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Smoothing Yield Curve Data by Least Squares and Concavity or Convexity

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***Abstract*—A yield curve represents the relationship between market interest rates and time to maturity of debt securities. Typically it rises gradually, but occasionally it may decrease as the time to maturity increases. In practice, however, the observed yield curve data demonstrate wide variation so as typical shapes of concavity or convexity are usually lost. We address the question of smoothing the yield data by a least squares calculation that provides non-positive second divided differences in order to restore concavity and analogously for convexity. An illustrative example is worked out using the L2CXFT software by the second author which is quite suitable for this purpose. Our approach may be helpful to long-term decisions for investments in bonds.**

***Index Terms*—Bonds, Concavity, Divided differences, Least squares data smoothing, Yield curve.**

I. Introduction

YIELD curves represent the relationship between market interest rates and time to maturity of debt securities [5]. Yields in theory follow a smooth rising curve as the time to maturity increases. Often yield curves fluctuate around with time and can assume various shapes, as for instance concave, convex, sigmoid, flat or humped. However, observed data from yield curves demonstrate wide variation over time. Hence, concavity or convexity etc is usually lost.

In this paper we apply a method that restores concavity by minimizing the sum of the quadratic differences between the yields that can be computed from the yields actually measured subject to non-positive second divided differences.

The data are the measured yield values ϕ_i at time instances x_i, for $i = 1, 2, \ldots, n$, where x_i are in strictly ascending order. The method used, the Demetriou and Powell [4] optimization calculation, is a non-parametric model which seeks estimates y_i of the ϕ_i by minimizing the objective function

$$\sum_{i=1}^{n} (y_i - \phi_i)^2 \tag{1}$$

subject to the concavity constraints

$$y[x_{i-1}, x_i, x_{i+1}] \leq 0, \; i = 2, 3, \ldots, n-1, \tag{2}$$

where

$$y\left[x_{i-1}, x_i, x_{i+1}\right] = \frac{y_{i-1}}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})} + \frac{y_i}{(x_i - x_{i-1})(x_i - x_{i+1})} + \frac{y_{i+1}}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)} \tag{3}$$

is the ith second divided difference on $\{y_i : \; i = 1, 2 \ldots, n\}$. We note that if the data are exact values of a function that has continuous second derivative, then the second differences provide approximate values to the second derivative.

In Section II we give a short note on the yield curve as a function of time to maturity and a definition of the method of [4] for obtaining concave or convex estimates of the observed yield curve data. In Section III we apply this method to a dataset with yield data of government bonds which have been obtained by the Statistical Data Warehouse of European Central Bank and demonstrate the suitability of our approach for smoothing by concavity. In Section IV we present some conclusions and propose topics for further study.

Our approach is intended for use as a guide to the risk analyst for long-term decisions concerning investments in bonds.

II. Yield Curve Data and the Method that gives Concavity

A. The Yield Curve

Fixed-income securities are investment instruments where the borrower is obliged to make payments of a fixed amount on a fixed schedule. A vast bulk of money in these securities is committed to bonds. Bonds are usually analyzed by computing the yield to maturity, which is the interest rate that makes the present value of promised bond payments equal to current bond price [7], [10].

The yield curve is constructed from yields of various debt securities as a function of time to maturity. These yields follow a smooth curve, which rises gradually. In practice, however, the yield curve undulates and can assume various shapes as for instance sloping gradually downward or upward or following a sigmoid shape (see, for example, Luenberger [6]).

European Central Bank (ECB) maintains data for yield curves [5]. A typical scatter plot of a yield to maturity relationship to a dataset obtained from [5] is given in Fig. 1. Although, it is not very informative due to the variation of the data, the specific graph suggests that the yield curve follows a concave shape, where there is a range of ascend that is followed subsequently by a range of descent.

B. The Method of Calculation

In this subsection we state the quadratic programming method of [4] for calculating the solution of the problem of Section I.

The method depends on the Karush-Kuhn-Tucker conditions for the minimization of (1) subject to (2). The solution, a n-vector $\underline{y}$ say, occurs if and only if the constraints (2) are

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Fig. 1. Yield curve of government bonds during the period 02-05-2013 to 24-04-2014 to data obtained from ECB

satisfied and there exist non-positive Lagrange multipliers $\{\mu_i : i \in \mathcal{A}\}$ such that

$$\underline{y} - \underline{\phi} = \frac{1}{2}\sum_{i\in\mathcal{A}} \mu_i \underline{a}_i, \tag{4}$$

where $\mathcal{A} = \{i : y[x_{i-1}, x_i, x_{i+1}] = 0\}$ and where the n-vector $\underline{a}_i$ is the gradient of $y[x_{i-1}, x_i, x_{i+1}]$.

