Numerical Analysis of a Family of Second Order Time Stepping Methods for Boussinesq Equations

Aytekin Çıbık, Medine Demir and Songul Kaya

Abstract—**This report considers a family of second order time stepping schemes for Boussinesq system. The scheme uses the idea of curvature stabilization in which the discrete curvature of the solutions is added together with the linearized advective term at each time step. Unconditional stability and convergence of the method are established. Several numerical experiments are provided to demonstrate the accuracy and the efficiency of the method.**

Index Terms—**Boussinesq equations, curvature stabilization, linear extrapolation, error analysis.**

I. Introduction

The governing equations of natural convection is given by the Boussinesq system consisting of the incompressible NSE together with the heat transport equation as

$$\begin{aligned}\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p=Ri\langle 0,T\rangle+\mathbf{f}\ \text{in}\ \Omega,\\ \nabla\cdot\mathbf{u}=0\ \text{in}\ \Omega,\\ T_t+(\mathbf{u}\cdot\nabla)T-\kappa\Delta T=\gamma\ \text{in}\ \Omega,\\ \mathbf{u}(0,x)=\mathbf{u}_0\ \text{and}\ T(0,x)=T_0\ \text{in}\ \Omega,\\ \mathbf{u}=0\ \text{and}\ T=0\ \text{on}\ \partial\Omega.\end{aligned}\quad (1)$$

Here $\mathbf{u}$ is the flui velocity, $\mathbf{u}_0$, the initial velocity, p the pressure, T the temperature, T_0, the initial temperature, f represents the prescribed forcing, Ri the Richardson number, accounting for the gravitational force, $Ri\langle 0,T\rangle$ represents the vector, γ the heat source, ν the kinematic viscosity, which is inversely proportional to the Reynolds number $Re=O(\nu^{-1})$, $\kappa:=Re^{-1}Pr^{-1}$, where Pr is the Prandtl number and $\Omega\subset\mathbb{R}^d$, $d=(2,3)$, is a bounded region with Lipschitz continuous boundary $\partial\Omega$.

The main goal in computational flui dynamics is to develop efficient accurate and sufficientl stabilized method for the solution of the incompressible flui fl ws. The application of the standard finit element method remains inefficien for the numerical solution of the time dependent multiphysics problems that lead to oscillations and unstable modes. In general, the solution technique for the time-stepping methods is to combine the stabilization terms with the linearization of the advective term at each time step. There are many studies using such numerical method see, e.g [1]. Moreover, since there is a need of only one linear system solution at each time step, the linear extrapolation schemes have more advantages over the fully implicit schemes which are more expensive in terms of stability and accuracy. There have been many recent studies based on the Crank Nicholson with linear extrapolation (CNLE) such as [2] and two step backward differentiation formula BDF2 with linear extrapolation (BDF2LE) for different flui problems [3], [4]. For natural convection with different right hand side function of velocity eqaution in [5], the similar model has been considered.

The purpose of this study, is to obtain an accurate stabilization for a family of second order time stepping methods for the Boussinesq system by extending the ideas of [6] to the time dependent natural convection fl ws. As it is shown below, appropriate choices of parameters leads to well-known second order time stepping methods namely CNLE and BDF2LE. As explained in [6], the method based on curvature stabilization achieves a sufficien stabilization with optimal accuracy in time.

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II. Numerical Scheme

To defin the method precisely, we will approximate the solution of (1) by using the finit element method. Let $\mathbf{X}=(\mathbf{H}_0^1(\Omega))^d$, $Q=L_0^2(\Omega)$ be the velocity and pressure spaces and $W=H_0^1(\Omega)$ be the temperature space.

