Elastodynamic Analysis of a Mechanism with Flexible Links

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***Abstract*— The research aim is to perform a dynamic analysis of a mechanism with elastic elements. Thus, a crank-rod-slider mechanism it's considered for this elastodynamic analysis which is performed in two stages, respectively the modal analysis where the natural frequencies are established and Eigen vibrations forms computation, which suppose to establish a kinematic parameters in the oscillation motion of a deformable link.**

To obtain these parameters, it's used the finite elements method, in that situation when the link is meshed into finite elements. The static and dynamic components of the matrix are identified which are involved in the motion equation, including the nodal forces vector. Finally it is described the experimental setup of the mechanism in order to calculate the kinematic parameters which defines the elastic link vibrations. These results are analyzed and compared with the ones obtained through modal dynamic analysis.

***Index Terms*—crank-rod, elastic element, modal dynamic analysis, vibrations, experimental kinematics.**

I. Introduction

The traditional approach for mechanical system dynamic analysis was studied on the assumption that the mechanisms are composed from rigid bodies [1]. This assumption is not always good, because the mechanism links are elastic and can deflect when they are subjected on external loads or inertia forces. Moreover at high operation speed a linkage may support serious elastic deformations due to its inertia, so the operation becomes improper [2].

Several authors have recently proposed methods and developed models for mechanism dynamics, considering flexible links, rather that rigid links [3, 4].

In [5] it is proposed for the first time an analysis method in which the effect of links flexibility was studied by applying a structural dynamics stiffness method, considering the assumption of superposition (uncoupling) of larger rigid-body motion and a small elastic deformation. It was also studied the idea of kineto-elastodynamics, which is the analyisis of mechanisms motion considering the links deflection due to external loads or internal body forces. This method is based on a flexibility matrix approach of structural analysis. The developed method assumes linear superposition for small elastic deformation due to the inertia forces that appears in a rigid body motion of the system.

The finite elements method is used by several authors to study the mechanism with flexible links. In this method are considered the geometrically nonlinear relations related to the element deformations in terms of the nodal position and orientation coordinates. The motion equations are formulated in terms of mechanism generalized coordinates mixed sets with rigid links and deformation mode coordinates (characteristic to the elastic links) [6].

Also finite element method was used for the optimal design of flexible mechanisms. The purpose of this was to obtain a minimum weight design. The design algorithm presented in [7] is formulated in terms of the Kuhn-Tucker optimization criteria, which assumes two damping factors in order to obtain solution convergence.

Modeling based on finite element method is presented in [8, 9]. In this study the kinematic links are meshed with beam finite elements. A Lagrange formulation of the finite element was adopted for the deformable elements connected in multibody systems. For taking into account the constraints imposed by the joints, the conditions for connection between elements were defined using a Boolean constant matrix [10].

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II. Mathematical models for elastodynamic analysis

A kinematic chain with n rigid solids it is considered, connected by n-1 kinematic joints, as presented in figure 1.

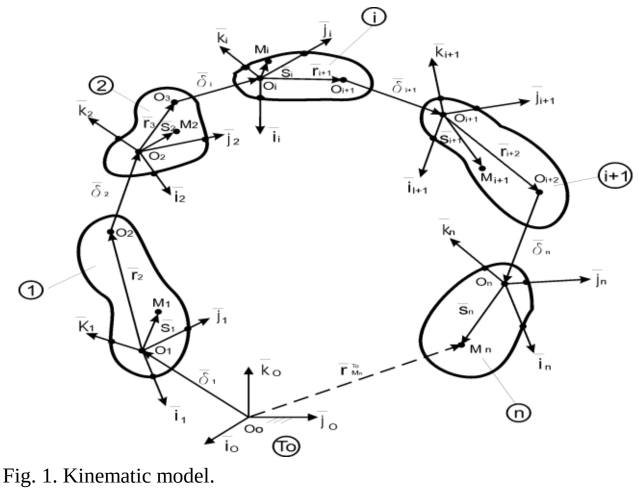

Fig. 1. Kinematic model.

The following notations were used:

$T_i\left(\vec{x}_i, \vec{y}_i, \vec{z}_i\right)$-reference system attached to element *i*; $T_0\left(\vec{x}_0, \vec{y}_0, \vec{z}_0\right)$-fixed reference system; $\vec{\delta}_i$-relative translation vector, between elements *i* and *i-1*, expressed in T_{i-1} reference system; $\bar{r}_i$-position vector of point *O'ᵢ*, expressed in T_{i-1} reference system, from where its starts the relative translation;

$$\bar{r}_i = \left\{r_i^x, r_i^y, r_i^z\right\}_{i-1} = \left\{r_i\right\}^T \left\{\vec{W}_{i-1}\right\} \tag{1}$$

$$\vec{\delta}_i = \left\{\delta_i^x, \delta_i^y, \delta_i^z\right\}_{i-1} = \left\{\delta_i\right\}^T \left\{\vec{W}_{i-1}\right\} \tag{2}$$

Also the following connectivity order was considered:

The transformation matrix coordinates, for crossing from one reference system to another, has the following form:

$$\left\{\vec{W}_{i-1}\right\} = \left[A_{oi-1}\right]\left\{\vec{W}_o\right\} \tag{3}$$

In this context, (1) and (2) become:

$$\bar{r}_i = \{r_i\}^T\left\{\vec{W}_{i-1}\right\} = \{r_i\}^T\left[A_{oi-1}\right]\left\{\vec{W}_o\right\} \tag{4}$$

$$\vec{\delta}_i = \{\delta_i\}^T\left\{\vec{W}_{i-1}\right\} = \{\delta_i\}^T\left[A_{oi-1}\right]\left\{\vec{W}_o\right\} \tag{5}$$

For the above mathematical expressions the geometric model of an elastic solid will be considered and this is shown in figure 2.

