon\cdot\overline{V}\ d\mathbf{x}\\ + \int_{\Omega_r^\varepsilon}(-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)E^\varepsilon\cdot\overline{V}\ d\mathbf{x}\\ = \int_{\Gamma_d}(\nabla\times E^\varepsilon\times e_2)\cdot\overline{V}_T\ d\sigma\\ = \int_{\Gamma_d} -i\omega H_d\times e_2\cdot\overline{V}_T\ d\sigma\end{array}\right. \tag{57}$$

where $\overline{V}$ is the complex conjugate of V and $V_T = (e_2\times V)\times e_2$. We introduce the sesquilinear form depending on parameters η and ε:

$$\left\{\begin{array}{l}\text{For } E^\varepsilon, V \in \mathbf{X}^\varepsilon(\Omega),\\ a^{\varepsilon,\eta}(E^\varepsilon, V) = \int_\Omega \nabla\times E^\varepsilon\cdot\nabla\times\overline{V}\ d\mathbf{x}\\ + \sum_{i=a,r,c}\int_{\Omega_i^\varepsilon}(-\omega^2\varepsilon^\eta\epsilon_i + i\omega\Sigma_i^\varepsilon)\ E^\varepsilon\cdot\overline{V}\ d\mathbf{x}.\end{array}\right. \tag{58}$$

Hence, the weak formulation of (36), (39) and (40) that we will use is the following, is:

$$\left\{\begin{array}{l}\text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \ \text{ such as }\ \forall\ \ V \in \mathbf{X}^\varepsilon(\Omega)\ \text{ we have :}\\ a^{\varepsilon,\eta}(E^\varepsilon, V) = -i\omega\int_{\Gamma_d} H_d\times e_2\cdot\overline{V}_T\ d\sigma.\end{array}\right. \tag{59}$$

Integrating by parts in the variational formulation (57), we find the following transmission problem:

$$\left\{\begin{array}{ll}\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta + i\ \omega\ \Sigma_a^\varepsilon)E^\varepsilon = 0 & \text{in } \ \Omega_a^\varepsilon,\\ \nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon_r + i\ \omega\ \Sigma_r^\varepsilon)E^\varepsilon = 0 & \text{in } \ \Omega_r^\varepsilon,\\ \nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon_c + i\ \omega\ \Sigma_c^\varepsilon)E^\varepsilon = 0 & \text{in } \ \Omega_c^\varepsilon.\\ E^\varepsilon_{|\Omega_a^\varepsilon}\times e_2 = E^\varepsilon_{|\Omega_r^\varepsilon}\times n_{|\Omega_r^\varepsilon} \ \text{ on } \ \Gamma_{ra}, & \\ E^\varepsilon_{|\Omega_r^\varepsilon}\times n_{|\Omega_r^\varepsilon} = E^\varepsilon_{|\Omega_c^\varepsilon}\times n_{|\Omega_c^\varepsilon} \ \text{ on } \ \Gamma_{cr}^\varepsilon, & \\ \nabla\times E^\varepsilon_{|\Omega_a^\varepsilon}\times e_2 = \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon}\times n_{|\Omega_r^\varepsilon} \ \text{ on } \ \Gamma_{ra}, & \\ \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon}\times n_{|\Omega_r^\varepsilon} = \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon}\times n_{|\Omega_c^\varepsilon} \ \text{ on } \ \Gamma_{cr}^\varepsilon, & \end{array}\right. \tag{60}$$

where e_2 is the unit outward normal to Ω_a^ε, $n_{|\Omega_r^\varepsilon}$ is the unit outward normal to Ω_r^ε and $n_{|\Omega_c^\varepsilon}$ is the unit outward normal to Ω_c^ε. We refer to [5] for the proof that transmission problem (60) is equivalent to ((36), (39), (40), (42)).

