On the Singular Integral Equation Connected with the Stokes Gravity Waves

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Abstract— The nonlinear singular integral equation associated with the Stokes gravity waves in the incompressible Euler fluid is studied. The existence of the solution is proved and the approximate solution is constructed by means of Maple.

Index Terms— Singular Integral Equation, Stokes Gravity Waves

I. INTRODUCTION

Stokes gravity wave is a periodic surface wave of permanent form on an invisced fluid layer of constant mean depth. In this case viscosity and compression plays an insignificant role and the fluid is assumed to be incompressible Euler fluid [1], [2], [3], [4]. The wavelength is small as compared with the mean depth. It is assumed, that the bot-tom of the reservoir is flat and the motion is two-dimensional. In the early work of the author the singular integral equation for the Stokes free boundary was obtained [5]. In the present work it is assumed, that the period of the wave is rather small. The integral equation is simplified and the exis-tence of the solution is proved by means of Muskhelishvili theory and Shauder's fixed point principle [11], [12]. The approximate profile of the wave is constructed by using Maple.

II. STATEMENT OF THE PROBLEM

The coordinate system xOy moving with the wave is chosen. The axis Ox passes along the bottom and the axis Oy passes through the maximum point of the wave. Mathematically the problem is stated as follows [3]

STOKES PROBLEM. Find the periodic curve $\Gamma: y = y(x)$ such that, if f is a conformal mapping of the area $D = \{0 < t < y(x)\}$ on the strip $\{0 < \psi < q, q = const\}$, $f(\pm\infty) = \infty$, then the following condition holds

$$\frac{1}{2}\left|f'(z)\right|^2 + gy = A, \quad A = const, \qquad (1)$$

where $f(z) == \varphi + i\psi,,\ z = x + iy$, is a complex potential, φ is a speed potential, ψ is a stream function, $f'(z)$ is a complex speed, A and q are the definite positive constants, g is a gravity acceleration.

Here we consider the Stokes Problem for the symmetric periodic peaked waves with the period $2\omega_1$ and with the condition $f'(2\omega_1 + iq) = 0$. The case $f'(z) \neq 0$, was considered by different authors [1], [2], [3], [4], [6], [7], [8], [9].

III. SOLUTION OF THE PROBLEM

In the work of the author [5] the Stokes problem was reduced to the following singular integral equation

$$u(t_0) = 3g \int_0^{t_0} \sin\left[\ \frac{1}{3\pi} \int_0^{2\omega_1} [\ln u(t)]\, K(t,\tau)\, dt\ \right] d\tau, \quad (2)$$

$$t_0 \in [0, 2\omega_1],$$

$$K(t,\tau) = [\varsigma(t-\tau) - \varsigma(t+\tau) - \varsigma(t-\tau-i\omega_2) + \varsigma(t+\tau-i\omega_2)],$$

where $u(t_0)$ is an unknown function of the Muskhelishvili-Kveselava H^* class [11], $\ln z$ is the branch for which $\ln 1 = 0$, ς is the Weierstrss "zeta-function" for the fundamental periods $2\omega_1$ and $2i\omega_2$, $\omega_2 = q$ [10].

Weierstrass "zeta-function" is representable by the series [10]

$$\varsigma(z) = \frac{\delta_1 z}{2\omega_1} + \frac{\pi}{2\omega_1} ctg\, \frac{\pi z}{2\omega_1} +$$
$$\frac{2\pi}{\omega_1} \sum_{r=1}^{\infty} \frac{h^{2r}}{1-h^{2r}} \sin\frac{r\pi z}{\omega_1}; h = \exp\frac{-\pi\omega_2}{\omega_1}; \qquad (3)$$

and has the following properties:

1. It is a meromorphic function with the simple poles
$$T_{mn} = 2m\omega_1 + 2ni\omega_2; \quad m,n = 0,\pm1,\pm2,\ldots$$

2. $\varsigma(-z) = -\varsigma(z)$.