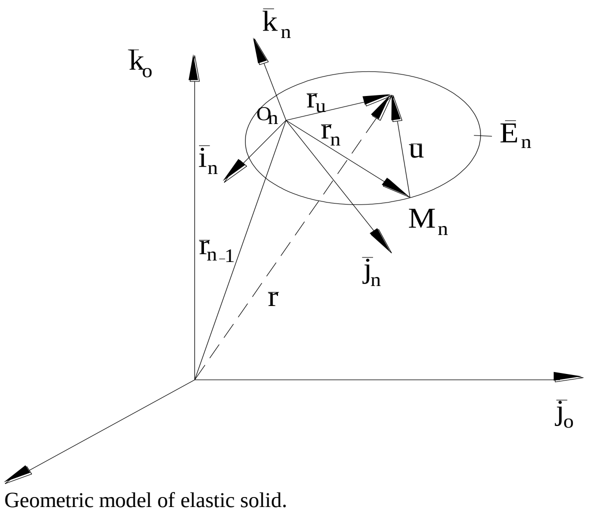

Fig.2. Geometric model of elastic solid.

On solid E_n will consider the point M_n and $u(M,t)$ the elastic displacement of point M_n. Thus it can be written:

$$\vec{r} = \vec{r}_{n-1} + \vec{r}_u \tag{6}$$

$$\vec{r}_u = \vec{r}_n + \vec{u} \tag{7}$$

$$\vec{r}_{n-1} = \sum_{i=1}^{n}\left(\left\{r_i\right\}^T + \left\{\delta_i\right\}^T\right)\left[A_{0i-1}\right]\left\{\vec{W}_0\right\} \tag{8}$$

Where: $\left[A_{0i-1}\right]$ is the coordinate's transformation matrix.

The elastic displacement vector is:

$$\vec{u} = \left\{u\right\}^T \left\{\vec{w}_n\right\} \tag{9}$$

$$\left\{u\right\} = \left[N\right]\left\{d\right\} \tag{10}$$

Where: $\left[N\right]$-form functions matrix, or interpolation polynomial matrix; $\left\{d\right\}$-nodal displacement vector;

Thus the following form will be obtained:

$$\left\{\vec{w}_n\right\} = \left[A_{0,n}\right]\left\{\vec{w}_0\right\} \tag{11}$$

Taking into account (11) it will be obtained the following form:

$$\vec{u} = \left\{d\right\}^T \left[N\right]^T \left[A_{0,n}\right]\left\{\vec{w}_0\right\} \tag{12}$$

$$\vec{r}_n = \left\{r_n\right\}^T \left\{\vec{w}_n\right\} = \left\{r_n\right\}^T \left[A_{0,n}\right]\left\{\vec{w}_0\right\} \tag{13}$$

$$\begin{aligned}\vec{r}_u &= \left\{r_n\right\}^T\left[A_{0,n}\right]\left\{\vec{w}_0\right\} + \left\{d\right\}^T[N]^T\left[A_{0,n}\right]\left\{\vec{w}_0\right\} = \\ &= \left(\left\{r_n\right\}^T + \left\{d\right\}^T[N]^T\right)\left[A_{0,n}\right]\left\{\vec{w}_0\right\}\end{aligned} \tag{14}$$

By introducing Eq. (9) and (4) in Eq. (8) we obtain:

$$\vec{r} = \left(\begin{array}{l}\sum_{i=1}^{n}\left(\left\{r_i\right\}^T + \left\{\delta_i\right\}^T\right)\left[A_{0i-1}\right] + \\ + \left(\left\{r_n\right\}^T + \left\{d\right\}^T[N]^T\right)\left[A_{0,n}\right]\end{array}\right)\left\{\vec{w}_0\right\} \tag{15}$$

If we take into account the generalized coordinates:

$$\left\{q_i\right\}^T = \left\{\left\{r_i\right\}^T \; \left\{\delta_i\right\}^T\right\} = \left\{\left\{r_i\right\} \left\{\delta_i\right\}\right\}^T \tag{16}$$

Thus it will be obtained:

$$\vec{r} = \left(\begin{array}{l}\sum_{i=1}^{n}\left\{q_i\right\}^T\left[A_{0i-1}\right] + \\ + \left(\left\{r_n\right\}^T + \left\{d\right\}^T[N]^T\right)\left[A_{0,n}\right]\end{array}\right)\left\{\vec{w}_0\right\} \tag{17}$$

The motion of a linear system, for an element "*e*" is described by the system of equations as:

$$\left\{\ddot{d}\right\}^T\left[M_e\right] + \left\{\dot{d}\right\}^T\left[C_e\right] + \left\{d\right\}^T\left[K_e\right] = \left[F_e\right]^T \tag{18}$$

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The structure matrices which were involved in equation (18) were established in accordance with the procedure described in [11].

The mass matrix is:

$$\left[M^{(e)}\right]=\iiint\limits_{V^{(e)}}\rho_e m_e\left[N\right]^T\left[N\right]dV \tag{19}$$

The amortization matrix with static and dynamic terms is:

$$\left[C^{(e)}\right]=\iiint\limits_{V^{(e)}}\left[N\right]^T\left[\mu\right]\left[N\right]dV+2\iiint\limits_{V^{(e)}}\rho_e\left[N\right]^T\left(\left[\tilde{\omega}_{0n}\right]\left[\tilde{\omega}_{0n}\right]+\left[\tilde{\varepsilon}_{0n}\right]\right)\left[N\right]dV= =\left[C_s^{(e)}\right]+\left[C_d^{(e)}\right] \tag{20}$$

The nodal elemental force is:

$$\left\{F^{(e)}\right\}=\iiint\limits_{V^{(e)}}\left\{N\right\}^T\left[F\right]dV-\iiint\limits_{V^{(e)}}\rho_e\left[N\right]^T\Bigg\{\left(\left[\tilde{\varepsilon}_{0,n}\right]+\left[\tilde{\omega}_{0,n}\right]\left[\tilde{\omega}_{0,n}\right]\right)\left\{r_n\right\}+ +\sum_{i=1}^{n}\left[A_{n-1}\right]\left(\left\{\ddot{q}_i\right\}+2\left[\tilde{\omega}_{0,i-1}\right]\left\{\dot{q}_i\right\}+ +\left[\tilde{\varepsilon}_{0,i-1}\right]\left\{q_i\right\}+\left[\tilde{\omega}_{0,i-1}\right]\left[\tilde{\omega}_{0,i-1}\right]\left\{q_i\right\}\right)\Bigg\}dV \tag{21}$$

Rigidity matrix, with static and dynamic components, is:

$$\left[K^{(e)}\right]=\iiint\limits_{V^{(e)}}\left[B\right]^T\left[D\right]\left[B\right]dV+\iiint\limits_{V^{(e)}}\rho_e\left[N\right]^T\left(\left[\tilde{\omega}_{0n}\right]\left[\tilde{\omega}_{0n}\right]+\left[\varepsilon_{0,n}\right]\right)\left[N\right]dV= =\left[K_s^{(e)}\right]+\left[K_d^{(e)}\right] \tag{22}$$

III. Modal dynamic analysis of a mechanism with elastic links

This analysis was focused on a crank-rod-slider mechanism, with an elastic rod. In this context the rod was considered as being composed from two beam finite elements, as it can be remarked in figure 3.