C. Regularized Maxwell equations for the electric field

The sesquilinear form $a^{\varepsilon,\eta}$ is not coercive on $\mathbf{X}^\varepsilon(\Omega)$, so we regularize it adding terms involving the divergence of E^ε in Ω_a^ε, Ω_r^ε and Ω_c^ε. Thanks to the additional terms, existence and uniqueness of the regularized variational formulation solution will be established by the Lax-Milgram theory. Let

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s be an arbitrary positive number, we define the regularized formulation of problem (59):

$$\left\{ \begin{array}{l} \text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \text{ such that for any } V \in \mathbf{X}^\varepsilon(\Omega) \\ a_R^{\varepsilon,\eta}(E^\varepsilon, V) = a^{\varepsilon,\eta}(E^\varepsilon, V) + s \int_{\Omega_a^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x} \\ +s \int_{\Omega_r^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x} + s \int_{\Omega_c^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x} \\ = -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot \overline{V}_T\ d\sigma. \end{array} \right. \tag{61}$$

For any $\varepsilon > 0$ and any $\eta \geq 0$, sesquilinear form $a_R^{\varepsilon,\eta}(.,.)$ is continuous over $\mathbf{X}^\varepsilon(\Omega)$ thanks to the continuity conditions. We will show that it is also coercive. The following proposition was inspired by article [7] Lemma 1.1.

Proposition 1: For any $\varepsilon > 0$, for any $\eta \geq 0$ and for any $s > 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, there exists a positive constant $\mathcal{C}_0$ depending on $\varepsilon_r, \varepsilon_c, s, \omega$ but not on ε such that $\forall\ E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega)$:

$$\Re[\exp(-i\tfrac{\pi}{4})\ a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)] \geq \mathcal{C}_0 \|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)}. \tag{62}$$

The proof is in [5].

D. Existence, uniqueness and estimate

Theorem 2: For any $\varepsilon > 0$, for any $\eta \geq 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, there exists a unique solution of (60) or ((36), (39), (40), (42)).

Proof: It is obvious that any solution of (60) or of ((36), (39), (40),(42)) is also solution to (61). Indeed, since from (60) or from ((36), (39), (40),(42)) we have $\nabla\cdot E^\varepsilon_{|\Omega_a^\varepsilon} = 0$, $\nabla\cdot E^\varepsilon_{|\Omega_r^\varepsilon} = 0$, $\nabla\cdot E^\varepsilon_{|\Omega_c^\varepsilon} = 0$, the additional terms $s \int_{\Omega_a^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x} + s \int_{\Omega_r^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x} + s \int_{\Omega_c^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot \overline{V}\ d\mathbf{x}$ cancel in (61).

Uniqueness follows from that if E_1^ε and E_2^ε are two solutions to (36) with the boundary condition (40) their difference $e^\varepsilon = E_2^\varepsilon - E_1^\varepsilon$ satisfies the problem (36) with (40). Then it comes

$$\begin{aligned} \int_\Omega |\nabla\times e^\varepsilon|^2\ d\mathbf{x} &+ \int_{\Omega_a^\varepsilon} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)|e^\varepsilon|^2\ d\mathbf{x} \\ &+ \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon)|e^\varepsilon|^2\ d\mathbf{x} \\ &+ \int_{\Omega_r^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)|e^\varepsilon|^2\ d\mathbf{x} \\ &= 0. \end{aligned} \tag{63}$$

Taking the imaginary part of the expression we get $\int_{\Omega_a^\varepsilon} \omega\Sigma_a^\varepsilon|e^\varepsilon|^2\ d\mathbf{x} + \int_{\Omega_c^\varepsilon} \omega\Sigma_c^\varepsilon|e^\varepsilon|^2\ d\mathbf{x} + \int_{\Omega_r^\varepsilon} \omega\Sigma_r^\varepsilon|e^\varepsilon|^2\ d\mathbf{x} = 0$ and then $e^\varepsilon = 0$.

Let us consider the reciprocal assertion, according to the same proof of S. Hassani, S. Nicaise, A. Maghnouji in [8], we define $H_0^1(\Omega_c^\varepsilon, \Delta)$ the subspace of $\psi \in H_0^1(\Omega_c^\varepsilon)$ such that $\Delta(\psi) \in L^2(\Omega_c^\varepsilon)$.