3. $\varsigma(z)$ is a double quasi-periodic function, i.e.

$$\varsigma(z + 2\omega_1) = \varsigma(z) + \delta_1,$$
$$\varsigma(z + 2i\omega_2) = \varsigma(z) + \delta_2,$$

where δ_1 and δ_2, are the addends of

$$2\varsigma(\omega_1) = \delta_1, \quad 2\varsigma(i\omega_2) = \delta_2,$$
$$i\omega_2\delta_1 - \omega_1\delta_2 = \pi i.$$

4. $[\ln \sigma(z)]' = \varsigma(z)$, where $\sigma(z)$ is the Weierstrass "sigma-function"

Having found $u(t_0)$, one period of the profile of the Stokes wave will be given by the formula

$$f_0(t_0) = \frac{1}{2g}(2A - u^{2/3}(t_0)). \qquad (4)$$

As the function $u(t_0)$ is symmetric, by using the properties of "zeta-function" we can rewrite the equation (2) in the form

$$u(t_0) = 3g\int_0^{t_0} \sin\left[\frac{1}{6\pi}A[u(\tau)]\right]d\tau, \qquad (5)$$
$$t_0 \in [0,\omega_1],$$

where

$$A[u(\tau)] = \int_0^{\omega_1}[\ln u(t)]\, K(t,\tau)\, dt\, \bigg]d\tau; \qquad (6)$$

$$K(t,\tau) = [\varsigma(t-\tau) - \varsigma(t+\tau) -$$
$$1/2\varsigma(t-\tau-i\omega_2) + 1/2\varsigma(t+\tau-i\omega_2)] - \qquad (7)$$
$$1/2\varsigma(t-\tau+i\omega_2) + 1/2\varsigma(t+\tau+i\omega_2)].$$

Assume that the solution of (5) is representable in the form

$$u(t_0) = v(t_0)u_0(t_0), \qquad (8)$$

$$u_0(t_0) = \frac{-1}{\cos^2\frac{\pi t_0}{2\omega_1}}\cdot\left(\sin\frac{\pi t_0}{2\omega_1}\right)\ln\left(\sin\frac{\pi t_0}{2\omega_1}\right),(9)$$

where $v(t_0)$ is an unknown positive function of Holder class.

By the representation (7) it is obvious $A[u(0)] = 0$.
We now suppose, that ω_1 is rather small and taking into account the properties of "zeta-function", the formulas (3),

(8), (9) and, we can rewrite equation (5) in the following form

$$v(t_0)u_0(t_0) = \frac{g}{2\pi}\int_0^{\omega_1}\ln v(t)\, K_0(t_0,t)dt + f(t_0);$$

$$f(t_0) = \frac{g}{2\pi}\int_0^{\omega_1}\ln u_0(t)\, K_0(t_0,t)dt, t_0 \in [0,\omega_1],(10)$$

where

$$K_0(t_0,t) = 1/2\times$$
$$\ln\left|\frac{\sigma(t_0-t-i\omega_2)\sigma(t_0-t+i\omega_2)}{\sigma^2(t_0-t)\sigma^2(t_0+t)}\times\right. \qquad (11)$$
$$\left.\frac{\sigma(t_0+t+i\omega_2)\sigma(t_0+t-i\omega_2)\sigma^4(t)}{\sigma^2(t-i\omega_2)\sigma^2(t+i\omega_2)}\right|,$$

$\sigma(z)$ is the Weierstrass "sigma-function". According to the properties of "sigma-function" the kernel $K_0(t_0,t)$ is weakly singular.

(10) is the integral equation with respect to $v(t_0)$. Let us rewrite it as

$$v(t_0) = \frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln v(t)\, K_0(t_0,t)dt + f^*(t_0); (12)$$

$$f^*(t_0) = \frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln u_0(t)\, K_0(t_0,t)dt, t_0 \in [0,\omega_1]$$

