Proceedings of the World Congress on Engineering 2018 Vol I
WCE 2018, July 4-6, 2018, London, U.K.

A New One-Dimensional Electrostatic Model for Membrane MEMS Devices

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***Abstract*—In this work, the authors present a new formalization of stationary $1D$-membrane MEMS in terms of profile of the membrane in which the electric field magnitude E is considered proportional to the curvature of the membrane achieving results of existence by fixed point approach and, after, establishing conditions of uniqueness. At the end, the obtained results have been confirmed by some numerical tests.**

***Index Terms*—MEMS, NEMS, electrostatic actuation, boundary semi-linear elliptic problems, fixed-point approach.**

I. Introduction to the Problem

In the last few years, micro dimensional engineering applications are more and more oriented towards low cost solutions where actuators/sensors play a key role because representing the link between the physical nature of the problem and the machine language. In such a context, static and dynamic Micro-Electro-Mechanical-Systems (MEMSs) represents a real conquest of micro engineering supported by analytical-numerical modeling that is increasingly closer to reality [1], [2]. However, a lot of theoretical models does not allow to get explicit solutions so that one is content to find conditions ensuring existence and uniqueness of the solution or to solve the problem numerically [3]. From a theoretical point of view, Scientific Community is busily engaged in the study of coupled systems (such as magnetically actuated systems, thermal-elastic systems [4], [5]) while, from the application point of view, research has even gone into micro applications of biomedical interest [6], wave propagation in micro-domains with fixed and moving boundary and so on [7]. Recently, regarding stationary and dynamical MEMS, existence/uniqueness/regularity results have been carried out by near operator theory even in presence of nonlinear singularities [8], [9], [10]. There, a dimensionless MEMS device is considered composed by two metallic plates (one fixed, one deformable but clumped at its boundary), and after voltage application, the deformable plate deflect towards the fixed plate. The model os the above mentioned MEMS can be written as follows:

$$\begin{cases} \nu\Delta^2 v = (\varrho\int_\Omega |\nabla v|^2 d\xi + \varsigma)\Delta v + \\ \qquad +\lambda_1 g_1(\xi))((1-v)^\vartheta\left(1+\alpha\int_\Omega \frac{d\xi}{(1-v)^{\vartheta-1}}\right)^{-1}\Big) \\ v = \Delta v - d v_\nu = 0, \quad \xi\in\partial\Omega, \quad d\geq 0 \\ 0<v<1, \quad \xi\in\Omega \end{cases}$$

where: 1) g_1 is a bounded function which carries dielectric properties of the material; 2) λ_1 is the applied voltage; 3) the positive parameters $\nu, \varrho, \varsigma, \alpha$ are related to the electric and mechanic properties of the material and 4) ϑ takes into account more general electrostatic potentials. In the case of zero deformable plate thickness and neglecting inertial effects as well as non-local effects ($\vartheta = 2$[1], $\nu = 1, \varrho = 0, \varsigma = 0$, and $\alpha = 0$) (1) is simplified as follows [11]:

$$\begin{cases} \Delta^2 v(\xi) = \lambda_1 g_1(\xi)([1-v(\xi)]^2)^{-1} \\ 0 < v(\xi) < 1 \ \text{ in } \Omega, \\ v = \Delta v - d v_\nu, \ \text{ on } \partial\Omega, \ d\geq 0 \end{cases} \tag{1}$$

In this paper, starting from (1), a new $1D$ model in which the bottom plate is replaced by a thin membrane attached to the edge, so that (1) is particularized as the following elliptical semi-linear model:

$$\begin{cases} v'' = -g_2(\xi)\lambda_1((1-v(\xi)^2)^{-1} & \text{in } \Omega = [-A, A] \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

in which λ_1 (related to the applied voltage) can be also expressed in terms of $|\mathbf{E}|$ and, since $\mathbf{E}$ on the membrane is locally normal to the tangent of the membrane, $|\mathbf{E}|$ can be considered proportional to the curvature C of the membrane in order to carry out a model in which the singularity $1 - v(\xi)$ is not directly involved. The paper is organized as follows. Some engineering preliminary results about the models related to membrane MEMS are presented in Section II so that, in Section III, the authors propose the new approach structured in the Dirichlet's form, written in its integral formulation, taking into account the safety distance τ [2] achieving existence and uniqueness conditions for it (Sections IV and V). Finally, some numerical considerations complete the work supporting the proposed approach (Section VI).

II. The Starting Electrostatic $1D$ membrane MEMS Model

To get the well-known dimensionless $1D$ membrane MEMS model, in $\mathbb{R}^3$ we consider a system of Cartesian axes $O'\xi'\eta'\zeta'$ in which an electrostatic-elastic system whose length is $2A$, formed by a pair of parallel plates, of which one fixed and the other one elastic (but fixed at the edges), placed at a mutual distance h orthogonally to the axis ζ' takes place. An electrostatic voltage V is applied on it[3] so that the electrostatic potential ϕ satisfies the Lapalce's equation $\Delta\phi = 0$ inside the zone bounded by the plates[4]. In these

Manuscript received March 21, 2018; revised April 4, 2018.

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[1]Usually, $\vartheta \geq 2$.

[2]Distance of the top of the membrane profile from the upper plate.

[3]V on the elastic plate and $V = 0$ on the fixed plate.

[4]$\phi = V$ on the elastic plate while $\phi = 0$ on the fixed one.