The quadratic programming algorithm generates a finite sequence of subsets $\mathcal{A}$ until termination and for each subset calculates $\underline{y}$ to minimize (1) subject to the equality constraints

$$\underline{y}^T \underline{a}_i = 0, \ i \in \mathcal{A}. \tag{5}$$

For each $\mathcal{A}$, equation (4) defines unique Lagrange multipliers. During the iteration course, the non-positivity of the multipliers is tested and either the algorithm terminates or a new iteration begins. For details and proofs on this calculation, one may consult [4]. For sensitivity issues see [3].

The equality constrained minimization problem (1)-(5) forms an important part of the calculation, because it is solved very efficiently by a reduction to an equivalent unconstrained one with fewer variables due to a linear B-spline representation. Moreover, this representation offers a condensed form of the solution which may be useful to further calculations, like differentiation, integration and storage economization. If $y(x)$, $x_1 \le x \le x_n$ is the piecewise linear interpolant to the points $\{(x_i, y_i) : i = 1, 2, \dots, n\}$, then $y(x)$ has its knots in the set $\{x_i : i \in \{1, 2, \dots, n\} \setminus \mathcal{A}\}$. Indeed, the equation $y[x_{i-1}, x_i, x_{i+1}] = 0$, when $i \in \mathcal{A}$, implies the collinearity of the points (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}), but if $y[x_{i-1}, x_i, x_{i+1}] > 0$, then i is the index of a knot of $y(x)$. Thus the knots of $y(x)$ are determined from the abscissae x_i due to set $\mathcal{A}$. Let $j = n - 1 - |\mathcal{A}|$, let $\{\xi_p : p = 1, \dots, j-1\}$ be the interior knots of $y(x)$ in ascending order, let also $\xi_{-1} = \xi_0 = x_1$ and $\xi_j = \xi_{j+1} = x_n$, and let $\{B_p : p = 0, 1, \dots, j\}$ be a basis of normalized linear B-splines that are defined on $\{x_i : i = 1, 2, \dots, n\}$ and satisfy the equations $B_p(\xi_p) = 1$ and $B_p(\xi_q) = 1$, $p \ne q$ (see, for example, de Boor [1]):

$$B_p(x) = \begin{cases} (x - \xi_{p-1})/(\xi_p - \xi_{p-1}), & \xi_{p-1} \le x \le \xi_p \\ (\xi_{p+1} - x)/(\xi_{p+1} - \xi_p), & \xi_p \le x \le \xi_{p+1} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Then $y(x)$ may be written uniquely in the form

$$y(x) = \sum_{p=0}^{j} \sigma_p B_p(x), \ x_1 \le x \le x_n, \tag{7}$$

where the coefficients $\{\sigma_p : p = 0, 1, \dots, j\}$ are the values of $y(x)$ at the knots and are calculated by solving the normal equations associated with the minimization of (1) subject to (5), after substituting (7) into (1). The intermediate components of $\underline{y}$ are found by linear interpolation to the spline coefficients due to (6) and (7).

The software package L2CXFT of Demetriou [2] (Algorithm 742 in Collected Algorithms of ACM, http://calgo.acm.org/) implements the quadratic programming method and includes several useful extensions to the algorithm in [4]. The method may well be applied if nonnegative divided differences are employed in (2) in order to obtain an analogous best convex fit to the data.

III. An Application of the Method in Real Yield Curve Data

A. The Data

In this section, we apply the method of Section II-B to a dataset obtained from the Statistical Data Warehouse of ECB

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TABLE I
BEST LEAST SQUARES CONCAVE FIT TO YIELD DATA DURING THE PERIOD 01-05-2013 TO 24-04-2014