Using the representation (3) and properties of "zeta-function" we obtain

$$K_0(t_0,t) = K_1(t_0,t) + K_2(t_0,t),$$

where

$$K_0(t_0,t) = \ln\frac{\sin^2\frac{\pi t}{2\omega_1}}{\left|\sin^2\frac{\pi t_0}{2\omega_1} - \sin^2\frac{\pi t}{2\omega_1}\right|}, \qquad (13)$$

$$K_2(t_0,t) = \frac{2\delta_1 t_0^2}{\omega_1} +$$

$$1/2\ln\frac{\sin\frac{\pi(t_0+t-i\omega_2)}{2\omega_1}\sin\frac{\pi(t_0+t+i\omega_2)}{2\omega_1}}{\sin^2\frac{\pi(t-i\omega_2)}{2\omega_1}}\times$$

$$\frac{\sin\frac{\pi(t_0+t-i\omega_2)}{2\omega_1}\sin\frac{\pi(t_0+t+i\omega_2)}{2\omega_1}}{\sin^2\frac{\pi(t+i\omega_2)}{2\omega_1}}+ \quad (14)$$

$$\sum_{r=1}^{\infty}\frac{8}{r}\frac{h^{2r}}{1-h^{2r}}\sin^2\frac{r\pi t_0}{\omega_1}\cos\frac{r\pi t}{\omega_1}\sin^2\frac{r\pi i\omega_2}{2\omega_1}.$$

The kernel $K_1(t_0,t)$ has the logarithmic singularity, the kernel $K_2(t_0,t)$ is continuous on the set $[0,\omega_1]\times[0,\omega_1]$ and

$$\lim_{t_0\to 0}\frac{K_2(t_0,t)}{t_0}=0.$$

Consequently

$$\lim_{t_0\to 0}\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln u_0(t)K_2(t_0,t)dt=0. \quad (15)$$

Now, let us consider the first term of the formula

$$f^*(t_0)=\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln u_0(t)K_1(t_0,t)dt+$$
$$\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln u_0(t)K_2(t_0,t)dt. \quad (16)$$

Taking into account formula (13) and inserting $t'=\sin\frac{\pi t}{2\omega_1}$ into first term of (16) we obtain

$$\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}\ln u_0(t)K_1(t_0,t)dt=$$
$$\frac{g\omega_1}{2\pi u_0(t_0)}\int_0^1[-\ln t+\ln|\ln t|-\ln 2(1-t)]\times \quad (17)$$
$$\frac{1}{\sqrt{1-t^2}}\ln\frac{t^2}{|t^2-(t_0')^2|}dt, t_0'=\sin\frac{\pi t_0}{2\omega_1}.$$

By using the representations (15), (16), (17) and properties of the Cauchy type integrals [11], it is easy to conclude that the function f^* is Hölder continuous in $[0,\omega_1]$, belongs to the class H_ε^* at the point $t_0=0$ and

$$\lim_{t_0\to 0}f^*(t_0)=\frac{2g\omega_1}{\pi^2}\ln(\sqrt{2}+1)\equiv A^*. \quad (18)$$

Now, let us suppose

$$\ln\frac{v(t_0)}{A^*}\approx\frac{v(t_0)}{A^*}-1. \quad (19)$$

Inserting (19) into the right-hand side of (12) we obtain

$$v(t_0)=\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}[\frac{v(t)}{A^*}-1+\ln A^*]K_0(t_0,t)dt+$$
$$f^*(t_0)\equiv B[v(t_0)]; \quad t_0\in[0,\omega_1]. \quad (20)$$

Hence, instead of equation (12) we consider the approximate equation (20). Let us analyze this equation.

Let be $v\in S\subset C[0,\omega_1], |v|\le M_0; M_0>0,$ is a bounded set of functions, from (20) according to the properties of the kernel $K_0(t_0,t)$ and formula (18) we obtain

$$|\frac{g}{2\pi u_0(t_0)}\int_0^{\omega_1}[\frac{v(t)}{A^*}-1+\ln A^*]K_0(t_0,t)dt+ \quad (21)$$
$$f^*(t_0)|\le\frac{g}{2\pi}\left[\frac{M_0}{A^*}+1+|\ln A^*|\right][M_1+M_2],$$

where

$$M_1=\max|\int_0^{\omega_1}\frac{1}{u_0(t_0)}K_0(t_0,t)dt|,$$
$$M_2=\max|f^*(t_0)|, t_0\in[0,\omega_1].$$