ISBN: 978-988-14047-9-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

conditions the deflection w' of the elastic plate satisfies the equation [1][5]:

$$-\vartheta\Delta_{\perp}w' + D\Delta_{\perp}^2 w' = -0.5\epsilon_0|\nabla\phi|^2 \quad (3)$$

in which: a) ϑ is the mechanical tension in the plate; b) D is its the flexural rigidity and, finally, 3) ϵ_o is the permittivity of free space. Taking into account the scaling factors $u = w'/h$, $\Phi = \phi/V$, $\xi = \xi'/2A$, $\eta = \eta'/2A$ and $\zeta = \zeta'/h$ and denoting by $\delta = D/((2A)^2\vartheta)$ as the relative importance of tension and rigidity and $\epsilon = h/(2A)$ as the aspect ration of the system, Eq. (3) becomes a system of nonlinear coupled partial differential equations as follows[6]:

$$\begin{cases} \epsilon^2\Delta_{\perp}\Phi + \Phi_{\zeta\zeta} = 0 \\ -\Delta_{\perp}u + \delta\Delta_{\perp}^2 u = -\lambda^2(\epsilon^2|\nabla_{\perp}\Phi|^2 + (\Phi_{\zeta})^2) \\ \Phi = 1 \text{ on elastic plate}; \quad \Phi = 0 \text{ on fixed plate} \end{cases} \quad (4)$$

where the ratio of a reference electrostatic force to a reference elastic force, is evaluable by:

$$\lambda_1 = \lambda^2 = \epsilon_0 V^2(2A)^2(2h^3\vartheta)^{-1} = \varrho V^2 \quad (5)$$

while

$$\varrho = \epsilon_0(2A)^2(2h^3\vartheta)^{-1} \quad (6)$$

takes into account the electro-mechanical properties of the membrane material[7]. Since both thickness and width of the device are negligible with respect to its length $2A$, (4) can be simplified when $\epsilon \to 0$ reducing the first equation of (4) to $\frac{\partial^2\Phi}{\partial\zeta^2} = 0$ whose solution $\Phi = \frac{\zeta}{u}$ can be substituted into the second equation of (4) obtaining the non-linear equation[8]:

$$-\Delta_{\perp}u + \delta\Delta_{\perp}^2 u = -\lambda^2 u^{-2} \quad (8)$$

solvable for only cases with simple geometries. If the deformable plate is replaced by a deformable membrane anchored along the edge of the lower face of a fixed plate supporting the deformable membrane so that (8) is still valid[9]. In addition, modern technologies allow us to exploit very performant materials whose flexural rigidity D can be considered negligible, getting a further simplification of (8) ($\delta = 0$) obtaining, in stationary deflection conditions and reversing the orientation of ζ so that the membrane in the rest position lies on $\zeta = 0$, the following semi-linear elliptic form [1], [12]:

$$\begin{cases} v'' = -\lambda^2(1-v)^{-2} \text{ in } \Omega = [-A_1, A_1] \\ v(-A_1) = v(-0.5) = v(A_1) = v(0.5) = 0 \end{cases} \quad (9)$$

where $u = 1 + v$ and A_1 is the dimensionless quantity A.

[5] $\Delta_{\perp}$ denotes the laplacian operator with respect to ξ' and η'.

[6] Φ_{ζ} and $\Phi_{\zeta\zeta}$ represent the first and second order partial derivative of Φ with respect to ζ respectively.

[7] Experimentally and in dimensionless conditions

$$\varrho_1 = \epsilon_0(2\vartheta)^{-1} > 10^{12}. \quad (7)$$

[8] Decoupled from the equation of the potential.

[9] But with different values of the electro-mechanical parameters.

III. The New Proposed Approach: $|\mathbf{E}|$ in terms of Curvature of the Membrane

Taking into account that λ^2 in (9), by means of (5), is proportional to V^2, then $\lambda^2(1-v)^{-2} \propto |\mathbf{E}|^{\mathbf{2}}$ and we can rewrite (9) as:

$$\begin{cases} -v'' = \varrho|\mathbf{E}|^2 \text{ in } \Omega = [-A_1, A_1] \\ v(-A_1) = v(A_1) = 0 \end{cases} \quad (10)$$

$|\mathbf{E}|^{\mathbf{2}}$ in (10) represents the square of the electric field magnitude and since the lines of force of $\mathbf{E}$ are orthogonal [13], point by point, to the tangent to the surface of the membrane, we can expressed $|\underline{\mathrm{E}}|$ as the product between the curvature C of the membrane deflection and a coefficient of proportionality κ. Testing this approach on an hemispherical benchmark well-known in literature [13] it was highlighted the following functional dependence:

$$|\mathbf{E}(\xi)| = \kappa(\xi, v(\xi), \lambda)C(\xi, v(\xi)) \quad (11)$$

where

$$\kappa(\xi, u(\xi), \lambda) = \lambda(1 - u(\xi) - \tau)^{-1} \quad (12)$$

with $\kappa(\xi, u(\xi), \lambda) \in C^0([-A_1, A_1] \times [0, 1) \times [\overline{\lambda}, T])$ [10], and τ critical distance equal to $\lambda(\epsilon_t)^{-1}$ with ϵ_t dielectric strength of the material constituting the membrane, even when the deflection v assumes its maximum deformation (we choice to exploit τ because in this wa the deflection of the membrane touches the upper plate of the device (situation mathematically representable by a singularity)[11]. In conclusion, problem (10) can be structured as below:

$$\begin{cases} -v'' = \varrho_1\kappa^2(\xi, v(\xi), \lambda)C^2(\xi, v(\xi)) = \\ \qquad = \varrho_1\lambda^2 C^2(\xi, v(\xi))(1 - v(\xi) - \tau)^{-2} \text{ in } \Omega \\ v(-A_1) = v(A_1) = 0; \quad 0 < v(\xi) < 1 - \tau. \end{cases} \quad (13)$$