date	i	x_i	ϕ_i	y_i	Δy	$\Delta^2 y$	date	i	x_i	ϕ_i	y_i	Δy	$\Delta^2 y$
2/5/2013	1	41396	4.0884	4.0743			30/7/2013	64	41485	4.6800	4.4823	0.0034	0
3/5/2013	2	41397	4.0951	4.0871	0.0128	0	31/7/2013	65	41486	4.6212	4.4857	0.0034	0
6/5/2013	3	41400	4.1021	4.1255	0.0128	0	1/8/2013	66	41487	4.5221	4.4891	0.0034	0
7/5/2013	4	41401	4.1323	4.1383	0.0128	0	2/8/2013	67	41488	4.4446	4.4925	0.0034	0
8/5/2013	5	41402	4.1270	4.1511	0.0128	0	5/8/2013	68	41491	4.4652	4.5028	0.0034	0
9/5/2013	6	41403	4.1548	4.1639	0.0128	0	6/8/2013	69	41492	4.4006	4.5062	0.0034	0
10/5/2013	7	41404	4.2282	4.1767	0.0128	-0.00110	7/8/2013	70	41493	4.4878	4.5096	0.0034	0
13/5/2013	8	41407	4.1974	4.2019	0.0084	0	8/8/2013	71	41494	4.4102	4.5130	0.0034	0
14/5/2013	9	41408	4.1713	4.2103	0.0084	0	9/8/2013	72	41495	4.4459	4.5164	0.0034	0
15/5/2013	10	41409	4.2448	4.2187	0.0084	0	12/8/2013	73	41498	4.4578	4.5266	0.0034	0
16/5/2013	11	41410	4.2639	4.2271	0.0084	-0.00249	13/8/2013	74	41499	4.5745	4.5300	0.0034	0
17/5/2013	12	41411	4.2539	4.2305	0.0034	0	14/8/2013	75	41500	4.5705	4.5334	0.0034	0
20/5/2013	13	41414	4.2567	4.2407	0.0034	0	15/8/2013	76	41501	4.6112	4.5368	0.0034	0
21/5/2013	14	41415	4.2981	4.2441	0.0034	0	16/8/2013	77	41502	4.4428	4.5402	0.0034	0
22/5/2013	15	41416	4.2659	4.2475	0.0034	0	19/8/2013	78	41505	4.4821	4.5504	0.0034	0
23/5/2013	16	41417	4.2454	4.2509	0.0034	0	20/8/2013	79	41506	4.4517	4.5538	0.0034	0
24/5/2013	17	41418	4.2193	4.2543	0.0034	0	21/8/2013	80	41507	4.5667	4.5572	0.0034	0
27/5/2013	18	41421	4.2790	4.2645	0.0034	0	22/8/2013	81	41508	4.5826	4.5606	0.0034	0
28/5/2013	19	41422	4.2840	4.2679	0.0034	0	23/8/2013	82	41509	4.5986	4.5640	0.0034	0
29/5/2013	20	41423	4.3038	4.2713	0.0034	0	26/8/2013	83	41512	4.5783	4.5742	0.0034	0
30/5/2013	21	41424	4.3266	4.2747	0.0034	0	27/8/2013	84	41513	4.4437	4.5776	0.0034	0
31/5/2013	22	41425	4.3352	4.2781	0.0034	0	28/8/2013	85	41514	4.5419	4.5810	0.0034	0
3/6/2013	23	41428	4.3017	4.2883	0.0034	0	29/8/2013	86	41515	4.5196	4.5844	0.0034	0
4/6/2013	24	41429	4.3392	4.2918	0.0034	0	30/8/2013	87	41516	4.5803	4.5878	0.0034	0
5/6/2013	25	41430	4.3231	4.2952	0.0034	0	2/9/2013	88	41519	4.5514	4.5980	0.0034	0
6/6/2013	26	41431	4.3641	4.2986	0.0034	0	3/9/2013	89	41520	4.5816	4.6015	0.0034	0
7/6/2013	27	41432	4.2447	4.3020	0.0034	0	4/9/2013	90	41521	4.5979	4.6049	0.0034	0
10/6/2013	28	41435	4.3520	4.3122	0.0034	0	5/9/2013	91	41522	4.7160	4.6083	0.0034	0
11/6/2013	29	41436	4.1893	4.3156	0.0034	0	6/9/2013	92	41523	4.6220	4.6117	0.0034	0
12/6/2013	30	41437	4.2293	4.3190	0.0034	0	9/9/2013	93	41526	4.6896	4.6219	0.0034	0
13/6/2013	31	41438	4.2192	4.3224	0.0034	0	10/9/2013	94	41527	4.7537	4.6253	0.0034	0
14/6/2013	32	41439	4.1592	4.3258	0.0034	0	11/9/2013	95	41528	4.7089	4.6287	0.0034	0
17/6/2013	33	41442	4.2681	4.3360	0.0034	0	12/9/2013	96	41529	4.6799	4.6321	0.0034	0
18/6/2013	34	41443	4.3264	4.3394	0.0034	0	13/9/2013	97	41530	4.6902	4.6355	0.0034	-0.00020
19/6/2013	35	41444	4.2094	4.3428	0.0034	0	16/9/2013	98	41533	4.6873	4.6433	0.0026	-0.00069
20/6/2013	36	41445	4.3844	4.3462	0.0034	0	17/9/2013	99	41534	4.6471	4.6432	-0.0002	0
21/6/2013	37	41446	4.4196	4.3496	0.0034	0	18/9/2013	100	41535	4.6730	4.6430	-0.0002	0
24/6/2013	38	41449	4.4587	4.3598	0.0034	0	19/9/2013	101	41536	4.6783	4.6428	-0.0002	0
25/6/2013	39	41450	4.5046	4.3632	0.0034	0	20/9/2013	102	41537	4.7101	4.6427	-0.0002	0
26/6/2013	40	41451	4.4387	4.3666	0.0034	0	23/9/2013	103	41540	4.6947	4.6422	-0.0002	0
27/6/2013	41	41452	4.4299	4.3700	0.0034	0	24/9/2013	104	41541	4.6560	4.6420	-0.0002	0
28/6/2013	42	41453	4.4649	4.3734	0.0034	0	25/9/2013	105	41542	4.5954	4.6419	-0.0002	0
1/7/2013	43	41456	4.4199	4.3836	0.0034	0	26/9/2013	106	41543	4.6130	4.6417	-0.0002	0
2/7/2013	44	41457	4.4121	4.3870	0.0034	0	27/9/2013	107	41544	4.6043	4.6416	-0.0002	0
3/7/2013	45	41458	4.3854	4.3904	0.0034	0	30/9/2013	108	41547	4.6036	4.6411	-0.0002	0
4/7/2013	46	41459	4.3089	4.3938	0.0034	0	1/10/2013	109	41548	4.5883	4.6409	-0.0002	0
5/7/2013	47	41460	4.3298	4.3973	0.0034	0	2/10/2013	110	41549	4.5920	4.6408	-0.0002	0
8/7/2013	48	41463	4.3413	4.4075	0.0034	0	3/10/2013	111	41550	4.6272	4.6406	-0.0002	0
9/7/2013	49	41464	4.3379	4.4109	0.0034	0	4/10/2013	112	41551	4.6592	4.6404	-0.0002	0
10/7/2013	50	41465	4.4171	4.4143	0.0034	0	7/10/2013	113	41554	4.6385	4.6400	-0.0002	0
11/7/2013	51	41466	4.4370	4.4177	0.0034	0	8/10/2013	114	41555	4.6427	4.6398	-0.0002	0
12/7/2013	52	41467	4.3738	4.4211	0.0034	0	9/10/2013	115	41556	4.6281	4.6396	-0.0002	0
15/7/2013	53	41470	4.3710	4.4313	0.0034	0	10/10/2013	116	41557	4.6748	4.6395	-0.0002	0
16/7/2013	54	41471	4.3675	4.4347	0.0034	0	11/10/2013	117	41558	4.6397	4.6393	-0.0002	0
17/7/2013	55	41472	4.3707	4.4381	0.0034	0	14/10/2013	118	41561	4.6577	4.6388	-0.0002	0
18/7/2013	56	41473	4.5168	4.4415	0.0034	0	15/10/2013	119	41562	4.6726	4.6387	-0.0002	0
19/7/2013	57	41474	4.4974	4.4449	0.0034	0	16/10/2013	120	41563	4.7033	4.6385	-0.0002	0
22/7/2013	58	41477	4.4108	4.4551	0.0034	0	17/10/2013	121	41564	4.6912	4.6384	-0.0002	0
23/7/2013	59	41478	4.4140	4.4585	0.0034	0	18/10/2013	122	41565	4.6743	4.6382	-0.0002	0
24/7/2013	60	41479	4.4648	4.4619	0.0034	0	21/10/2013	123	41568	4.6805	4.6377	-0.0002	0
25/7/2013	61	41480	4.4773	4.4653	0.0034	0	22/10/2013	124	41569	4.6274	4.6376	-0.0002	0
26/7/2013	62	41481	4.4809	4.4687	0.0034	0	23/10/2013	125	41570	4.6041	4.6374	-0.0002	0
29/7/2013	63	41484	4.5379	4.4789	0.0034	0	24/10/2013	126	41571	4.5856	4.6372	-0.0002	0