Let $\mathbf{X}^h\subset\mathbf{X}, W^h\subset W, Q^h\subset Q$ be finit element spaces where the velocity and pressure spaces fulfil the inf-sup condition. The usual $L^2(\Omega)$ norm and the inner product is denoted by $\|.\|$ and $(\cdot,\cdot)$, respectively. Defin skew symmetric trilinear forms

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w})=\frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})-\frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}),\quad (2)$$

$$c^*(\mathbf{u},T,S)=\frac{1}{2}(\mathbf{u}\cdot\nabla T,S)-\frac{1}{2}(\mathbf{u}\cdot\nabla S,T)\quad (3)$$

Algorithm. We divide $[0,t]$ time interval into N equal subintervals and set time step size $\Delta t=t/N$. Denote the fully discrete solutions by

$$\mathbf{u}_{n+1}^h:=\mathbf{u}^h(t_{n+1}),\quad p_{n+1}^h:=p^h(t_{n+1})$$

$$T_{n+1}^h:=T^h(t_{n+1}),$$

for all $n=1,2,...,N-1$. Let the initial conditions $\mathbf{u}_0$ and T_0, the forcing function $\mathbf{f}$ and the heat source γ be given. Defin $\mathbf{u}_0^h$, $\mathbf{u}_{-1}^h$, T_0^h and T_{-1}^h as the nodal interpolants of $\mathbf{u}_0(\mathbf{x})$ and T_0, respectively. Then, given time step Δt and $\mathbf{u}_n,\mathbf{u}_{n-1},T_n$ and T_{n-1}, compute $\mathbf{u}_{n+1}\in\mathbf{X}^h, T_{n+1}\in W^h$

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and $p_{n+1} \in Q^h$ satisfying

$$\Big(\frac{(\theta+\frac{1}{2})\mathbf{u}^h_{n+1} - 2\theta\mathbf{u}^h_n + (\theta-\frac{1}{2})\mathbf{u}^h_{n-1}}{\Delta t}, \mathbf{v}^h\Big)$$
$$+\nu\Big(\theta\frac{(\nu+\varepsilon)}{\nu}\nabla\mathbf{u}^h_{n+1} + \Big(1-\theta\frac{\nu+2\varepsilon}{\nu}\Big)\nabla\mathbf{u}^h_n$$
$$+\theta\frac{\varepsilon}{\nu}\nabla\mathbf{u}^h_{n-1}, \nabla\mathbf{v}^h\Big) + b^*\Big((\theta+1)\mathbf{u}^h_n - \theta\mathbf{u}^h_{n-1},$$
$$\theta\frac{(\nu+\varepsilon)}{\nu}\mathbf{u}^h_{n+1} + \Big(1-\theta\frac{\nu+2\varepsilon}{\nu}\Big)\mathbf{u}^h_n + \theta\frac{\varepsilon}{\nu}\mathbf{u}^h_{n-1}, \mathbf{v}^h\Big)$$
$$-\Big(\theta\frac{(\nu+\varepsilon)}{\nu}p^h_{n+1} + \Big(1-\theta\frac{\nu+2\varepsilon}{\nu}\Big)p^h_n + \theta\frac{\varepsilon}{\nu}p^h_{n-1}, \nabla\cdot\mathbf{v}^h\Big)$$
$$= Ri\big(\langle 0, (\theta+1)T^h_n - \theta T^h_{n-1}\rangle, \mathbf{v}^h\big) + \big(\mathbf{f}_{n+\theta}, \mathbf{v}^h\big), \quad (4)$$
$$\big(\nabla\cdot\mathbf{u}^h, q^h\big) = 0, \quad (5)$$
$$\Big(\frac{(\theta+\frac{1}{2})T^h_{n+1} - 2\theta T^h_n + (\theta-\frac{1}{2})T^h_{n-1}}{\Delta t}, S^h\Big)$$
$$+\kappa\Big(\theta\frac{(\kappa+\varepsilon_1)}{\kappa}\nabla T u^h_{n+1} + \Big(1-\theta\frac{\kappa+2\varepsilon_1}{\kappa}\Big)\nabla T^h_n$$
$$+\theta\frac{\varepsilon_1}{\kappa}\nabla T^h_{n-1}, \nabla S^h\Big) + c^*\Big((\theta+1)\mathbf{u}^h_n - \theta\mathbf{u}^h_{n-1},$$
$$\theta\frac{(\kappa+\varepsilon_1)}{\kappa}T^h_{n+1} + \Big(1-\theta\frac{\kappa+2\varepsilon_1}{\kappa}\Big)T^h_n\theta\frac{\varepsilon_1}{\kappa}T^h_{n-1}, S^h\Big)$$
$$= \big(\gamma_{n+\theta}, S^h\big) \quad (6)$$