Substituting $C(\xi, v(\xi)) = |v''(\xi)|(1+|v'(\xi)|^2)^{-3/2}$, the well-known $1D$ formulation of curvature C [14], in (13) we can write the following interesting equation:

$$v''(\xi) + \varrho\kappa^2(\xi, v(\xi), \lambda)|v''(\xi)|^2(1 + (v'(\xi))^2)^{-3} = 0 \quad (14)$$

from which, since $v(\xi) > 0$, we obtain[12]:

$$1 + \varrho_1\kappa^2(\xi, v(\xi), \lambda)(v''(\xi))(1 + (v'(\xi))^2)^{-3} = 0 \quad (15)$$

so that (13) assumes the final expression:

$$\begin{cases} v''(\xi) = -(1 + (v'(\xi))^2)^3(\varrho_1\kappa^2(\xi, v(\xi), \lambda))^{-1} \text{ in } \Omega \\ v(-A_1) = v(A_1) = 0 \\ 0 < v(\xi) < 1 - \tau. \end{cases} \quad (16)$$

However, problem (16) can be consider it as a particular case of the following Dirichlet's problem:

$$\begin{cases} v''(\xi) + f(x, v(\xi), v'(\xi)) = 0 \text{ in } \Omega = [A_1, -A_1] \\ v(-A_1) = v(A_1) = 0 \\ 0 < v < \nu \quad v \in C^2(\Omega) \end{cases} \quad (17)$$

[10] Where $\overline{\lambda}^2$ is the minimum voltage to apply to the device to win the inertia of the membrane and T^2 is the maximum admissible voltage.

[11] It is clear that, if $\epsilon_t \to \infty$, model (9) is restored.

[12] $v''(\xi) = 0$ represents a physical impossible occurrence because its solution is $v(\xi) = mx + b$ with m arbitrary constant giving us a linear defection of the membrane when $|\mathbf{E}| = 0$

ISBN: 978-988-14047-9-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R})$ and $\nu = 1 - \tau$. Considering $f(\xi, v(\xi), v'(\xi)) = (1 + (v'(\xi))^2)^3(\varrho_1\kappa^2(\xi, v(\xi), \lambda))^{-1}$, problem (17) can be written as follows:

$$\begin{cases} v'' = -(1 + (v'(\xi))^2)^3)(\varrho_1\kappa^2(\xi, v(\xi), \lambda)^{-1} = \\ \quad = -(1 + (v'(\xi))^2)^3(\nu - v(\xi))^2)(\varrho_1\lambda^2)^{-1} \text{ in } \Omega \\ v(-A_1) = v(A_1) = 0; \;\; 0 < v < \nu \end{cases} \tag{18}$$

with $v \in C^2(\Omega)$ ([13]), $\kappa = \kappa(\xi, v(\xi), \lambda) \in C^0(\Omega \times [0,1], [\overline{\lambda}, T])$ and $\kappa = \lambda(\nu - v(\xi))^{-1}$. It seems that (18) has not the singularity shown in (9) (or in (13) when $v = 1-\tau$). In this case, from (18), we would achieved the condition $v''(\xi) = 0$ producing the impossible condition that $|\mathbf{E}| = 0$ that is linear deflection.

IV. A Result of Existence for *Problem I*

We start premising the definitions of two particular functional spaces useful for the following of the paper for both formulating the problem under study in terms of integral equations and achieving results of existence and uniqueness of solution.

Definition 1. Let S and S_1 be the functional spaces so defined in $\Omega = [-A_1, A_1]$:

$$S = \{C_0^2(\Omega) : \; 0 < v(\xi) < \nu, \; |v'(\xi)| < M < +\infty\} \tag{19}$$

$$S_1 = \{C_0^1(\Omega) : \; 0 < v(\xi) < \nu, \;\; |v'(\xi)| < M < +\infty\}^{\,14} \tag{20}$$

Problem (17), by differentiation procedure, can be translated in its integral formulation exploiting a Green's function $\Xi(\xi, \iota)$ [14]. That means:

$$v(\xi) = \int_{-A_1}^{A_1} \Xi(\xi, \iota) f(\iota, v(\iota), v'(\iota)) d\iota, \;\; 0 < v < \nu \tag{21}$$

from which it is permissible we can write:

$$v'(\xi) = \int_{-A_1}^{A_1} \Xi_x(\xi, \iota) f(\iota, v(\iota), v'(\iota)) d\iota \tag{22}$$

so that (18) can be rewritten as:

$$v(\xi) = \int_{-A_1}^{A_1} \Xi(\xi, \iota)(1 + (v'(\iota))^2)^3)(\varrho_1\kappa^2(\iota, v(\iota), \lambda))^{-1} d\iota. \tag{23}$$

In this paper the existence of the solution of the equation $\Pi(v) = s$, with $v \in S_1$, is proved by means of a procedure based on Schauder-Tychonoff fixed point approach on the operator $s = \Pi(v)$ from S to S. In fact, we define the positive operator Π as:

$$\Pi(v(\xi)) = \int_{-A_1}^{A_1} \Xi(\xi, \iota)((1 + (v'(\iota))^2)^3(\varrho_1\kappa^2(\iota, v(\iota), \lambda)^{-1} d\iota \tag{24}$$

from which it makes sense to write:

$$\Pi'(v(\xi)) = \int_{-A_1}^{A_1} \Xi_x(\xi, \iota)((1 + (v'(\iota))^2)^3(\varrho_1\kappa^2(\iota, v(\iota), \lambda))^{-1} d\iota. \tag{25}$$

[13] As well-known, this assumption is physically plausible because membrane tears are not allowed and slopes vary continuously.

