Historical Review of Topological or Differential Invariants and Application to Engineering Sciences and Physics

Philippe Durand

***Abstract*—The notion of topological invariants is very old. Since long used in pure mathematics, it is now widely used in engineering science, applied mathematics and theoretical physics. We propose here revisiting this notion and giving examples that have advanced the engineering sciences but also mathematical physics.**

Keywords: Invariants, Kron method, moduli spaces, index theorem, mathematical physics.

I. Introduction

THe search for invariants dates back to the work of Euler and Poincaré. An invariant of a topological space is easily defined, if two topological spaces X and Y are homeomorphic if I denotes an invariant, necessarily $I(X) = I(Y)$. the converse is obviously false, and this motivates the search for invariant sufficiently sophisticated to separate different spaces. An invariant can be numerical, it can also be an algebraic structure. Euler was interested in topological manifolds of dimension 1, namely, graphs, to solve the problem of the Konigsberg bridge. He defines the notion of Eulerian graphs. He also discovered a formula to characterize topologically graphs or polyhedra: the Euler characteristic. Poincaré generalizes this invariant, which will become the characteristic of Euler Poincaré for topological manifolds of higher dimensions, this invariant is insufficient to distinguish the sphere S^1 from the sphere S^3, it is on the other hand sufficient to classify all the compact surfaces orientable ... A notion derived from graph theory is the notion of k-connectivity: A graph is k-connected when we can disconnect it by removing k-edges (this is a "discrete" version of the homotopy theory introduced by Poincaré). In defining the fundamental group, then the groups of higher homotopies, Poincaré does no more than translate the notion of k-connectivity. in the context of topological spaces. In algebraic topology, we say that a topological space is k-connected, if all groups of homotopies are zero until $i = k - 1$. So the S^{k+1} sphere or the space $\mathbf{R}^{k+2} - \{0\}$ are k-connected. We can notice that to disconnect the sphere of dimension $k + 1$, we must remove a sphere of dimension k:. On the other hand, in topology a space is contractile when all his higher homotopy groups are null, so, maybe we could to invent an equivalent notion for graph theory (to convince oneself that a complete graph could do the trick ...). In the first part, we give an application to tensor analysis of electrical networks was developed by Gabriel Kron. In the following, we revisit the applications to mathematical physics and field theory. Atiyah at the beginning of the sixties upsets the world of mathematics to the index theorem. We discuss the machinery for defining new invariants and its applications in physics.

So, first of all,, we consider an example, applied to electrical networks of invariant for manifolds of dimension 1: the invariant of Kron: two equations that connect, in the graph of an electric circuit, the number of nodes, branches and meshes, independent between us, after, which must be considered to put into equation an electrical network in the space of the meshes. In the second part of the paper, we give a reminder of some simple invariants that can be considered for topological manifolds. The third part, finally, shows that mathematical physics has provided new invariants. to better understand symplectic manifold but also three- and four-dimensional manifolds.

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II. Tensorial analysis of networks: Kron method

Gabriel Kron, inspired by Einstein's work on general relativity, proposes to study electrical machines from the angle of tensor analysis [1]. An electrical circuit can then be decomposed into nodes (vertices of a graph), edges then meshes.

The most classical invariant to which we think in graph theory is the characteristic of Euler Poincaré. For a graph, we can consider the number of cycles decreased by the number of vertices and increased by a number of edges. This invariant is not interesting for the study of the electrical circuits because it does not distinguish among the vertices, edges, cycles, how many are independent. only those, will be taken into account for transformed currents in the space of the meshes in the method of Kron.

A. Kron Invariant

We consider the vector space of the formal chains constituted by the nodes:$n^1, n^2, ...n^N$.

Similarly, we consider the vector space of the branches generated by $\mathcal{B}$: $b_1, b_2..., b_B$.

we consider linear map: δ de $\mathcal{B}$ dans $\mathcal{N}$ define by:

$\delta(b_i) =\varepsilon_j.n^j$ with $\varepsilon_j = 1$ if the end of b_i is n^j

$\varepsilon_j = -1$ the origin of b_i is n^j

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$\varepsilon_j = 0$ if the end of b_i is origin of b_i
For example for the graph whose branches are:
b_1: origin n^1 and the end n^2
b_2: origin n^2 and the end n^3
b_3: origin n^1 and the end n^3
b_4: origin n^2 and the end n^2

The matrix of linear map is given by:

$$G = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The fundamental relation of linear algebra gives:

$dim(\mathcal{B}) = dimker(\delta) + dim(Im(\delta)$
This relation is the first relation of Kron, in fact the kernels of δ is the vector space $\mathcal{M}$ of meshes of dimension M. The image of δ, the vector space $\mathcal{P}$ of pairs of nodes, of dimension P of or the dimensional relation:

$$B = M + P \tag{1}$$

We also have the relation:
$dimIm(\delta) = dim(\mathcal{N}) - dim(\mathcal{N}/Im(\delta))$,

the last part of this equality is the quotient of the set of nodes, by the nodes that go in pairs. This gives the number of connected components of the graph: the number of subnetworks. This is the second relation of Kron

$$P = N - S \tag{2}$$

these two quantities are topological invariants because it depends only on the dimension of the spaces and sub-spaces vector considered. In a previous paper, we use, starting from the singular homology, finer topological invariants, to find topologically the law of the mesh and that of the nodes [2],[4]

B. Example

Figure 1 shows an example of circuit :two networks such that each one is controlled by the other. The second network is powered by the voltage Vdc(t) reported from the first network, and the load current of the second network i_s is injected in the first network depending on a command law.

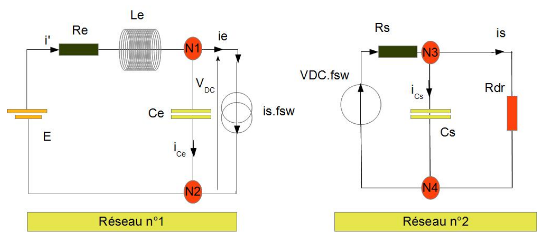

Fig. 1. Network with two connected components

The second network includes a generator E_2, given by: $E_2 = Vdc * fsw$. The visible elements in the graph given Figure 5 are the topological following character :

- 4 physical nodes n1,..,n4 ($\rightarrow N = 4$)
- 5 branches b1,..,b5 ($\rightarrow B = 5$)
- 3 meshes m1,m2,m3 ($\rightarrow M = 3$)
- 2 networks R1, R2 ($\rightarrow R = 2$)
- 2 nodes pair ($\rightarrow P = 2$)

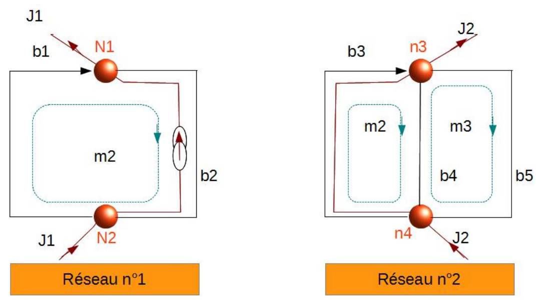

Fig. 2. Topology of previous network

we choosing arbitrarily the initial node 1 on our first network, , we start of this Node worm node 2, we have an return of Node 2 to Node 1.We construct by this return , the first couple $P1$ who will wear the current source J1, and will be in final, our current injected in the first network coming from the second network. We verify the relationship for node pair: $P = N - S = 4 - 2 = 2$ and meshes: $M = B - N + S = 5 - 4 + 2 = 3$. As in our first Network, we choosing arbitrarily on our second network the node n3, as reference from depart. We depart of this Node worm Node 4, we have an return from Node n4 to Node 3 , we construct with this return, the second couple "P1", who will wear normally the current source J2, but all along our study we assume that J2 is null, because it is rattached to a branch which comported not a current source. The good number of nodes, edges, pairs of nodes and mesh provided by the invariant of Kron makes it possible to transpose the electrical study of the circuit in the space of the meshes. It is one of the main objectives of the analysis tensorial network (TAN)

III. Invariants of topological manifolds

The search for invariants of topological and differentiable manifolds is a complicated subject. Although it is completely solved in dimension smaller than 2, and thanks to the cobordism for the varieties, of large dimensions, the case of the intermediate dimensions three and four is more complicated. It turns out that these dimensions are interesting because they intervene in the gauge theories. The topological manifolds with dimensions 4 are classified thanks to the quadratic form of intersection: $H_2(X, Z) \times H_2(X, Z) \rightarrow Z$, because simple connectivity and the Poincaré duality shows that in dimension four the only non-trivial homology groups are those of dimension 2: Only $H_2(X)$ is non zero and to contribute at the homology. This is exactly what has been demonstrated Michael Freedman in 1982
if we set I_X intersection form of X, we have:

1) $I_{S^4} = 0$: there are not two non-trivial cycles ($H_2(S^4, Z) = 0$).