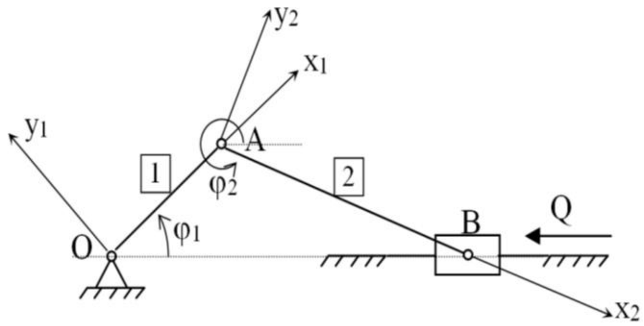

Fig. 3 Crank rod mechanism.

The input data: $l_1=l_{OA}=r=55$ mm; $l_2=l_{AB}=280$ mm; Q=20 N; $n_1=100$ rot/min;

The mechanism links are manufactured from steel, with the following mechanical characteristics: E=2, $1\cdot10^5$ MPa – longitudinal elasticity module; ρ=7, $7\cdot10^{-9}$ $N\cdot s^2/mm^4$- specific mass; ν=0, 28 – transversal contraction coefficient.

The mechanism connecting rod is meshed in two beam finite elements as it is shown in figure 4. The kinematic parameters of the connecting rod are:

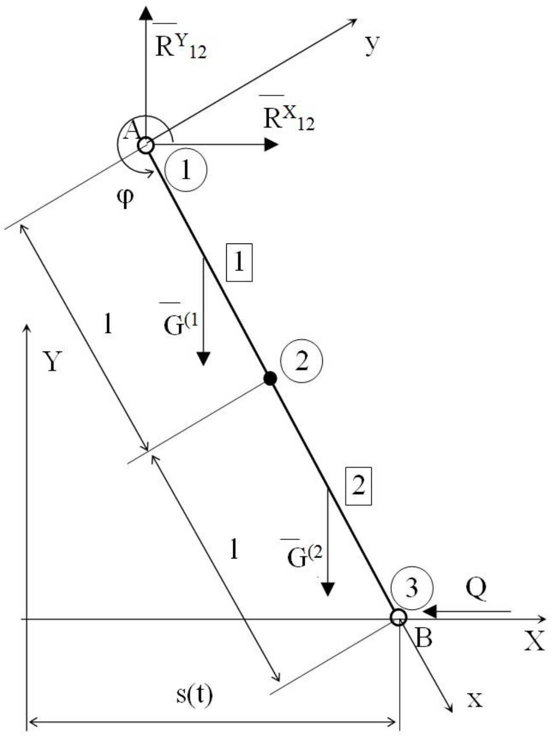

Fig. 4. The meshed rod.

$$\sin\phi_2=-\lambda\cdot\sin\phi_1 \tag{23}$$

$$\omega_2=-\lambda\cdot\omega_1\cdot\cos\phi_1\left[1+\frac{1}{2}\cdot\lambda^2\cdot\sin\phi_1\right] \tag{24}$$

Where: $\lambda=\dfrac{l_{rod}}{l_{crank}}$; ω_2 is the angular velocity of the mechanism connecting rod;

A. Nodal forces evaluation

The mechanism kinetostatic analysis gave the following expressions for the connecting forces:

$$R_{12}^X=Q \tag{25}$$

$$R_{12}^Y=\frac{\left(-Q\times Y_A-G_2\times X_{C2}+G_2\times X_B\right)}{X_B-X_A} \tag{26}$$

Nodal forces corresponding to the finite elements are computed as in (27) for fixed reference system and in (28) for local reference system.

$$\left[F_1\right]=r^{(1)}gA^{(1)}\int_0^l\left[N\right]^T dx \tag{27}$$

$$\left[f_1\right]=\left[F_1\right]\times\begin{bmatrix}-\sin f_2\\-\cos f_2\end{bmatrix} \tag{28}$$

In the same way was computed for the link 2:

$$\left[F_2\right]=\rho^{(2)}gA^{(2)}\int_l^{2l}\left[N\right]^T dx \tag{29}$$

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$$[f_2]=[F_{21}]\times\begin{bmatrix}-\sin f_2\\ -\cos f_2\end{bmatrix} \quad (30)$$

In the node no:1 acts the reaction force $\vec{R}_{12}$, with two components $\vec{R}_{12}^{X}$ and $\vec{R}_{12}^{Y}$:

$$\left[R_{12}^{local}\right]=\begin{bmatrix}R_{12}^X\cos\phi_2+R_{12}^Y\sin\phi_2\\ -R_{12}^X\sin\phi_2+R_{12}^Y\cos\phi_2\end{bmatrix} \quad (31)$$

In the node 3 acts the technologic force *Q*. The slider will be considered as a concentrated mass m_c, attached to the rod.