for all $(\mathbf{v}^h, S^h, q^h) \in (\mathbf{X}^h, W^h, Q^h)$.

III. Numerical Experiments

In this section, we perform two numerical tests in order to show the efficiency of proposed method. We first test the numerical convergence rates with a known analytical solution. Then, we provide the so-called Marsigli's flow example to prove that the method captures flow patterns by using a coarse mesh discretization. All computations are carried out with the finite element software package FreeFem++ [7]. In all numerical studies, the Taylor-Hood finite elements for velocity and pressure spaces and piecewise quadratics for temperature were used on uniform triangular grids. In order to see the effect of stabilization parameters, we sometime make comparison with usual BDF2LE method, which is obtained through picking $\epsilon = \epsilon_1 = 0$ (unstabilized case) in continuous case of (4)-(6) and taking $\theta = 1$, which gives

$$\frac{(\theta+\frac{1}{2})u_{n+1} - 2\theta u_n + (\theta-\frac{1}{2})u_{n-1}}{\Delta t} - \theta\nu\Delta u_{n+1}$$
$$-(\nu-\theta\nu)\Delta u_n + ((\theta+1)u_n - \theta u_{n-1})\cdot\nabla(\theta u_{n+1}$$
$$+(1-\theta u_n)) + \theta\nabla p_{n+1} + (1-\theta)\nabla p_n$$
$$= Ri(\langle 0, ((\theta+1)T_n - \theta T_{n-1})\rangle + f_{n+\theta} \quad (7)$$
$$\nabla\cdot u_{n+1} = 0 \quad (8)$$
$$\frac{(\theta+\frac{1}{2})T_{n+1} - 2\theta T_n + (\theta-\frac{1}{2})T_{n-1}}{\Delta t} - \theta\kappa\Delta T_{n+1}$$
$$-(\kappa-\theta\kappa)\Delta T_n + ((\theta+1)u_n - \theta u_{n-1})\cdot\nabla(\theta T_{n+1}$$
$$+(1-\theta T_n)) = \gamma_{n+\theta}. \quad (9)$$

We note that, the similar results are also obtained with CNLE in which $\theta = 1/2$ and $\epsilon = \epsilon_1 = 0$.

A. Numerical convergence study

In this subsection, we show that the theoretical orders of the errors are also obtained through a numerical simulation. In order to do so, we pick the known-solution

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} \cos(y) \\ \sin(x) \end{pmatrix}\exp(t), \\ p &= (x-y)(1+t), \\ T &= \sin(x+y)\exp(1-t). \end{aligned} \quad (10)$$

with the parameters $Pr = Re = Ri = \kappa = 1$ and the right hand side f, γ functions are chosen such that (10) satisfies (1). We will present computational results with $\epsilon = \epsilon_1 = 0$ (no stabilization), $\theta = 1$ and $\epsilon = \epsilon_1 = 1$ (with stabilization). The final time and the time step size are chosen as $t = 10^{-4}$ and $\Delta t = t/8$. To test the spatial convergence, we fix the time step size and calculate the errors for varying h and consider the velocity errors in the discrete norm $L^2(0, T; \mathbf{H}^1(\Omega))$

$$\|\text{u} - u^h\|_{2,1} = \left\{\Delta t\sum_{n=1}^{N}\|\mathbf{u}(t^n) - \mathbf{u}^h_n\|^2\right\}^{1/2}.$$