For our purposes, we exploit the following Green's function defined as follows [15]:

$$\Xi(\xi, \iota) = \begin{cases} (\iota + A_1)(A_1 - \xi)(2A_1)^{-1} & -A_1 \le \iota \le \xi \\ (A_1 - \iota)(\xi + A_1)(2A_1)^{-1} & \xi \le \iota \le A_1 \end{cases} \tag{26}$$

allowing us to write:

$$\Xi_x(\xi, \iota) = \begin{cases} -(\iota + A_1)(2A_1)^{-1} & -A_1 \le \iota \le \xi \\ (A_1 - \iota)(2A_1)^{-1} & \xi \le \iota \le A_1. \end{cases} \tag{27}$$

We underline the following properties of the function $\Xi(\xi, \iota)$ of which we take into account in our proof:

a. $\Xi(\xi, \iota) \ge 0$ and continuous;
b. $max(\Xi(\xi, \iota)) = \Xi(\xi = \iota, \iota = 0) = A_1/2$ so that

$$0 \le \Xi(\xi, \iota) \le 0.5A_1 \quad \forall \xi, \iota \in \Omega; \tag{28}$$

c. $\int_{-A_1}^{A_1} \Xi(\xi, \iota) d\iota = 0.5(A_1 - \xi)(\xi + A_1) \le 0.5(A_1^2)$;
d. $\left| \int_{-A_1}^{A_1} \Xi_x(\xi, \iota) d\iota \right| \le \int_{-A_1}^{A_1} |\Xi_x(\xi, \iota)| d\iota \le A_1$;
e. $\forall \xi, \iota \in (\Omega \times \Omega)$ the following limitation holds:

$$\Xi_x(\xi, \iota) \le 0.5. \tag{29}$$

In order to demonstrate the existence of at least a solution for problem (18) and its uniqueness as well, we start by to present the following Lemma which demonstration is reported in appendix A.

Lemma IV.1. *Consider the operator* $\Pi(v)$ *defined by* (24)*. It is an operator from* S *to* S*.*

This lemma, that we use to prove our existence results, gives us the following important inequality that proves the fact that M depends on ϱ (properties of the material of the membrane)

$$1 + M^6 < (M\varrho_1\overline{\lambda}^2)(4\nu A_1)^{-1} \tag{30}$$

By exploiting the results achieved by this Lemma, we are ready to prove the existence of at least a solution for problem (18).

Theorem IV.2. *Problem (18) admits at least one solution in* S*.*

The proof is reported in appendix B.

V. On the uniqueness of the Solution for problem (18)

This theorem, whose proof is reported in appendix C, proves that problem (18), $\forall M > 0$ and then for each kind of material constituting the membrane, admits unique solution. Morover, we can prove:

Theorem V.1. *The solution* v *of the problem (18) is unique* $\forall M > 0$*; moreover* $v \in C^\infty(\Omega)$*, it is is symmetric with respect to the origin and*

$$\forall \xi \in \Omega, \;\; |v'(\xi)| \le |v'(A_1)| = |v'(-A_1)|. \tag{31}$$

VI. Some Numerical Verifications

Algebraic system (40), as above proved, can be reduced to the inequality (30) and this reduction is confirmed by means of some numerical evaluations reported in this section also confirming that (40) admits numerical solutions overlapping with the analytical ones. Specifically, from the orders of amplitude point of view, we can say that, setting $\overline{\lambda} = 1$ and $A_1 = 0.5$ in (39), H is smaller than a quantity whose order of amplitude is at least 10^2 so that it is correct to consider the following algebraic system of inequalities:

$$\begin{cases} 1 + M^6 < (H\varrho_1\overline{\lambda}^{-2})(4\nu A_1)^{-1} < (10^2 10^{12}\overline{\lambda}^{-2})(2\nu)^{-1} \\ 1 + M^6 < (\varrho_1\overline{\lambda}^{-2})(4A_1^2)^{-1} < 10^{12}\overline{\lambda}^{-2}. \end{cases} \tag{32}$$

In addition, since $\nu < 1$ and $(\varrho_1\overline{\lambda}^{-2})(4A_1^2)^{-1} < (H\varrho_1\overline{\lambda}^{-2})(4\nu A_1)^{-1}$ we can write that (40) is equivalent to (30). And again, reformulating (32) as the following system:

$$\begin{cases} f_1(M) = (M10^{12}\overline{\lambda}^{-2})(2\nu)^{-1} - (M^6+1) > 0 \\ f_2(M) = 10^{12}\overline{\lambda}^{-2}(1-\tau) - (M^6+1) > 0 \end{cases} \tag{33}$$

where $f_1(M)$ and $f_2(M)$ are two artificial function defined for our purposes. In particular, since (33) must be verified, by means of the Newton-Raphson's procedure (tolerance 10^{-3}) we firstly seek their zeros values ($f_1(M^*) = 0$ and $f_2(M^*) = 0$) and then, consider them as the sup of the set of the value of M that verifies the inequalities in (33). Table I show the results of the tests demonstrating that, in dimensionless conditions, numerical results are comparable with the analytical ones (in particular numerical results are lower than the analytical ones). For example, for $\overline{\lambda} = 1.01$, we choice a range of values of M in which the Newton-Raphson's algorithm is applicable for both $f_1(M)$ and $f_2(M)$ (in this case we consider $[230, 250]$ and $[90, 110]$) obtaining then their M^* (labeled by M_1^* and M_2^* respectively) so that, to guarantee the existence of the solution of the problem, we must choice $sup|M| = min(M_1^*, M_2^*)$ and, for safety advantage, we can conclude that $sup|M| = min(M_1^*, M_2^*)_{numerical} = 98.2$ corresponding, in dimensionless point of view, just a bit higher to 86 degrees.