For the finite element no.:1, the nodal forces in the local reference system are:

$$f_s^{(1)}=\begin{bmatrix}R_{12}^X\cos\phi_2+R_{12}^Y\sin\phi_2-\frac{1}{2}\rho gAl\sin\phi_2\\ -R_{12}^X\sin\phi_2+R_{12}^Y\cos\phi_2-\frac{1}{2}\rho gAl\cos\phi_2\\ -\frac{1}{12}\rho gAl^2\cos\phi_2\\ -\frac{1}{2}\rho gAl\sin\phi_2\\ -\frac{1}{2}\rho gAl\cos\phi_2\\ \frac{1}{12}\rho gAl^2\cos\phi_2\end{bmatrix} \quad (32)$$

$$f_s^{(2)}=\begin{bmatrix}\frac{1}{2}\rho gAl\sin\phi_2\\ -\frac{3}{2}\rho gAl\cos\phi_2\\ -\frac{7}{12}\rho gAl^2\cos\phi_2\\ -Q\cos\phi_2-\frac{3}{2}\rho gAl\sin\phi_2\\ Q\sin\phi_2+\frac{1}{2}\rho gAl\cos\phi_2\\ -\frac{17}{12}\rho gAl^2\cos\phi_2\end{bmatrix} \quad (33)$$

The forces dynamic components are computed as:

$$\left[F_d^l\right]=-\iiint_{V^{(l)}}\rho^{(l)}[N]^T\left\{\begin{pmatrix}[\varepsilon_{02}]+[\omega_{02}]\cdot[\omega_{02}]\cdot\{x\}+\\ +[A_{12}][\omega_{1,0}][\omega_{1,0}]\{r_2\}\end{pmatrix}\right\}dv \quad (34)$$

For the finite element no.:1 the dynamic force is computed with the following mathematical expression:

$$F_d^{(1)}=-\rho A\int_0^l\left\{[N]^T\begin{bmatrix}-(\omega_1-\omega_2)^2 & \varepsilon_1-\varepsilon_2 & 0\\ \varepsilon_2-\varepsilon_1 & -(\omega_1-\omega_2)^2 & 0\\ 0 & 0 & 0\end{bmatrix}\{x\}+\begin{bmatrix}-\cos\phi_2\omega_1^2 l_2\\ -\sin\phi_2\omega_1^2 l_2\\ 0\end{bmatrix}\right\}dx \quad (35)$$

For the finite element no.:2 the dynamic force has the following form:

$$F_d^{(2)}=-\rho A\int_l^{2l}\left\{[N]^T\begin{bmatrix}-(\omega_1-\omega_2)^2 & \varepsilon_1-\varepsilon_2 & 0\\ \varepsilon_2-\varepsilon_1 & -(\omega_1-\omega_2)^2 & 0\\ 0 & 0 & 0\end{bmatrix}\{x\}+\begin{bmatrix}-\cos\varphi_2\omega_1^2 l_2\\ -\sin\varphi_2\omega_1^2 l_2\\ 0\end{bmatrix}\right\}dx \quad (36)$$

In these conditions the finite element nodal force no.:1 is:

$$\left[f^{(1)}\right]=\left[F_s^{(1)}\right]+\left[F_d^{(1)}\right] \quad (37)$$

For the finite element no.: 2 the nodal force is:

$$\left[f^{(2)}\right]=\left[F_s^{(2)}\right]+\left[F_d^{(2)}\right] \quad (38)$$

By crossing from the local reference system to a fixed reference system, the coordinate transformation matrix is defined as:

$$[T]=\begin{bmatrix}\cos\phi_2 & \sin\phi_2 & 0 & 0 & 0 & 0\\ -\sin\phi_2 & \cos\phi_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos\phi_2 & \sin\phi_2 & 0\\ 0 & 0 & 0 & -\sin\phi_2 & \cos\phi_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \quad (39)$$

The nodal force for the finite element no.:1, on the fixed reference system is computed as: $\left[F^{(1)}\right]=[T]^T\left[f^{(1)}\right]$, and for element no.:2 with relation: $\left[F^{(2)}\right]=[T]^T\left[f^{(2)}\right]$.

The nodal forces assembled vector in fixed global reference system is $[F(t)]$.

B. Matrix evaluation from the motion equation

On this frame is computed the matrix that was involve in the general dynamics equation. For element no: 1, when $x\in[0,l]$, the damping matrix in the local coordinate system is:

$$\left[C^{(e)}\right]=\left[C_s^{(e)}\right]+\left[C_d^{(e)}\right] \quad (40)$$

Elemental stiffness matrix for element no: 1 is:

$$\left[k_s^{(e1)}\right]=\iiint_{v^{(e)}}[B]^T[D][B]dv=EA\int_0^e[B]^T[B]dx \quad (41)$$

The assembled stiffness matrix, for element no: 1, in local reference system, is expressed by (41). The stiffness matrix in dynamics conditions is computed as:

$$\left[K_d^{(e1)}\right]=\iiint_{v^{(e)}}\rho^{(e)}[N]^T\left(\left[\dot{\tilde{\omega}}_{02}\right]+\left[\tilde{\omega}_{02}\right]\left[\tilde{\omega}_{02}\right]\right)[N]dv=$$

$$=\rho^{(e)}A\int_0^l[N]^T\begin{bmatrix}-(\omega_1-\omega_2)^2 & \varepsilon_1-\varepsilon_2 & 0\\ -(\varepsilon_1-\varepsilon_2) & -(\omega_1-\omega_2)^2 & 0\\ 0 & 0 & 0\end{bmatrix}[N]dx \quad (42)$$

The total elemental stiffness matrix for element no.:1 is:

$$\left[K^{(l1)}\right]=\left[K_s^{(l1)}\right]+\left[K_d^{(e1)}\right] \tag{43}$$

The stiffness matrix, for element no.:1, in global reference system is computed as:

$$\left[K^{(1)}\right]=\left[T\right]^T\left[K^{(l1)}\right]\left[T\right] \tag{44}$$

The stiffness matrix for element no.: 2, in dynamic conditions is computed as:

$$\left[K_d^{(e2)}\right]=\iiint_{v^{(e)}}\rho^{(e)}\left[N\right]^T\left(\left[\dot{\tilde{\omega}}_{02}\right]+\left[\tilde{\omega}_{02}\right]\left[\tilde{\omega}_{02}\right]\right)\left[N\right]dv=$$
$$=\rho^{(e)}A\int_l^{2l}\left[N\right]^T\begin{bmatrix}-(\omega_1-\omega_2)^2 & \varepsilon_1-\varepsilon_2 & 0\\ -(\varepsilon_1-\varepsilon_2) & -(\omega_1-\omega_2)^2 & 0\\ 0 & 0 & 0\end{bmatrix}\left[N\right]dx \tag{45}$$