The results with different ϵ and ϵ_1 values for spatial errors and error rates are given in Table I and Table II. One can see that the order of convergence of $\|u - u^h\|_{2,1}$ is around 2 for all simulations, which is an optimal order for both BDF2LE and for the proposed method. We also fix the

TABLE I
Spatial errors and rates of convergence for $\epsilon = \epsilon_1 = 0$.

h	$\|u - u^h\|_{2,1}$	Rate	$\|T - T^h\|_{2,1}$	Rate
1/4	0.0007139	–	0.0005043	–
1/8	0.0001778	2.005	0.0001259	2.002
1/16	4.360e-5	2.027	3.068e-5	2.036
1/32	1.131e-5	1.946	7.446e-6	2.042
1/64	2.937e-6	1.945	1.847e-6	2.015
1/128	7.529e-7	1.963	4.616e-7	2.000

TABLE II
Spatial errors and rates of convergence for $\epsilon = \epsilon_1 = 1$.

h	$\|u - u^h\|_{2,1}$	Rate	$\|T - T^h\|_{2,1}$	Rate
1/4	0.0007131	–	0.0005045	–
1/8	0.0001771	2.009	0.0001256	2.005
1/16	4.352e-5	2.025	3.065e-5	2.035
1/32	1.125e-5	1.959	7.440e-6	2.040
1/64	2.934e-6	1.941	1.841e-6	2.015
1/128	7.525e-7	1.963	4.611e-7	2.002

mesh size to $h = 1/128$ to see the temporal errors and the convergence rates by using different time steps with an end time of $t = 1$. The results are given in Table III and Table IV for $\epsilon = \epsilon_1 = 0$ and $\epsilon = \epsilon_1 = 1$, respectively. As expected, we observe a second order convergence in time. However, the velocity error rates becomes better for the stabilized case as Δt decreases and also the rates for the temperature error is far more better then the no stabilization case compared with the proposed method.

B. Marsigli's Flow Experiment

As another numerical test, we apply the proposed method to so-called Marsigli flow. This flow experiment was firstly

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TABLE III
TEMPORAL ERRORS AND RATES OF CONVERGENCE FOR $\epsilon = \epsilon_1 = 0$.

Δt	$\|u - u^h\|_{2,1}$	Rate	$\|T - T^h\|_{2,1}$	Rate
1	1.951e-2	–	6.572e-2	–
1/2	3.483e-3	2.48	3.417e-2	1.01
1/4	7.539e-4	2.20	1.221e-2	1.49
1/8	1.763e-4	2.09	3.618e-3	1.75
1/16	4.354e-5	2.01	9.838e-4	1.88
1/32	1.356e-5	1.78	2.565e-4	1.94

TABLE IV
TEMPORAL ERRORS AND RATES OF CONVERGENCE FOR $\epsilon = \epsilon_1 = 1$.

Δt	$\|u - u^h\|_{2,1}$	Rate	$\|T - T^h\|_{2,1}$	Rate
1	2.333e-2	–	7.005e-1	–
1/2	8.980e-3	1.37	1.991e-1	1.81
1/4	3.043e-4	1.51	5.230e-2	1.92
1/8	7.682e-4	1.98	1.151e-2	2.18
1/16	1.913e-4	2.00	2.610e-3	2.14
1/32	4.855e-5	1.99	6.212e-4	2.07

motivated by the undercurrent in Bosphorus by Marsigli in 1681. The problem set up was given in [8]. It is hard to capture correct flow patterns such as the velocity and the temperature contours for this flow example. A direct numerical simulation is known to fail even for finer meshes [9]. Our main goal in this numerical experiment is to demonstrate the correct flow patterns by comparing the results of [8] in which a fourth order finite difference scheme is used for the Boussinesq equations. Since we assume the Boussinesq approximation, the density differences could be seen as temperature differences and so the Boussinesq equations we solve could be used as a model problem here. We simulate the problem in a rectangular box in the domain $(0, 1) \times (0, 8)$. No slip velocity boundary conditions are used and the temperature gradients are taken to be zero at all boundary. Our initial temperature is a piecewise function given by