(continued)

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(Table I)

date	i	x_i	ϕ_i	y_i	Δy	$\Delta^2 y$	date	i	x_i	ϕ_{ii}	y_i	Δy	$\Delta^2 y$
25/10/2013	127	41572	4.5857	4.6371	-0.0002	0	27/1/2014	190	41666	4.3744	4.4214	-0.0050	0
28/10/2013	128	41575	4.5512	4.6366	-0.0002	0	28/1/2014	191	41667	4.4485	4.4164	-0.0050	0
29/10/2013	129	41576	4.5673	4.6364	-0.0002	0	29/1/2014	192	41668	4.4296	4.4113	-0.0050	0
30/10/2013	130	41577	4.5698	4.6363	-0.0002	0	30/1/2014	193	41669	4.3522	4.4063	-0.0050	0
31/10/2013	131	41578	4.5582	4.6361	-0.0002	0	31/1/2014	194	41670	4.3267	4.4012	-0.0050	0
1/11/2013	132	41579	4.5754	4.6360	-0.0002	0	3/2/2014	195	41673	4.3315	4.3861	-0.0050	0
4/11/2013	133	41582	4.5686	4.6355	-0.0002	0	4/2/2014	196	41674	4.3673	4.3811	-0.0050	0
5/11/2013	134	41583	4.6069	4.6353	-0.0002	0	5/2/2014	197	41675	4.3808	4.3760	-0.0050	0
6/11/2013	135	41584	4.6301	4.6352	-0.0002	0	6/2/2014	198	41676	4.3809	4.3710	-0.0050	0
7/11/2013	136	41585	4.6438	4.6350	-0.0002	0	7/2/2014	199	41677	4.3435	4.3660	-0.0050	0
8/11/2013	137	41586	4.6707	4.6348	-0.0002	0	10/2/2014	200	41680	4.3516	4.3508	-0.0050	0
11/11/2013	138	41589	4.6573	4.6344	-0.0002	0	11/2/2014	201	41681	4.3374	4.3458	-0.0050	0
12/11/2013	139	41590	4.6836	4.6342	-0.0002	0	12/2/2014	202	41682	4.3403	4.3408	-0.0050	0
13/11/2013	140	41591	4.6818	4.6340	-0.0002	0	13/2/2014	203	41683	4.3453	4.3357	-0.0050	0
14/11/2013	141	41592	4.6560	4.6339	-0.0002	0	14/2/2014	204	41684	4.3328	4.3307	-0.0050	0
15/11/2013	142	41593	4.6480	4.6337	-0.0002	0	17/2/2014	205	41687	4.3325	4.3156	-0.0050	0
18/11/2013	143	41596	4.6439	4.6332	-0.0002	0	18/2/2014	206	41688	4.3191	4.3105	-0.0050	0
19/11/2013	144	41597	4.6355	4.6331	-0.0002	0	19/2/2014	207	41689	4.3192	4.3055	-0.0050	0
20/11/2013	145	41598	4.6436	4.6329	-0.0002	0	20/2/2014	208	41690	4.3458	4.3004	-0.0050	0
21/11/2013	146	41599	4.6801	4.6328	-0.0002	0	21/2/2014	209	41691	4.3312	4.2954	-0.0050	0
22/11/2013	147	41600	4.6462	4.6326	-0.0002	0	24/2/2014	210	41694	4.3252	4.2803	-0.0050	0
25/11/2013	148	41603	4.6333	4.6321	-0.0002	0	25/2/2014	211	41695	4.2813	4.2752	-0.0050	0
26/11/2013	149	41604	4.6511	4.6320	-0.0002	0	26/2/2014	212	41696	4.2193	4.2702	-0.0050	0
27/11/2013	150	41605	4.6565	4.6318	-0.0002	-0.00113	27/2/2014	213	41697	4.2263	4.2652	-0.0050	0
28/11/2013	151	41606	4.6411	4.6294	-0.0024	0	28/2/2014	214	41698	4.2343	4.2601	-0.0050	0
29/11/2013	152	41607	4.6228	4.6270	-0.0024	0	3/3/2014	215	41701	4.1944	4.2450	-0.0050	0
2/12/2013	153	41610	4.6329	4.6197	-0.0024	0	4/3/2014	216	41702	4.2025	4.2400	-0.0050	0
3/12/2013	154	41611	4.6056	4.6173	-0.0024	0	5/3/2014	217	41703	4.2108	4.2349	-0.0050	0
4/12/2013	155	41612	4.6403	4.6149	-0.0024	0	6/3/2014	218	41704	4.2258	4.2299	-0.0050	0
5/12/2013	156	41613	4.6754	4.6125	-0.0024	0	7/3/2014	219	41705	4.2014	4.2248	-0.0050	0