2) $I_{S^2 \times S^2}, = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: $H_2(S^2 \times S^2, Z) = Z \oplus Z$: There are two cycles in general position $A = S^2 \times pt$, $B = pt \times S^2$ et $\langle A, B\rangle = \langle A, B\rangle = 1$, $\langle A, A\rangle = \langle B, B\rangle = 0$.
3) $I_{M\sharp N} = \begin{pmatrix} I_M & 0 \\ 0 & I_N \end{pmatrix}$:$H_2(M\sharp N, Z) = H_2(M, Z) \oplus H_2(N, Z)$, ($M\sharp N$ is the connected sum of two 4 dimension manifolds.

S. Donaldson [6], in the 80's, E. Witten [7], in the 90's, use new techniques from gauge theories to define new invariants for manifolds of dimension four in the differentiable category. At the same time, Gromov defined new invariants for symplectic manifolds: Holomorphic curves.[3]. There were indeed very few global invariants in symplectic geometry. In the following we will look at how mathematics inspired physics and vice versa, how theories of gauges inspired mathematics for the definition of new invariants

IV. Deformation invariants From mathematical physic

A. *The theorem of the index, and quantum field theories*

The theorem of the index was demonstrated at the beginning of the sixties by M. Atiyah, then revisited by many mathematicians and physicists. It is, in a way, a smooth version of Riemann Roch's theorem demonstrating for a long time for the algebraic curves then extended to the algebraic varieties by A.Grothendieck. The index theorem says that the index of a certain elliptic operator on a variety can be calculated taking into account the topology of the manifold, a quick review is given in [5].An important application of the index , or Riemann-Roch's formula for complex or algebraic varieties, is the determination of the dimension of a moduli space. In physics, classical field theory is based on the data of a Lagrangian that takes into account the theories considered (gravitation, theories of gauges). Lagrangian density is a function on one or more fields and its first derivatives:

$$\mathcal{L} = \mathcal{L}(\varphi_1, \varphi_2, ..., \partial_\mu\varphi_1, \partial_\mu\varphi_2...) \tag{3}$$

Classical action is the integral of the classical Lagrangian density on space $S = \int \mathcal{L} d^{n+1}x$
This quantity verifie the Principle of least action. The quantification of these theories leads to the formalism of the path integral: function of partition given by:

$$\mathcal{Z} = \int e^{-S(\varphi)} \mathcal{D}\varphi \tag{4}$$

And correlation functions :

$$< \varphi_1(x_1), ..., \varphi_n(x_n) >= \int \varphi_1(x_1)...\varphi_n(x_n) e^{-S(\varphi)} \mathcal{D}\varphi \tag{5}$$

Not all configurations are interesting in the path integral. Witten showed that by introducing supersymmetry concept, these path integrals could be localized on particular configurations: the instantons spaces. In mathematics, this is called moduli spaces. In the case of a gauge theory in dimension 4 whose main bundle is modeled by the group $SU(2)$, we have the theory of S. Donaldson.

B. *Strategy for the search of invariants*

To determine new invariants related to the topological field theory, it is necessary to:

1) Define a moduli space (instanton space)
2) Compactification of this space
3) Linearization and define ellipic complex
4) Calculate dimension of the moduli space using Riemann-Roch, or index Theorem
5) Add constraints, for the appropriate dimension of the moduli space.
6) We can count instantons.

V. Exemple: Symplectic geometry and correlation function for strings

A. *A toy model*

In symplectic geometry, there are very few local invariants. This is due to Darboux's theorem which assumes that locally all symplectic manifolds are similar, unlike the Riemanian varieties that can be separated locally by the curvature. A strategy, due to M. Gromov, for constructing invariants is to consider sub varieties such as, for example, holomorphic curves (function from Riemann surface to a symplectic manifold); There are parameterized curves: $u : (\Sigma, j) \to (Y, J)$,checking the conditions of Cauchy Riemann: $du \circ j = J \circ du$, where j and J are almost complex structures respectively on Σ and Y, and modelized a sigma-model in quantum field theory. Counting the holomorphic functions passing through marked pointson a Riemann surface, makes it possible to determine the correlation functions in superstring theory, the so-called invariants of Gromov Witten. Indeed E. Witten showed that a holomorphic function represents an instanton among all complex parametric curves. these parametric curves represent the evolution of a strings in space-time, in theoretical physics. A toy model, consists in defining the moduli space of the planar curves: (function :$\mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ of given degree (this degree corresponds to a class of cohomology in $H_2(Y, \mathbf{Z})$. For example, for degree one:

$$\mathcal{M} = \{u/u : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})/PGL(2, \mathbf{C}) \tag{6}$$