The stiffness matrix for element no.: 2 has the following form:

$$\left[K^{(l2)}\right]=\left[K_s^{(l2)}\right]+\left[K_d^{(e2)}\right] \tag{46}$$

The stiffness matrix for finite element no.: 2, in global reference system can be computed as:

$$\left[K^{(2)}\right]=\left[T\right]^T\left[K^{(l2)}\right]\left[T\right] \tag{47}$$

The mass matrix for element no: 1 has the following form:

$$M^{(l1)}=\iiint_{V^{(e)}}\rho^{(1)}\left[N\right]^T\left[N\right]dv=\rho^{(1)}A\int_0^l\left[N\right]^T\left[N\right]dx \tag{48}$$

The mass matrix for element no.: 1, expressed in global reference system is:

$$\left[M^{(1)}\right]=\left[T\right]^T\left[M^{(l1)}\right]\left[T\right] \tag{49}$$

The mass matrix, for element no.: 2, in local reference system, can be expressed through (50).

$$\left[M^{(l2)}\right]=\iiint_{V^{(e)}}\rho^{(1)}\left[N\right]^T\left[N\right]dv=\rho^{(1)}A\int_l^{2l}\left[N\right]^T\left[N\right]dx \tag{50}$$

The mass matrix for element no.: 2, expressed in global reference system has the following mathematical expression:

$$\left[M^{(2)}\right]=\left[T\right]^T\left[M^{(l2)}\right]\left[T\right] \tag{51}$$

The assembled stiffness matrix $\left[K\right]$, for the entire rod, expressed in global reference system is represented through (52). The mass matrix, damping matrix and stiffness matrix, assembled are computed upon defining the boundary conditions (d_{7x}= s(t); q_{8y}=0).

$$\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & K_{15}^{(1)} & K_{16}^{(1)} & 0 & 0 & 0\\
K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & K_{25}^{(1)} & K_{26}^{(1)} & 0 & 0 & 0\\
K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} & K_{35}^{(1)} & K_{36}^{(1)} & 0 & 0 & 0\\
K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)}+K_{11}^{(2)} & K_{45}^{(1)}+K_{12}^{(2)} & K_{46}^{(1)}+K_{13}^{(2)} & K_{14}^{(2)} & K_{15}^{(2)} & K_{16}^{(2)}\\
K_{51}^{(1)} & K_{52}^{(1)} & K_{53}^{(1)} & K_{54}^{(1)}+K_{21}^{(2)} & K_{55}^{(1)}+K_{12}^{(2)} & K_{56}^{(1)}+K_{23}^{(2)} & K_{24}^{(2)} & K_{25}^{(2)} & K_{26}^{(2)}\\
K_{61}^{(1)} & K_{62}^{(1)} & K_{63}^{(1)} & K_{64}^{(1)}+K_{31}^{(2)} & K_{65}^{(1)}+K_{32}^{(12)} & K_{66}^{(1)}+K_{33}^{(2)} & K_{34}^{(2)} & K_{35}^{(2)} & K_{36}^{(2)}\\
0 & 0 & 0 & K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} & K_{45}^{(2)} & K_{46}^{(2)}\\
0 & 0 & 0 & K_{51}^{(2)} & K_{52}^{(2)} & K_{53}^{(2)} & K_{54}^{(2)} & K_{55}^{(2)} & K_{56}^{(2)}\\
0 & 0 & 0 & K_{61}^{(2)} & K_{62}^{(2)} & K_{63}^{(2)} & K_{64}^{(2)} & K_{65}^{(2)} & K_{66}^{(2)}
\end{bmatrix} \tag{52}$$

IV. Modal Dynamic Analysis

The motion equation for the dynamic analysis, without damping is:

$$\left[M\right]\left\{\ddot{d}\right\}+k\left\{d\right\}=\left\{F\left(t\right)\right\} \tag{53}$$

Introducing the boundary conditions, the matrix [K] and [M] have the dimensions 8x8. The boundary conditions are:

$$t=0\quad\begin{cases}\left\{d\right\}=\left\{d_0\right\}\\ \left\{\dot{d}\right\}=\left\{\dot{d}_0\right\}\end{cases} \tag{54}$$

The natural frequencies are computed with (55):

$$\det\left[\left[K\right]-p^2\left[M\right]\right]=0 \tag{55}$$

Computed values for the natural frequencies are:
$f_i = \varphi_i / 2\pi$ (cycles / second);
f_1=411,3230552 [Hz] ; f_2=1580,215521 [Hz];
f_3=2776,688816 [Hz];f_4=11139,76175 [Hz];
f_5=9,124618253 [Hz];f_6=0,1837169438 [Hz];
f_7=712,9568269 [Hz]; f_8=0.1667817193 [Hz];
The vibration modes are computed with the relation:

$$\left(\left[K\right]-p^2\left[M\right]\right)\left\{\varphi\right\}=0 \tag{56}$$

The numerical results obtained with Maple software, for the vibration forms are shown in figure 5. Considering the (56), and also the substitution $\left\{d\right\}=\left\{\varphi\right\}\left\{a\left(t\right)\right\}$, the following mathematical expressions are obtained:

$$\left[M\right]\left\{\ddot{d}\right\}+\left[K\right]\left\{d\right\}=\left\{F\left(t\right)\right\} \tag{57}$$

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$$[\varphi][M][\varphi]=[I] \tag{58}$$

$$[\varphi]^T[K][\varphi]=\left[\omega^2\right]_d \tag{59}$$

Differentiating depending on time the (59) will be obtained:

$$\left\{\dot{d}\right\}=\{\varphi\}\left\{\dot{a}(t)\right\} \ - \ \text{velocity}$$
$$\tag{60}$$
$$\left\{\ddot{d}\right\}=\{\varphi\}\left\{\ddot{a}(t)\right\} \ - \ \text{accelerations}$$
$$\tag{61}$$