6/12/2013	157	41614	4.6328	4.6100	-0.0024	0	10/3/2014	220	41708	4.1118	4.2097	-0.0050	0
9/12/2013	158	41617	4.6164	4.6028	-0.0024	0	11/3/2014	221	41709	4.2354	4.2047	-0.0050	0
10/12/2013	159	41618	4.6018	4.6004	-0.0024	0	12/3/2014	222	41710	4.2495	4.1996	-0.0050	0
11/12/2013	160	41619	4.5679	4.5980	-0.0024	0	13/3/2014	223	41711	4.1918	4.1946	-0.0050	0
12/12/2013	161	41620	4.5746	4.5955	-0.0024	0	14/3/2014	224	41712	4.1700	4.1896	-0.0050	0
13/12/2013	162	41621	4.5580	4.5931	-0.0024	0	17/3/2014	225	41715	4.1652	4.1744	-0.0050	0
16/12/2013	163	41624	4.5877	4.5859	-0.0024	0	18/3/2014	226	41716	4.1078	4.1694	-0.0050	0
17/12/2013	164	41625	4.5849	4.5835	-0.0024	0	19/3/2014	227	41717	4.1272	4.1644	-0.0050	0
18/12/2013	165	41626	4.5462	4.5810	-0.0024	0	20/3/2014	228	41718	4.1437	4.1593	-0.0050	0
19/12/2013	166	41627	4.5256	4.5786	-0.0024	0	21/3/2014	229	41719	4.1099	4.1543	-0.0050	0
20/12/2013	167	41628	4.5255	4.5762	-0.0024	0	24/3/2014	230	41722	4.1238	4.1392	-0.0050	0
23/12/2013	168	41631	4.5630	4.5690	-0.0024	0	25/3/2014	231	41723	4.0552	4.1341	-0.0050	0
24/12/2013	169	41632	4.5582	4.5665	-0.0024	0	26/3/2014	232	41724	4.1078	4.1291	-0.0050	0
27/12/2013	170	41635	4.5755	4.5593	-0.0024	0	27/3/2014	233	41725	4.0937	4.1240	-0.0050	0
30/12/2013	171	41638	4.5430	4.5520	-0.0024	0	28/3/2014	234	41726	4.0788	4.1190	-0.0050	0
31/12/2013	172	41639	4.5342	4.5496	-0.0024	0	31/3/2014	235	41729	4.0913	4.1039	-0.0050	0
2/1/2014	173	41641	4.5922	4.5448	-0.0024	0	1/4/2014	236	41730	4.0894	4.0988	-0.0050	0
3/1/2014	174	41642	4.6355	4.5424	-0.0024	-0.00066	2/4/2014	237	41731	4.1256	4.0938	-0.0050	0
6/1/2014	175	41645	4.6178	4.5273	-0.0050	0	3/4/2014	238	41732	4.1184	4.0888	-0.0050	0
7/1/2014	176	41646	4.5852	4.5222	-0.0050	0	4/4/2014	239	41733	4.1273	4.0837	-0.0050	0
8/1/2014	177	41647	4.5611	4.5172	-0.0050	0	7/4/2014	240	41736	4.1254	4.0686	-0.0050	0
9/1/2014	178	41648	4.5502	4.5121	-0.0050	0	8/4/2014	241	41737	4.1358	4.0636	-0.0050	0
10/1/2014	179	41649	4.5756	4.5071	-0.0050	0	9/4/2014	242	41738	4.1283	4.0585	-0.0050	0
13/1/2014	180	41652	4.5730	4.4920	-0.0050	0	10/4/2014	243	41739	4.0964	4.0535	-0.0050	0
14/1/2014	181	41653	4.5129	4.4869	-0.0050	0	11/4/2014	244	41740	4.0737	4.0484	-0.0050	0
15/1/2014	182	41654	4.5002	4.4819	-0.0050	0	14/4/2014	245	41743	4.0739	4.0333	-0.0050	0
16/1/2014	183	41655	4.5044	4.4769	-0.0050	0	15/4/2014	246	41744	4.0632	4.0283	-0.0050	0
17/1/2014	184	41656	4.4695	4.4718	-0.0050	0	16/4/2014	247	41745	4.0217	4.0232	-0.0050	0
20/1/2014	185	41659	4.4271	4.4567	-0.0050	0	17/4/2014	248	41746	4.0069	4.0182	-0.0050	0
21/1/2014	186	41660	4.3794	4.4517	-0.0050	0	22/4/2014	249	41751	4.0282	3.9930	-0.0050	0
22/1/2014	187	41661	4.3935	4.4466	-0.0050	0	23/4/2014	250	41752	3.9965	3.9880	-0.0050	0
23/1/2014	188	41662	4.3990	4.4416	-0.0050	0	24/4/2014	251	41753	3.9869	3.9829	-0.0050	-
24/1/2014	189	41663	4.3618	4.4365	-0.0050	0							