TABLE I: Comparison between numerical and analytical results. The exploited Newton-Raphson procedures has been applied wit a tolerance equal to 10^3.

$\overline{\lambda}$	M_1^* numerical	M_2^* numerical	M_1^* analytical	M_2^* analytical
1	229.3	98.2	229.4	98.6
1.01	241.1	102.1	241.4	101.7
1.02	243.6	108	243.8	108.1
1.03	249.1	1123	249.4	112.7
1.04	252.8	116.3	253.6	116.7
1.05	254.9	117.1	255.5	118.1

VII. Appendix A

Proof of Lemma IV.1

Let be $\Omega = [-A_1, A_1]$. To prove our existence result, firstly we propose to consider the norm of the operator $||\Pi(v(\xi))||_{C^2(\Omega)}$:

$$\begin{aligned} ||\Pi(v(\xi))||_{C^2(\Omega)} = sup_{\xi\in\Omega}|\Pi(v(\xi))| + \\ + sup_{\xi\in\Omega}|\Pi'(v(\xi))| + + sup_{\xi\in\Omega}|\Pi''(v(\xi))| < +\infty. \end{aligned} \tag{34}$$

By the structure of $\Xi(\xi, \iota)$, $\Pi(v) \geq 0$ and, in particular, $\Pi(v(-A_1)) = \Pi(v(A_1)) = 0$. In addition, since (12) holds, we can write $\kappa(\xi, v(\xi), \lambda) > 1$ in $[-A_1, A_1]$. This condition is physically true because $|\mathbf{E}|$, to deform the membrane, must win its inertia of deformation and then the coefficient of proportionality $\kappa(\xi, v(\xi), \lambda)$ must be greater than 1. Now we can assume $\overline{\lambda} > 0$, a minimum voltage necessary to win the inertia of the membrane for which we have $\overline{\lambda} < \lambda < sup\{\lambda\} < +\infty$ and $1/\lambda^2 < +\infty$. So, considering (28), the following chain of inequalities can be considered:

$$\begin{aligned} 0 \leq |\Pi(v(\xi))| \leq sup_{\xi\in\Omega}|\Pi(v(\xi))| = \\ = sup_{\xi\in\Omega}\left|\int_\Omega \Xi(\xi,\iota)(\varrho_1\kappa^2)^{-1}(1+(v'(\iota))^2)^3)d\iota\right| = \\ \leq (\varrho_1\lambda^2)^{-1} sup_{\xi\in\Omega}\left|\int_{-A_1}^{\xi}(2A_1)^{-1}(\iota+A_1)(A_1-\xi)\right. \\ \left.(1+(v'(\iota))^2)^3(\nu-v(\iota))^2 d\iota\right| + \\ +(\varrho_1\lambda^2)^{-1} sup_{\xi\in\Omega}\left|\int_{\xi}^{A_1}(2A_1)^{-1}(A_1-\iota)(\xi+A_1)\right. \\ \left.(1+(v'(\iota))^2)^3(\nu-v(\iota))^2 d\iota\right| = \\ = \nu(\varrho_1\lambda^2)^{-1}\left\{ sup_{\xi\in\Omega}\left|\int_{-A_1}^{\xi}(2A_1)^{-1}(\iota+A_1)(A_1-\xi)\right.\right. \\ (1+(v'(\iota))^2)^3 d\iota + \\ \left.\left. + \int_{\xi}^{A_1}(2A_1)^{-1}(A_1-\iota)(\xi+A_1)(1+(v'(\iota))^2)^3 d\iota\right|\right\} \leq \\ \leq 4\nu(\varrho_1\lambda^2)^{-1}(1+M^6) \\ sup_{\xi\in\Omega}\left\{\int_{-A_1}^{\xi}(2A_1)^{-1}(\iota+A_1)(A_1-\xi)d\iota + \right. \\ \left. + \int_{\xi}^{A_1}(2A_1)^{-1}(A_1-\iota)(\xi+A_1)d\iota\right\} \leq \\ \leq 4\nu(\varrho_1\lambda^2)^{-1}(1+M^6)A_1^2 < +\infty. \end{aligned} \tag{35}$$

And again, we can write:

$$\begin{aligned} sup_{\xi\in\Omega}|\Pi'(v(\xi))| = \\ sup_{\xi\in\Omega}\left|\int_\Omega \Xi_x(\xi,\iota)(\varrho_1\kappa^2)^{-1}(1+(v'(\iota))^2)^3)d\iota\right| = \\ = (\varrho_1\lambda^2)^{-1} sup_{\xi\in\Omega}\left|\int_{-A_1}^{x} -(2A_1)^{-1}(\iota+A_1)(1+(v'(\iota))^2)^3\right. \\ (\nu-v(\iota))^2 d\iota + \int_x^{A_1} -(2A_1)^{-1}(A_1-\iota) \\ \left.(1+(v'(\iota))^2)^3(\nu-v(\iota))^2 d\iota\right| \leq \\ \leq 4\nu(\varrho_1\lambda^2)^{-1}(1+M^6) sup_{\xi\in\Omega}\left|\int_{-A_1}^{x} -(2A_1)^{-1}(\iota+A_1)d\iota + \right. \\ \left.\int_x^{A_1}(2A_1)^{-1}(A_1-\iota)d\iota\right| \leq 4\nu(\varrho_1\lambda^2)^{-1}(1+M^6)A_1 < +\infty. \end{aligned} \tag{36}$$