Let E^ε be the solution of the regularized formulation (61). In (61) we take a test function $V = \nabla\psi$ where $\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$, extended by zero outside Ω_c^ε. We get:

$$\begin{aligned} &\int_{\Omega_c^\varepsilon} s\nabla\cdot E^\varepsilon \nabla\cdot(\nabla\psi)\ d\mathbf{x} + \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_c \\ &+ i\omega\Sigma_c^\varepsilon) E^\varepsilon \cdot \nabla\psi\ d\mathbf{x} = 0. \end{aligned} \tag{64}$$

By Green's formula, $\forall\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$, we obtain:

$$\int_{\Omega_c^\varepsilon} \nabla\cdot E^\varepsilon (\Delta\psi + \tfrac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}\ \psi)\ \ d\mathbf{x} = 0. \tag{65}$$

Thus, if we choose s such that $\frac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_c^\varepsilon)$: the Laplacian operator in Ω_c^ε with Dirichlet condition on its boundary, then for all $\varphi \in L(\Omega_c^\varepsilon)^2$ there exists $\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$ solution of

$$\Delta\psi + \tfrac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}\ \psi = \varphi. \tag{66}$$

Then, we conclude that

$$\nabla \cdot E^\varepsilon_{|\Omega_c^\varepsilon} = 0. \tag{67}$$

A similar argument in Ω_a^ε yields $\nabla\cdot E^\varepsilon_{|\Omega_a^\varepsilon} = 0$ for s such that $\frac{\omega^2\varepsilon^\eta - i\omega\Sigma_a^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_a^\varepsilon)$. In the same way, we obtain in Ω_r^ε, $\nabla\cdot E^\varepsilon_{|\Omega_r^\varepsilon} = 0$ with s such that $\frac{\omega^2\varepsilon^\eta\epsilon_r - i\omega\Sigma_r^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_r^\varepsilon)$.

So, (61) becomes (57). Applying Green's formula, we find (36). ■

Theorem 3: Under the assumptions of Theorem 2, $E^\varepsilon \in X^\varepsilon(\Omega)$ solution of (61) satisfies

$$\|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)} \leq C \tag{68}$$

with $C = \frac{C_{\gamma_t} C_{\gamma_T}}{\mathcal{C}_0} \|H_d\|_{H(curl,\Omega)}$

The propositions and theorems above has been proved in [5].

Proof: The sesquilinear form $a_R^{\varepsilon,\eta}(E^\varepsilon, V)$ is coercive, weak formulation (61) becomes:

$$\begin{aligned} \mathcal{C}_0 \|E^\varepsilon\|^2_{\mathbf{X}^\varepsilon(\Omega)} &\leq \Re(\exp(-i\tfrac{\pi}{4}) a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)) \\ &\leq |\exp(-i\tfrac{\pi}{4}) \cdot a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)| = |a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)| \\ &\leq |\int_{\Gamma_d} -i\omega H_d \times e_2 \cdot E_T^\varepsilon\ \ d\sigma| \\ &\leq \|H_d \times e_2\|_{H^{-\frac{1}{2}}(\text{div},\Gamma_d)} \|E_T^\varepsilon\|_{H^{-\frac{1}{2}}(\text{curl},\Gamma_d)} \\ &\leq C_{\gamma_t} C_{\gamma_T} \|H_d \times e_2\|_{\mathbf{H}(\text{curl},\Omega)} \|E^\varepsilon\|_{\mathbf{H}(\text{curl},\Omega)} \end{aligned} \tag{69}$$

where $E_T^\varepsilon = e_2 \times (E^\varepsilon \times e_2)$ and the continuous dependence of the trace norm with $C = \frac{C_{\gamma_t} C_{\gamma_T}}{\mathcal{C}_0} \|H_d\|_{\mathbf{H}(curl,\Omega)}$ gives:

$$\|E^\varepsilon\|^2_{\mathbf{X}^\varepsilon(\Omega)} \leq C \|E^\varepsilon\|_{\mathbf{H}(\text{curl},\Omega)} \leq C \|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)}. \tag{70}$$

■

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