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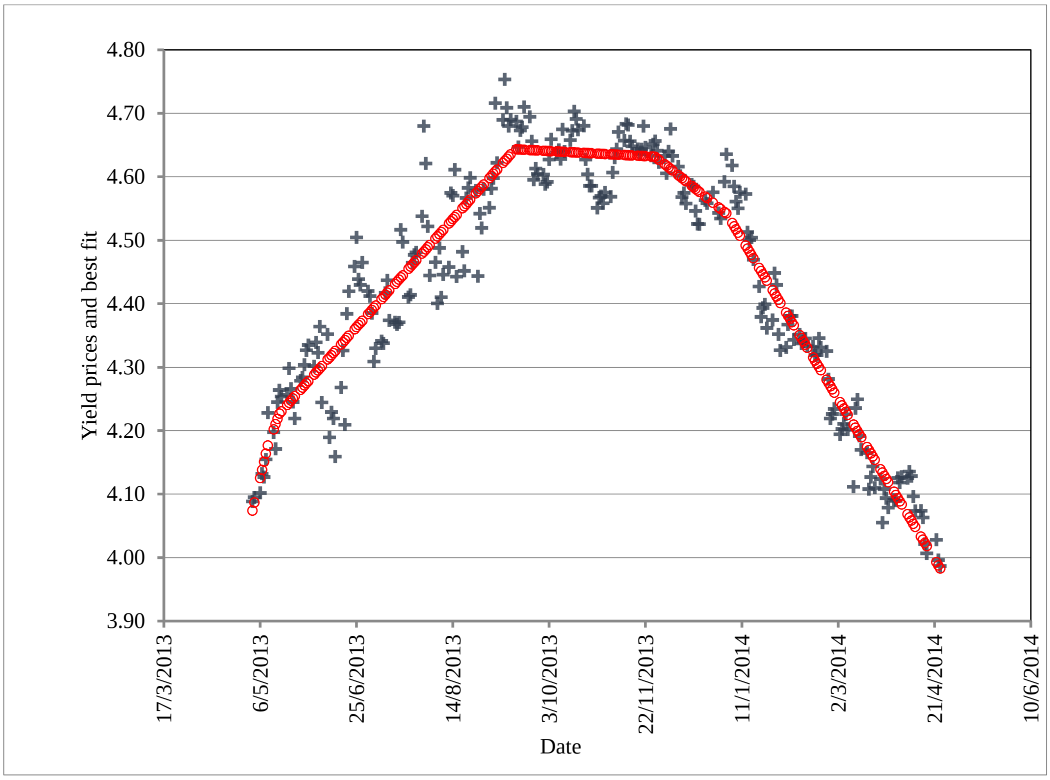