If the expression (57) will be amplified to left side, and considering the other relations (58) and (59) it can be written:

$$\left\{\ddot{a}(t)\right\}+\left[\omega^2\right]_d\{a(t)\}=[\varphi]^T\{F(t)\} \tag{62}$$

Where:

$$\left[\omega^2\right]_d=diag(p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2, p_7^2, p_8^2) \tag{63}$$

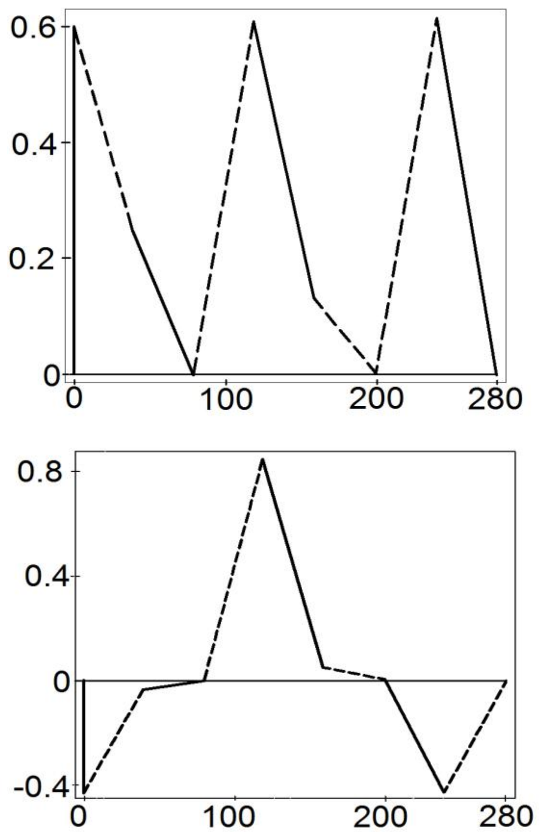

Fig. 5 Computed vibration forms (mode shape number 6 and 7)

The solution of (62) is:

$$\{a(t)\}=\{a_1(t) \quad a_2(t) \quad a_3(t) \quad a_4(t) \quad a_5(t) \quad a_6(t) \quad a_7(t) \quad a_8(t)\}^T$$
$$\left\{\ddot{a}(t)\right\}=\left\{\ddot{a}_1(t) \quad \ddot{a}_2(t) \quad \ddot{a}_3(t) \quad \ddot{a}_4(t) \quad \ddot{a}_5(t) \quad \ddot{a}_6(t) \quad \ddot{a}_7(t) \quad \ddot{a}_8(t)\right\}^T$$
$$\tag{64}$$

The initial conditions for (62) are defined as:

$$a_i(0)=0\,, \ i=1\div 7; \qquad \ddot{a}_i(0)=0, \quad i=1\div 7; \tag{65}$$

The defined boundary conditions are:
- Displacement upon y axis, in the node 3: d_{3y}=0.
- Displacement upon x axis, in the node 3: d_{3x}=s(t).

Considering the substitution $\{d\}=\{\varphi\}\{a(t)\}$, are computed the nodal displacement. Some variations of the elastic longitudinal and transversal displacement, on the rod length are computed with (66).

$$\vec{u}=[N]\cdot\vec{d} \tag{66}$$

Where:

$$[N]=\begin{bmatrix} N_{1x} & 0 & 0 & N_{2x}+N_{1x} & 0 & 0 & 0 \\ 0 & N_1 & N_2 & 0 & N_1+N_3 & N_4+N_2 & N_4 \end{bmatrix} \tag{67}$$

$\{d\}=\{d_{1x} \ d_{1y} \ \theta_1 \ d_{2x} \ d_{2y} \ \theta_2 \ \theta_3\}^T$

The computed nodal displacements are:
u[1,1] = N1x phi11 a1(t) + N1x phi12 a2(t) + N1x phi13 a3(t)+ N1x phi14 a4(t) + N1x phi15 a5(t) + N1x phi16 a6(t)+ N1x phi17 a8(t) + N2x phi41 a1(t) + N2x phi42 a2(t)+ N2x phi43 a3(t) + N2x phi44 a4(t) + N2x phi45 a5(t) + N2x phi46 a6(t) + N2x phi47 a8(t) + N1x phi41 a1(t)+ N1x phi42 a2(t) + N1x phi43 a3(t) + N1x phi44 a4(t) + N1x phi45 a5(t) + N1x phi46 a6(t) + N1x phi47 a8(t);

u[2,1] = N1 phi51 a1(t) + N1 phi56 a6(t) + N1 phi21 a1(t) + N1 phi22 a2(t) + N1 phi23 a3(t) + N1 phi24 a4(t)+ N1 phi25 a5(t) + N1 phi26 a6(t) + N1 phi27 a8(t)+ N2 phi31 a1(t) + N2 phi32 a2(t) + N2 phi33 a3(t)+ N2 phi34 a4(t) + N2 phi35 a5(t) + N2 phi36 a6(t) + N2 phi37 a8(t) + N3 phi51 a1(t) + N3 phi52 a2(t)+ N3 phi53 a3(t) + N3 phi54 a4(t) + N3 phi55 a5(t)+ N3 phi56 a6(t) + N3 phi57 a8(t) + N1 phi52 a2(t)+ N1 phi53 a3(t) + N1 phi54 a4(t) + N1 phi55 a5(t)+ N1 phi57 a8(t) + N4 phi61 a1(t) + N4 phi62 a2(t) + N4 phi63 a3(t) + N4 phi64 a4(t) + N4 phi65 a5(t) + N4 phi66 a6(t) + N4 phi67 a8(t) + N2 phi61 a1(t)+ N2 phi62 a2(t) + N2 phi63 a3(t) + N2 phi64 a4(t)+ N2 phi65 a5(t) + N2 phi66 a6(t) + N2 phi67 a8(t)+ N4 phi71 a1(t) + N4 phi72 a2(t) + N4 phi73 a3(t)+ N4 phi74 a4(t) + N4 phi75 a5(t) + N4 phi76 a6(t)+ N4 phi77 a8(t);

The obtained results by solving with Maple software, the presented mathematical model are shown in figures 6 to 10.