Moreover if we consider $sup_{\xi\in\Omega}|\Pi''(v(\xi))|$, taking into account (25), (27), (29), and that $|v'| \leq M$ and, $|1/\kappa^2| < 1$

are verified, the following chain of inequalities hold:

$$\begin{aligned} & sup_{\xi\in\Omega}|\Pi''(v(\xi))| = \\ & sup_{\xi\in\Omega}\left|\frac{d}{d\xi}\int_\Omega \Xi_x(\xi,\iota)(\varrho_1\kappa^2)^{-1}(1+(v'(\iota))^2)^3 d\iota\right| = \\ & \leq (2\varrho_1\lambda^2)^{-1}\left[sup_{\xi\in\Omega}\left|\frac{d}{d\xi}\int_{-A_1}^{\xi}(1+(v'(\iota))^2)^3)d\iota\right| + \right. \\ & \left. +sup_{\xi\in\Omega}\left|\frac{d}{d\xi}\int_{\xi}^{A_1}(1+(v'(\iota))^2)^3)d\iota\right|\right] = \\ & = (2\varrho_1\lambda^2)^{-1}\left[2sup_{\xi\in\Omega}(1+(v'(\iota)^2)^3\right] \leq \\ & \leq (2\varrho_1\lambda^2)^{-1}2(1+M^2)^3 = \\ & = (\varrho_1\lambda^2)^{-1}(1+M^2)^3 < +\infty. \end{aligned} \tag{37}$$

Finally, putting (35), (36) and (37) into (34), it so easy to conclude that:

$$\begin{aligned} & ||\Pi(v(\xi))||_{C^2(\Omega)} \leq 4\nu(\varrho_1\lambda^2)^{-1}(1+M^6)A_1^2 + \\ & +4\nu(\varrho_1\lambda^2)^{-1}(1+M^6)A_1 + (\varrho_1\lambda^2)^{-1}(1+M^2)^3 < +\infty. \end{aligned} \tag{38}$$

Then to prove that $\Pi(u) \in S$, we observe that by (37), $4\nu(\varrho_1\lambda^2)^{-1}(1+\mathrm{M})A_1^2 < \nu$ then

$$1+M^6 < (4A_1^2)^{-1}\varrho_1\overline{\lambda^2} \Rightarrow M < (4A_1^2)^{-1}(\varrho_1\overline{\lambda^2}-1)^{1/6} \tag{39}$$

and, since both (36) and (39) are verified, it allows us to write:

$$\begin{cases} 1+M^6 < M(4\nu A_1)^{-1}\varrho_1\overline{\lambda^2} \\ 1+M^6 < (4A_1^2)^{-1}\varrho_1\overline{\lambda^2}. \end{cases} \tag{40}$$

In this system we can observe that, if by contradiction $(4A_1^2)^{-1}\varrho_1\overline{\lambda^2} < M(4\nu A_1)^{-1}\varrho_1\overline{\lambda^2}$, then $M > \frac{\nu}{A_1} = 2\nu$ and, since $M = \frac{\zeta}{\xi}$ and $M' = \frac{\zeta'}{\xi'}$, it makes sense to write [15]:

$$M = \frac{\zeta}{\xi} = \frac{\zeta'}{h}\frac{2A}{\xi'} = H'\frac{2A}{h} > 2\nu. \tag{41}$$

However, $\nu = \frac{\nu'}{h}$ then, considering (41), $M' > \frac{\nu'}{A}$ and, if $A \to 0$, then $\frac{\nu'}{A} \to +\infty$ so that $M' = sup|v'| = +\infty$ (absurd). Then, $(4A_1^2)^{-1}\varrho_1\overline{\lambda^2} > M(4\nu A_1)^{-1}\varrho_1\overline{\lambda^2}$ so (40) is rewritable as:

$$1+M^6 < (4\nu A_1)^{-1}M\varrho_1\overline{\lambda^2} \tag{42}$$

and this imply that $\Pi(v) : S \to S$.

VIII. Appendix B

Proof of Theorem IV.2

If $\Omega = [-A_1, A_1]$,taking into account the previous results (Theorem IV.1) and compact immersions $C_0^2(\Omega) \hookrightarrow C_0^1(\Omega)$ and then that $S_1 \hookrightarrow S$ are verified, we apply the fixed-point theorem (Schauder-Tychonoff) to conclude that the problem $v = \Pi(s)$ admits at least a fixed point $v = \Pi(v)$ in S_1. In other terms, problem (18) admits at least a solution.

[15]See the scaling operations.

IX. Appendix C

Proof of Theorem V

To prove (31), from problem (18), we preliminarily observe that $v"(\xi) \leq 0$ with $\xi \in \Omega = [-A_1, A_1]$ (membrane concave with first derivative decreasing) and, again, we can write:

$$v"(\xi)([1+(v'(\xi))^2]^3)^{-1} = -(\varrho_1\overline{\lambda^2})^{-1}[1-\tau-v(\xi)]^2 \tag{43}$$

from which, multiplying by v' both member, it makes sens write:

$$\begin{aligned} & v"(\xi)\,v'(\xi)([1+(v'(\xi))^2]^3)^{-1} = \\ & = -(\varrho_1\overline{\lambda^2})^{-1}[1-\tau-v(\xi)]^2\,v'(\xi) = \\ & = -(\varrho_1\overline{\lambda^2})^{-1}(1-\tau)^2 v'(\xi) + \\ & +(\varrho_1\overline{\lambda^2})^{-1}(1-\tau)\frac{d}{d\xi}[v(\xi)]^2 - (3\varrho_1\overline{\lambda^2})^{-1}\frac{d}{d\xi}[v(\xi)]^3 \end{aligned} \tag{44}$$