Fig. 2. The best concave fit (circle) to the data (cross) presented in Table I

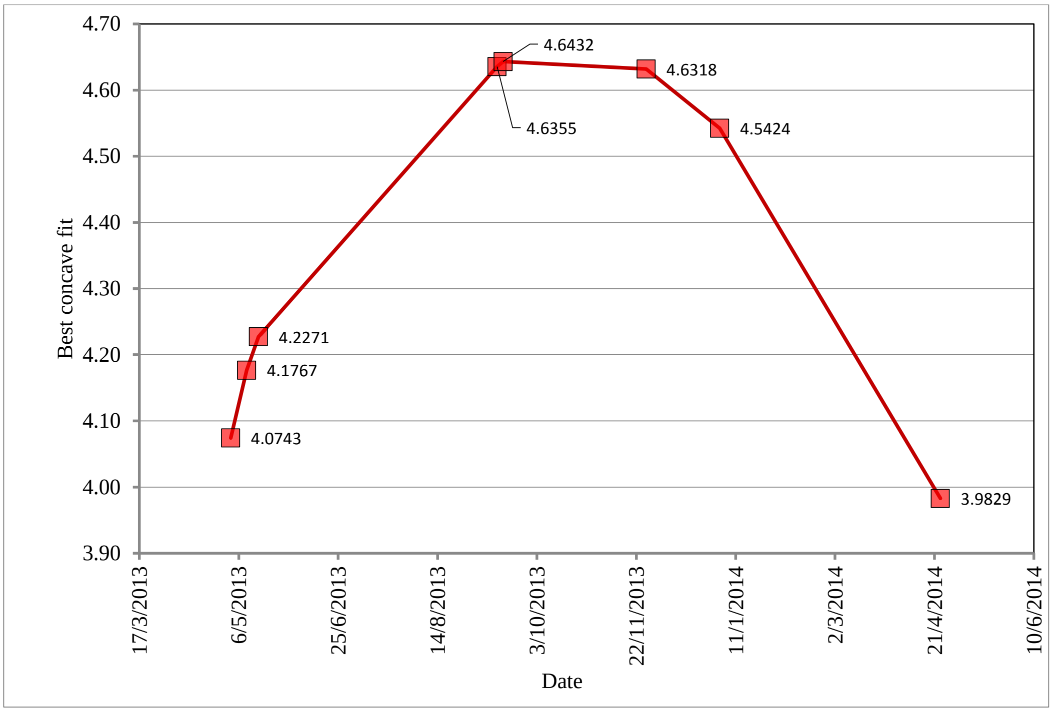

Fig. 3. The best linear spline fit (line) to the data of Fig. 2 together with the spline coefficients (square) presented in Table II

[5].

Data Warehouse website provides a sophisticated tool, which enables the user to retrieve information setting suitable parameters. We downloaded $n = 251$ observations of yield values of the government bonds issued from May 2nd 2013 to April 24th 2014, which are displayed in Fig. 1. The actual data values are presented in the columns labeled 'date' and 'ϕ_i' of Table I, while in column labeled 'i' a running integer counts the data components.

B. Calculating Concavities by Software L2CXFT

At a first glance the data exhibits a concave trend throughout the whole period with wide variations. Our attempt at fitting these data is going to demonstrate some important points in order to provide guidelines for the analyst.

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The actual smoothing calculation was carried out by supplying the data to the L2CXFT software, which is the main tool of our work.