Also, the following formula is valid

$$|B[v(t_1)]-B[v(t_2)]|\le\frac{g}{2\pi}\frac{M_0}{A^*}\times$$
$$\int_0^{\omega_1}|\frac{K_0(t_1,t)}{u_0(t_1)}-\frac{K_0(t_2,t)}{u_0(t_2)}|dt$$
$$+|f^*(t_1)-f^*(t_2)|; t_1,t_2\in[0,\omega_1]. \quad (22)$$

Taking into account the properties of the functions in (21) and (22)

$$f^*(t_0) \text{ and } \int_0^{\omega_1}\frac{K_0(t_0,t)}{u_0(t_0)}dt$$

and formula (18) we conclude that the set of functions S is uniformly bounded and uniformly continuous. Consequently, Arcella conditions holds and the operator B is completely continuous.

Also, according to the formula (18) and Muskhelishvili [11] and Mikhlin [13] the function on the right-hand side of (20) is Hölder continuous in $[0, \omega_1]$ and belongs to the class H_ε^* at the point $t_0 = 0$.

Hence, we have proved the following

THEOREM 1. The operator B on the right-hand side of (20) is completely continuous in the space $C[0, \omega_1]$, if the integral equation (20) has the continuous solution, it belongs to the Hölder class and $\lim_{t_0 \to 0} v(t_0) = A^*$.

Now, let us prove, that the solution of equation (20) exists.

Putting the notation $t = \omega_1 t_1$ in the integral of the right-hand side of (20) we can represent this integral equation as

$$v(t_0) = \frac{\omega_1 g}{2\pi u_0(t_0)} \int_0^1 [\frac{v(\omega_1 t)}{A^*} - 1 + \ln A^*] \times \quad (23)$$

$$K_0(t_0, \omega_1 t) dt + \frac{\omega_1 g}{2\pi u_0(t_0)} f^{**}(t_0) \equiv B[v(t_0)];$$

$$t_0 \in [0, \omega_1].$$

The following theorem is valid

THEOREM 2. There exists the solution of equation (23) of the class $C[0, \omega_1]$ in the ball

$$| v(t_0) - A^* | \leq \varepsilon_0; \ A^* > \varepsilon_0. \quad (24)$$

Taking into the account the representation (23) we obtain

$$| Bv(t_0) - A^* | =$$

$$\frac{\omega_1 g}{2\pi} [1 + \ln A^*] \times \int_0^1 | \frac{K_0(t_0, \omega_1 t)}{u_0(t_0)} | \, dt +$$

$$\frac{\omega_1 g}{2\pi} | \frac{f^{**}(t_0)}{u_0(t_0)} - \frac{4}{\pi} \ln(\sqrt{2} + 1) | \leq \varepsilon_0. \quad (25)$$

The formula (25) implies, that ε_0 satisfies the condition

$$\varepsilon_0 \geq \frac{\omega_1 g}{2\pi} [1 + \ln A^*] M_1^* + \frac{\omega_1 g}{2\pi} M_2^*, \quad (26)$$

$$M_1^* = \max \int_0^1 | \frac{K_0(t_0, \omega_1 t)}{u_0(t_0)} | \, dt, \quad t_0 \in [0, \omega_1],$$

$$M_1^* = \max | \frac{f^{**}(t_0)}{u_0(t_0)} - \frac{4}{\pi} \ln(\sqrt{2} + 1) |.$$

Hence, according to Shauder's fixed point principle, if ω_1 and ε_0 satisfy the condition (26) there exists the continuous solution of equation (23) and consequently of equation (20).

According to (5), (8), (9), (20), (23) and Theorem 2. we conclude, that the function given by the formulas (8), (9) is the solution of equation (2) in the ball (24).

IV. CONCLUSION

There exist the Stokes gravity periodic waves of the form

$$f_0(t_0) = \frac{A}{g} - \frac{1}{2g} \frac{v(t_0)}{\cos^{4/3} \frac{\pi t_0}{2\omega_1}} \cdot \left(\sin \frac{\pi t_0}{2\omega_1} \right)^{2/3} \times$$

$$\left(-\ln \left(\sin \frac{\pi t_0}{2\omega_1} \right) \right)^{2/3},$$

where $v(t_0)$ is a Hölder continues positive function, ω_1 is a rather small positive number.