but since

$$\begin{aligned} & v"(\xi)\,v'(\xi)([1+(v'(\xi))^2]^2)^{-1} = \\ & = -\frac{1}{4}\frac{d}{d\xi}(1+[v'(\xi)]^3)^2)^{-1} \end{aligned} \tag{45}$$

integrating (44), we obtain:

$$-\frac{1}{4}(1+[v'(A_1)]^2)^2)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^2)^{-1} = 0, \tag{46}$$

from which $|v'(-A_1)| = |v'(A_1)|$. And again, integrating (44) form $-A_1$ to t, and considering that $v(-A_1) = 0$, we can write:

$$\begin{aligned} & -\frac{1}{4}(1+[v'(t)]^2)^2)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^2)^{-1} = \\ & = -(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)^2 v'(t) + \\ & +(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)\frac{d}{dt}[v(t)]^2 - (3\varrho_1\overline{\lambda}^2)^{-1}\frac{d}{dt}[v(t)]^3 \end{aligned} \tag{47}$$

and then $\forall t \in [-A_1, A_1]$:

$$\begin{aligned} & -(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)^2 v(t) + \\ & +(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)[v(t)]^2 - (3\varrho_1\overline{\lambda}^2)^{-1}[v(t)]^3 = \\ & = (\varrho_1\overline{\lambda}^2)^{-1}v(t)\left\{(1-\tau)[v(t)-(1-\tau)] - \frac{1}{3}[v(t)]^2\right\} < 0 \end{aligned}$$

from which:

$$-\frac{1}{4}(1+[v'(t)]^2)^3)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^3)^{-1} < 0 \tag{48}$$

so that $\forall t \in \Omega$, $|v'(t)| < |v'(-A_1)|$.

To prove that problem (18) admits just one solution, we suppose by contradiction that, in S_1, it has two different solutions labeled by v_1 and v_2. From problem (18), by integration, we can write ($\forall t \in \Omega$):

$$v_1'(t) \leq M - (\varrho_1\overline{\lambda}^2)^{-1}\int_{-A_1}^{t}[1+(v_1'(\xi))^2]^3[1-\tau-v_1(\xi)]^2 d\xi$$

$$v_2'(t) \leq M - (\varrho_1\overline{\lambda}^2)^{-1}\int_{-A_1}^{t}[1+(v_2'(\xi))^2]^3[1-\tau-v_2(\xi)]^2 d\xi.$$

And integrating and subtracting on both members, we obtain $\forall t \in \Omega$:

$$\begin{aligned} & v_1'(t) - v_2'(t) = \\ & = (\varrho_1\overline{\lambda}^2)^{-1}\int_{-A_1}^{t}\{[1+(v_2'(\xi)^2]^3[1-\tau-v_2(\xi)]^2 - \\ & -[1+(v_1'(\xi))^2]^3[1-\tau-v_1(\xi)]^2\}d\xi. \end{aligned} \tag{49}$$

ISBN: 978-988-14047-9-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Now, needing to evaluate the term inside the integral, we construct two auxiliary functions F and g made as follows:

$$\begin{aligned} F(w,v) = [1+w^2]^3(1-\tau-v)^2, \\ g(t) = F(w_t, v_t) = \\ = F(tw_1 + (1-t)w_2, tv_1 + (1-t)v_2) \end{aligned} \tag{50}$$

in order that we can write:

$$g'(t) = F_w(w_t, v_t)(w_1 - w_2) + F_v(w_t, v_t)(v_1 - v_2)$$

and

$$\begin{aligned} g(1) = F(w_1, v_1) \\ g(0) = F(w_2, v_2) \\ g(1) - g(0) = g'(\xi),\ \xi \in (0,1). \end{aligned}$$

However, it is correct to write:

$$\begin{aligned} F_w(w_\xi, v_\xi) = 6[1+w_\xi^2]^2 w_\xi (1-\tau-v_\xi)^2 = \\ = 6\{1+[\xi w_1 + (1-\xi)w_2]^2\}^2[\xi w_1 + \\ +(1-\xi)w_2](1-\tau-v_\xi)^2 \leq \\ \leq 6\{\xi[1+w_1^2]^2 + (1-\xi)[1+w_2^2]^2\}[\xi w_1 + \\ +(1-\xi)w_2](1-\tau-v_\xi)^2. \end{aligned}$$

Considering, in addition, that $w_1 \leq M$, $w_2 \leq M$, $v_\xi \leq 1$, the following inequality holds:

$$|F_w(w_\xi, v_\xi)| \leq 24(1+M^2)^2 M \tag{51}$$

so that we obtain the important inequality:

$$\begin{aligned} |F_v(w_\xi, v_\xi)| = |-2[1+(w_\xi)^2]^3(1-\tau-v_\xi)| \leq \\ \leq 2|\xi(1+w_1^2)^3 + (1-\xi)(1+w_2^2)^3| \leq 4(1+M^2)^3 \end{aligned}$$

that, considering both (49) and the Poincaré's inequality the following chain of inequalities holds