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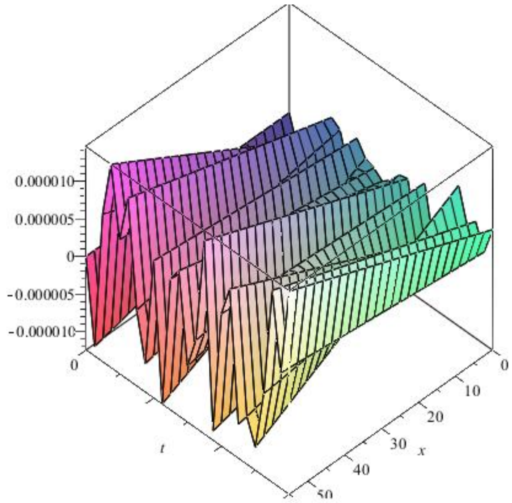

Fig. 6. Variation diagram of longitudinal elastic displacement.

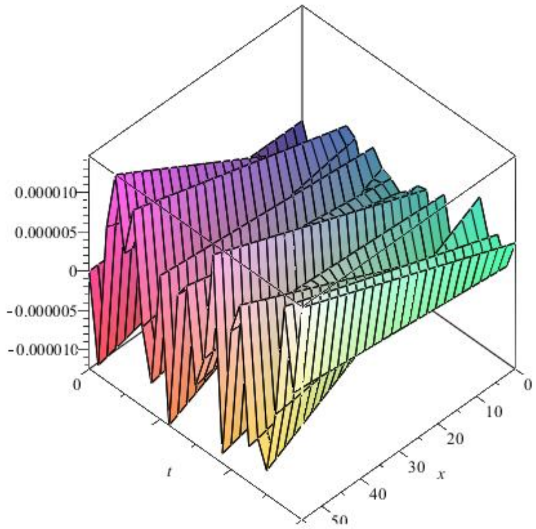

Fig. 7. Variation diagram of transversal elastic displacement.

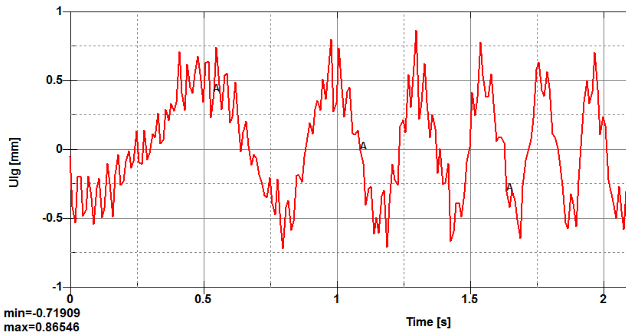

Fig. 8. Longitudinal acceleration diagram variation.

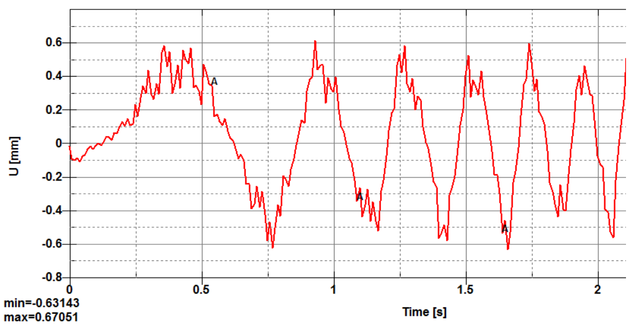

Fig. 9. Transversal acceleration diagram variation

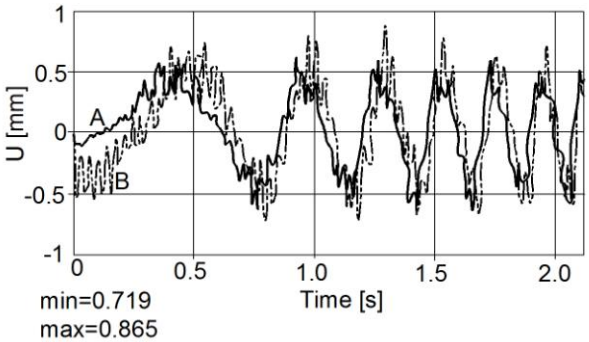

Fig. 10. The longitudinal and transversal accelerations superimposed diagram.

V. Experimental Analysis

Experimental setup was composed from a mechanical friction CVT (continuous variable transmission), a planetary gearbox and a crank –rod-slider mechanism, as is shown in figure. 11. On the rod and slider are placed transversal accelerometers, which these can be remarked in figure 12.

Fig. 11. Experimental setup.

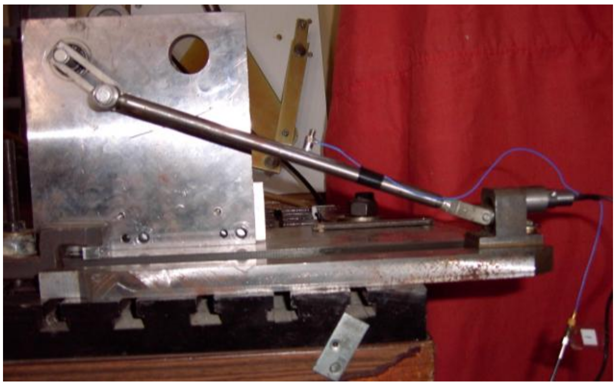

Fig. 12. Accelerometer mounted on the mechanism.

The measured parameters are:

1. Linear displacement of the slider, C_{lin} [m] with a displacement transducer, type LK15. This has the mobile core connected to the slider. The initial value of the registered displacement corresponds to the slider extreme position.
2. Angular displacement of the driving wheel C_{rot} [deg], with angular position transducer type RK5. This has the mobile core attached to the wheel shaft.

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3. Transversal and longitudinal accelerations of the rod Acc_vib[m/s^2], have been measured with accelerometers fitted rigidly on the mechanism kinematic links.

The obtained results are presented in figure 13 and figure 14.