$PGL(2, \mathbf{C})$ represents automorphism group of $\mathbf{P}^2(\mathbf{C})$, his dimension is three; the space of the applications u is of complex dimension 5, therefore, one finds again that the space of the complex lines has complex dimension 2 For example, for lines passing through two fixed points, we have another moduli space:

$$\mathcal{M}' = \{(u, z_1, z_2)/u : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C}), z_1 \neq z_2\}/PGL(2, \mathbf{C}) \tag{7}$$

with the constraint of passing through two points, we find that this space has the dimension 4 Because you add, two parameters (two points) each of them is an element of $\mathbf{P}^2(\mathbf{C})$. It is possible now to evaluate (u, z_1, z_2) in other words, construct:

$$ev : (u, z_1, z_2) \in \mathcal{M}' \to (u(z_1), u(z_2)) \in \mathbf{P}^2(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}) \tag{8}$$

Here we have the simplest example of what is called a Gromov-Witten invariant: evaluation from a degree one map u through two points give only one line.. . In physics

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this correlation function is called a propagator.
If we now choose a complex curve of degree 2, we define a conic, we can show that the moduli space considered has dimension 5: five points determine only one conic. In this case, the moduli space must be compactified: there is a sequence of conics which converges towards a couple of line for example...
Kontsevich [8] has demonstrated a recurrence formula for counting all planar complex curves of given degree and thereby solved an enumerative geometry conjecture. The consideration of mirror symmetry in string theory has made it possible to demonstrate other conjectures in simple cases.

B. Theorical model

we are now considering an application of a Riemann surface in any complex manifold. let ϕ an application of a Riemann surface in any complex manifold. Note respectively $\mathcal{M}_g, \mathcal{M}_{g,n}$ the space of the curves modules (actually riemann surfaces), and the space of curve with n marked points. Thee Riemann-Roch formula for Curve give:

$$\begin{aligned} dim_{\mathbf{C}} H^0(T\Sigma) - dim_{\mathbf{C}} H^1(T\Sigma) &= \int_\Sigma ch(T\Sigma) td(T\Sigma) \\ &= 3 - 3g \end{aligned} \tag{9}$$

If $\phi : \Sigma \to X$ is a map from Σ to X The Riemann Roch formula give:

$$\begin{aligned} & dim_{\mathbf{C}} H^0(\phi^* TX) - dim_{\mathbf{C}} H^1(\phi^* TX) \\ & = \int_\Sigma ch(\phi^* TX) td(\Sigma) \\ & = n(1-g) + \int_\Sigma \phi^* c_1(TX) \end{aligned} \tag{10}$$

The deformation invariant of the problem are obtained thanks to the short exact sequence.

$$0 \to T_\Sigma \to \phi^* T_X \to N_{\Sigma/X} \to 0 \tag{11}$$

The long exact sequence associated, gives the index of the complex: the dimension of the moduli space of the applications $\mathcal{M}_g(X, \beta, n)$, β degree of the map, n number of marked point on Σ :

Roughly, the first term manages the deformation of the Riemann surface, the second the deformation of the ϕ the surface of Rieman being fixed, and the third term the deformations of the application. The long exact sequence associated,combines the two previous formula [9] and [10] and compute the index of the complex: the dimension of the compactified moduli space of the applications $\overline{\mathcal{M}}_{g,n}(X, \beta)$ degree of the map, n number of marked point on Σ :

$$\begin{aligned} & dim_{virt} \overline{\mathcal{M}}_{g,n}(X, \beta) = \\ & (dim X)(1-g) + \int_{f_*(\Sigma)} c_1(TX) + 3g - 3 + n \end{aligned} \tag{12}$$

Taking care not to confuse real and complex dimensions, in the case of the plane curves of degree one (the straight lines), we retrieve the dimension of the space of module $\mathcal{M}'$ seen previously.

VI. Conclusion

Other invariants have been introduced in field theory adapted to gauge theories. In dimension 3: the action of Chern-Simons [11] involves topological invariants enriching those obtained by the knot theory. Donaldson then Witten [9], [10] quantifying the action of Yang-Mills also use module spaces, and define invariants on well-chosen spaces of connections. We hope with this quick survey demonstate the power of the topology whose field of application sweeps the sciences of the engineer up to theoretical physics.

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