$$\begin{aligned} |v_1'(t) - v_2'(t)| \leq \\ \leq 24(\varrho_1\overline{\lambda}^2)^{-1}(1+M^2)^2 M \int_{-A_1}^{t} |v_1'(\xi) - v_2'(\xi)| d\xi + \\ +4(\varrho_1\overline{\lambda}^2)^{-1}(1+M^2)^3 \int_{-A_1}^{t} |v_1(\xi) - v_2(\xi)| d\xi \leq \\ \leq 24(\varrho_1\overline{\lambda}^2)^{-1}(1+M^2)^2 M \int_{-A_1}^{t} |v_1'(\xi) - v_2'(\xi)| d\xi + \\ 8A_1(\varrho_1\overline{\lambda}^2)^{-1}(1+M^2)^3 \int_{-A_1}^{t} |v_1'(\xi) - v_2'(\xi)| d\xi = \\ \leq c(M, \overline{\lambda}, A_1, \varrho_1) \int_{-A_1}^{t} |v_1'(\xi) - v_2'(\xi)| d\xi. \end{aligned}$$

from which, by means of Gronwall's lemma [14], we can write that, $\forall t \in \Omega$, $|v_1'(t) - v_2'(t)| \leq 0$. In other words, $\forall t \in \Omega$, $v_1'(t) - v_2'(t) = 0$ from which $v_1 - v_2 = r \in \mathbb{R}$. But, since $v_1(-A_1) = v_2(-A_1) = v_1(A_1) = v_2(A_1) = 0$, then $v_1 = v_2$ holds.

To prove that v is symmetric with respect to the origin, we consider a solution v of the problem (18), and $\forall t \in \Omega$ we set $u(t) = v(-t)$ in order to make another solution of the problem (18) labeled by u). This is possible because $u'(t) = -v'(-t)$ and $u"(t) = v"(-t)$ that substituted in

$$v"(-t) = -(\varrho_1\overline{\lambda}^2)^{-1}([1+(v'(-t))^2]^3)(1-\tau-v(-t))$$

and taking into account that $u'(-A_1) = -v'(A_1) = v'(-A_1) \leq M$ and, because from unicity above proved, $v(t) = u(t)$, $\forall t \in \Omega$ so that we have $u(t) = u(-t)$ over Ω.

To prove that $v \in C^\infty(\Omega)$ is quite easy because $v \in C^2(\Omega)$ so that the second member of the equation belongs to $C^1(\Omega)$ and, by means of induction procedure, $v \in C^\infty(\Omega)$ holds.

References

[1] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS*, Chapman & Hall, CRC Press Company oca Raton London New York Washington D.C., 2003.

[2] M. Gad-el-Hak, *MEMS: Introduction and Fundamentals*, Chapman & Hall, CRC Taylor & Francis., 2005.

[3] G. Angiulli and A. Jannelli and F.C. Morabito and M. Versaci, "Reconstructing the Membrane Detection of a 1D Electrostatic-Driven MEMS Device by the Shooting Method: Convergence Analysis and Ghost Solutions Identification", *Comp. Appl. Math*, https://doi.org/10.1007/s40314-017-0564-4, 2018.

[4] M. Huja and M. Husak, "Thermal Microactuators for Optical Purpose," *Coding and Computing*, vol. 15, no. 1, pp. 137–142, 2001.

[5] L. Lin and M. Chiao, "Electro, Thermal and Elastic Characterization of Suspended Micro Beams," *Microelectronics Journal*, vol. 29, no. 4-5, pp. 104–121, 1998.

[6] F. Khoshnoud and C.W. Silva, "Recent Advances in MEMS Sensor Technology - Biomedical Application," *IEEE Instrumentation and Measurement Magazine*, vol. 15, no. 1, DOI: 10.1109/MIM.2012.6145254 , 2012.

[7] A.A.. Rogers and S. Samson and S. Kedia, "Far-Fiedl Evanescent Wave Propagation Using Coupled Subwavelength Gratings for a MEMS Sensors," *J. Opt. Soc. Am. A. Opt. Image Sci. Vis,*, vol. 26, no. 12, pp. 2526-2531 DOI: 10.1364/JOSAA.26.002526, 2009.

[8] D. Cassani and L. Fattorusso and A. Tarsia, "Nonlocal Singular Problems and Application to MEMS," *in Proceedings of WCE 2013.*

[9] D. Cassani and L. Fattorusso and A. Tarsia, "Nonlocal Dynamic Problems with Singular Nonlinearities and Application to MEMS, " *Progress in Nonlinear Differential Equations and their Applications*, vol. 85, pp. 185–206, 2014.

[10] D. Cassani and A. Tarsia, "Periodic Solutions to Nonlocal MEMS Equations, " *Discrete and Continuous Dynamical Systems - Serie S*, vol. 9, no. 3, pp. 631–642, 2016.

[11] D. Cassani and M. d'O and N. Ghoussoub,"On a Fourth Order Elliptic Problem with a Singular Nonlinearity, " *Nonlinear Studies- Serie S*, vol. 9, pp. 189–209, 2009.

[12] P. Di Barba and L. Fattorusso and M. Versaci, "Electrostatic Field in Terms of Geometric Curvature in Membrane MEMS Devices," *Communications in Applied and Industrial Mathematics*, vol. 8, pp. 165–184, DOI: 10.1515/caim-2017-0009, 2017.

[13] P.V. Reson Panat, "Contributions of MAxwell to Electromagnetism, " *Resonance* vol. 8, pp. 17–29, 2003.

[14] E. Giusti, *Analisi Matematica*, Vol. 2, Bollati Boringhieri, 2005.

[15] S. Lang, *Green's Functions on Riemann Surfaces*, *Introduction to Arakelov Theory*, Springer, New Yotk, NY, 1988.

ISBN: 978-988-14047-9-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)