As ε_0 from the condition (26) is rather small one of the approximate solutions of the equation (20) is $v(t_0) \approx A^*$ and consequently the approximate solution of the equation (2) of the type (8), (9) is

$$u(t_0) \approx \frac{-A^*}{\cos^2 \frac{\pi t_0}{2\omega_1}} \cdot \left(\sin \frac{\pi t_0}{2\omega_1} \right) \cdot \ln \left(\sin \frac{\pi t_0}{2\omega_1} \right),$$
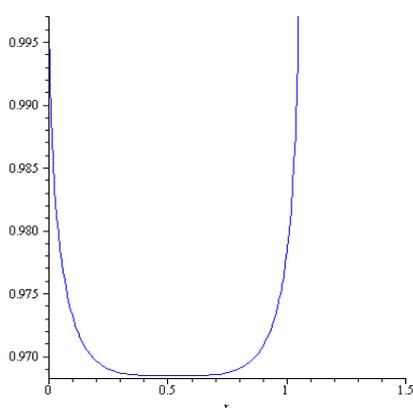
According to (4) the corresponding Stokes wave will be given by

$$f_0(t_0) = \frac{A}{g} - \frac{1}{2g} \frac{A^*}{\cos^{4/3} \frac{\pi t_0}{2\omega_1}} \cdot \left(\sin \frac{\pi t_0}{2\omega_1} \right)^{2/3} \times$$

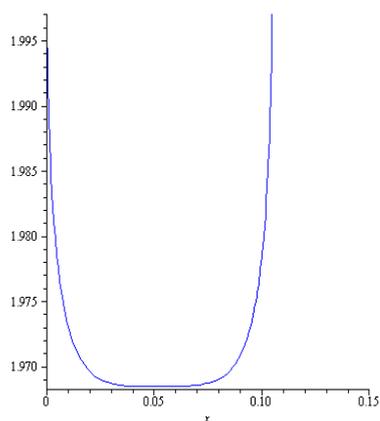$$\left(-\ln \left(\sin \frac{\pi t_0}{2\omega_1} \right) \right)^{2/3}. \quad (27)$$

Below, the graph of (27) is constructed by means of Maple-12 for the different parameters and is given in Figure 1, Figure 2 and Figure 3.
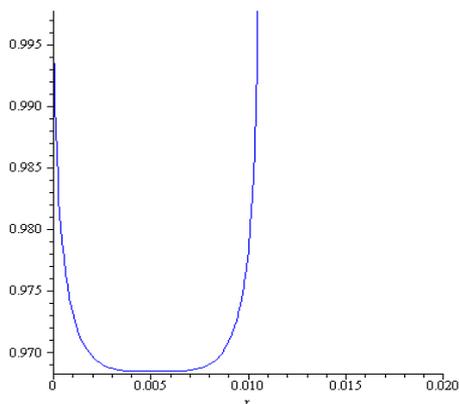
Fig.1. The graph of (27) in case of

$$A = g; \; A^* \approx 0.01; \omega_1 = 0.05$$

Fig.2. The graph of (27) in case of

$$A = g; \; A^* \approx 0.1; \omega_1 = 0.5.$$

Fig.3. The graph of (27) in case of

$$A = 2g; \; A^* \approx 0.01; \omega_1 = 0.05$$

APPENDIX

In the work of the author [6] by means of the conformal mapping method the Stokes problem was reduced to the nonlinear integral equation of the different form in a new variable

$$u(\xi) = -\frac{3g}{4\pi} \int_0^a |z'(t)| [\ln|z'(t)| - 2/3\ln u(t)] \times$$

$$K(t,\xi)dt; \;\; \xi \in [0,a],$$

$$K(t,\xi) = 2\ln \frac{\sqrt{b^2 - \xi^2} + \sqrt{b^2 - t^2}}{\sqrt{b^2 - a^2} + \sqrt{b^2 - t^2}} + \ln\left|\frac{a^2 - t^2}{t^2 - \xi^2}\right|,$$

$$z'(t) = C\frac{1}{\sqrt{a^2 - t^2}\sqrt{b^2 - t^2}}$$

where C is the definite constant, a and b are the arbitrary small positive constants. In [7] the approximate solution of this equation is given.

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