Initially we prepared the input file to the L2CXFT, which contains only two columns: the first column keeps the abscissae x_i, namely the dates of the observations, transformed to a series of consecutive numerical values. The 'DATEVALUE' function of Microsoft Excel^{TM} was used to convert a particular date to an integer. For example, the formula '=DATEVALUE("2/5/2013")' returns 41396, where the arguments represent the date 02/05/2013. The second column of the input file keeps the ϕ_i values which are the yield data at the respective x_i. In view of the trend of the data, L2CXFT was required to provide the best concave fit to the data. Indeed, we fed the data to the computer program and the best fit was calculated, where only few iterations were needed for termination. Fig. 2 displays the fit along with the raw data for comparison. Despite the data fluctuation, it is noticeable that the concave fit followed quite satisfactorily the trend of the data, while it exhibited clear linear relationships within the data range.

In Table I, besides 'date', i, x_i and ϕ_i, we have tabulated y_i (best fit), Δy (first difference, namely the quotient of $(y_{i+1} - y_i)/(x_{i+1} - x_i)$) and $\Delta^2 y$ (second divided difference centered at y_i, namely $y[x_{i-1}, x_i, x_{i+1}]$). The first differences are the rates of change of the estimated yield values having the values $\{0.0128, 0.0084, 0.0034, 0.0026, -0.0002, -0.0024, -0.0050\}$. We see that they decrease in the whole range of the data having positive values in the range $[x_1, x_{98}]$ and negative values in the range $[x_{99}, x_{251}]$. Hence, the maximum estimated yield is equal to 4.6433, which occurs at x_{98} =16/9/2013. Further, we note that if $y[x_{i-1}, x_i, x_{i+1}] < 0$ (seventh column) then x_i is a knot of the resultant fit. The first differences give the slopes of the line segments between the knots, that is on the intervals $[\xi_0, \xi_1]$, $[\xi_1, \xi_2]$ and so on.

The software package provided also the coefficients of the spline representation of the fit $y(x),\ x \in [x_1, x_{251}]$. Indeed, Table II gives the data indices of the spline knots, the corresponding date at the knot ξ_p, the B-spline coefficient $\sigma_p = y(\xi_p)$ which is a yield estimate, and the calculated difference $\Delta^2(y(\xi_p))$. Fig. 3 displays the piecewise linear concave fit by indicating the positions of the knots $\xi_p,\ p = 0, 1, \ldots, 7$ together with the end points that consists of only seven line segments. We see that the third and the fourth knots are at successive abscissae, which allows the fit to "catch" the steep directional change on the top range of the data and thus to be able to follow the concave data trend.

TABLE II
KNOTS, ESTIMATES OF BONDS AND CONCAVITY ESTIMATES

p	Data index at a knot	Date at knot ξ_p	Yield estimate $\sigma_p = y(\xi_p)$	Concavity $\Delta^2(y(\xi_p))$
0	1	2/5/2013	4.0743	-
1	7	10/5/2013	4.1767	–0.00110
2	11	16/5/2013	4.2271	–0.00249
3	97	13/9/2013	4.6355	–0.00020
4	98	16/9/2013	4.6432	–0.00069
5	150	27/11/2013	4.6318	–0.00113
6	174	3/1/2014	4.5424	–0.00066
7	251	24/4/2014	3.9829	-

IV. Conclusions

Yield data demonstrate wide variation over time. This paper has addressed the question of smoothing the yield data by minimizing the sum of the squares of the variations subject to non-positive second divided differences of the yield estimates. The calculation is performed by the particularly suitable software package L2CXFT. The user needs only to supply the data to the program, which subsequently calculates the best fit and its linear spline coefficients.

This method was applied to yield observations obtained by the Statistical Data Warehouse of the European Central Bank and the best concave polygonal fit to the dataset was calculated automatically by a fast and efficient calculation.

The technique was applied to the whole dataset which is useful to long-term decisions. It may be very helpful also to be applied to short data ranges in order to guide short-term investment decisions in bonds. The way of investing in bonds for short-term or long-term depends on the investment goals and time frames, as well as on the amount of risk which an investor is willing to take. Of course, one may well combine certain features of our analysis with suggested yield models, as for instance the model of [9] or other models, if some opportunity exists to analyze the data.

It is worthwhile to note that the piecewise linear fit that is obtained by our data fitting method, economizes on storage requirements of the yield over time dataset. Indeed, the method defines the time history of a dataset as a linear spline, defined in terms of the values contained in the columns labeled 'Data index at a knot', 'Date at a knot' and 'Yield estimate' of Table II, which require far less storage than the initial dataset. In our example, it requires two orders of magnitude less memory storage. Moreover, this remark suggests that storage economization would be useful to machine learning type algorithms (see, for example, [8]) for handling the large amounts of yield data that are generated daily worldwide.

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