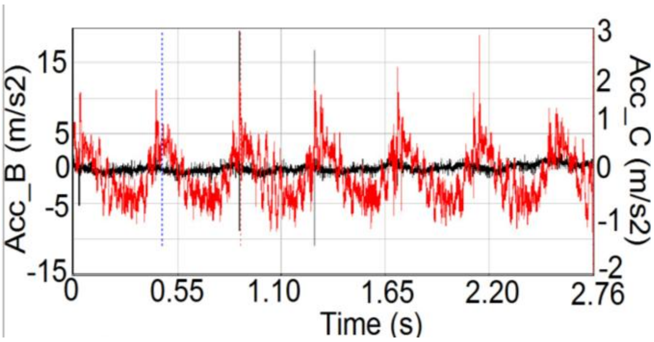

Fig. 13. Time variation of measured accelerations of the rod, for 144 rot/min RPM.

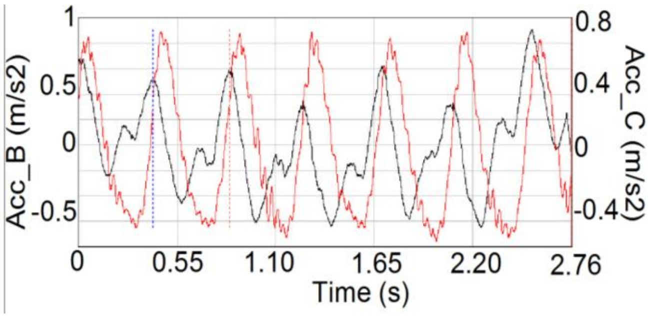

Fig. 14. Filtered time variation of measured accelerations of the rod, for 144 rot/min RPM. (Black-transversal accelerations, Magenta-longitudinal accelerations).

VI. Conclusions

Dynamic response of the system consists on two segments, one due to the initial conditions, that is rapidly damped and another due to the perturbing forces.

Using the finite elements method, the continuous system is replaced with a discrete system, with a finite number of degrees of freedom. The unknown problems are not the displacement functions *u (x, y, z, t)* anymore, but the nodal displacements *d(t)*.

The equations systems with partial derivatives (Lame equations from the elasticity theory) are become a system with differential equations. The matrix that occurs in the general motion equation is identified by starting from the idea of overlapping the rigid body motion upon the elastic body motion. The rigid body motion is introduced by a set of reference coordinates that define the location and orientation of the local reference system attached to each kinematic link.

Also, assume that the damping force is proportional with the velocity it is considered a differential system of equations with constant coefficients ([M], [K] and [C]).

Composing the two motions it is proposed a complete dynamic analysis with multiple practical applications.
The motion of mechanical system is described by the general equations (53), where the coefficients [M], [K] and [C] can be variable in time.

Introducing the modal matrix, composed by the Eigen modes it can be assured the differential equation system uncoupled, by obtaining a set of rank II differential equation. These are solved more rapidly than the initial system, with the remark that the frequencies of the Eigen modes are all computed.

There are developed lots of numerical models that follow a quick solving of differential equations. Some of these methods take into account the dynamic answer which can be obtained with a good precision, by considering only the first Eigen vectors [4].

The experimental setup allows the kinematic parameters establishment that characterize the kinematic links vibrations for different working conditions. It was considered the experimental analysis at 144 RPM and 20 N for the technological force. In this condition was studied the theoretical dynamic response, presented through this research. By comparing the results obtained theoretically and experimentally the following conclusions were obtained:
1. For the rod longitudinal accelerations, by analyzing Fig. 6 and 14, it was recorded a 1.16% error;
2. For the transversal accelerations, by analyzing figure 7 and figure 14, it was recorded a 2.67% error.

For an easier identification of kinematic parameters variation diagrams from figures 13 and 14, on the left axis it is represented the transversal acceleration and on right vertical axis it is represented the longitudinal acceleration, in [meters/second^2].

References

[1] Dumitru, N.; Malciu, R., Geonea, I., Differential Transmission for Robotizing a Powered Wheelchair, *Proceedings of the OPTIROB* 2010, 28-30 May 2010, pp. 47-51, ISBN 978-981-08-5840-7.
[2] G. Piras, W.L. Cleghorn, J.K. Mills, Dynamic finite-element analysis of a planar high-speed, high-precision parallel manipulator with flexible links, *Mechanism and Machine Theory*, Volume 40, Issue 7, July 2005, Pages 849–862.
[3] W.L Cleghorn, G Fenton, B Tabarrok, Finite element analysis of high-speed flexible mechanisms, *Mechanism and Machine Theory*, Volume 16, Issue 4, 1981, Pages 407-424.
[4] B.M Bahgat, K.D Willmert, Finite element vibrational analysis of planar mechanisms, *Mechanism and Machine Theory*, Volume 11, Issue 1, 1976, Pages 47-71.
[5] Ji Oh Song, J. Haug, Dynamic analysis of planar flexible mechanisms, *Computer Methods in Applied Mechanics and Engineering*, Volume 24, Issue 3, December 1980, Pages 359-381.
[6] B. Jonker, A finite element dynamic analysis of spatial mechanisms with flexible links, *Computer Methods in Applied Mechanics and Engineering*, Volume 76, Issue 1, November 1989, Pages 17-40.
[7] Z. Xianmi, S. Yunwen, Optimal design of flexible mechanisms with frequency constraints, *Mechanism and Machine Theory*, Volume 30, Issue 1, January 1995, Pages 131-139.
[8] N. Dumitru, M. Cherciu, A. Zuhair, Theoretical and experimental modelling of the dynamic response of the mechanisms with deformable kinematics elements, *Proceedings of IFToMM, Besancon, France*, Paper A-954.
[9] N. Dumitru, C. Copilusi, A. Zuhair, Dynamic Modeling of a Mobile Mechanical System with Deformable Elements, *Proceedings of the World Congress on Engineering 2009* Vol II WCE 2009, July 1 - 3, 2009, London, U.K.
[10] N. Dumitru, R. Malciu, M. Calbureanu, S. Dumitru, G. C. Marinescu, "Dynamic Analysis of a Mobile Mechanical System with Deformable Elements", *Advanced Materials Research*, Vols. 463-464, pp. 1242-1245, 2012.
[11] Farid Amirouch, Fundamentals of multybody dynamics- Theory and applications, ISBN 0-8176-4236-6, Ed. Birkhauser Boston, 2006.

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