$$T_0 = \begin{cases} 1.5 & x \leq 4.0 \\ 1.0 & x > 4.0 \end{cases} . \qquad (11)$$

and the initial velocity is zero. We take flow parameters as $Pr = 1, Re = 1000, Ri = 4$. We use a rather large time step size of $\Delta t = 0.02$ to calculate the behavior of temperature contours and velocity streamlines at dimensionless time instances $t = 2, 4, 6, 8$. The stabilization parameter ϵ is taken as of $O(\nu)$ and ϵ_1 is taken as of $O(\kappa)$.

The method we apply captures the correct patterns for all time instances for a very coarse mesh consisting of 13362 velocity degree of freedom and 6681 temperature degree of freedom. The resulting natural convection patterns are given in Figure 1 and Figure 2. As expected, fluids at different temperature mix at the interface and as time evolves the warmer fluid tends to spread out on the colder one. When these results are compared with [8] one can easily deduce the excellent agreement between the flow patterns even for this very coarse mesh. This comparison proves the promise of the method in this sense.

IV. CONCLUSION

The accuracy and the efficiency of the algorithm (4)-(6) are verified for two numerical tests.

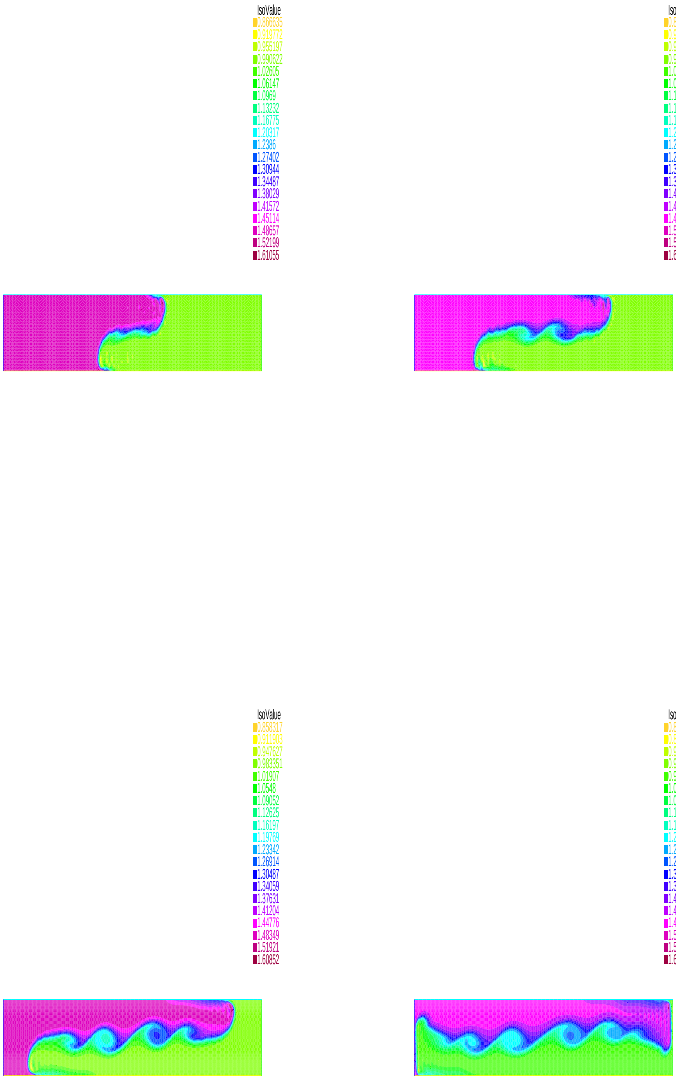

Fig. 1. Temperature iso-contours at $t = 2, 4, 6, 8$, respectively

REFERENCES

[1] P. Gresho and R. Sani, *Incompressible flow and the finite element method Vol. 1: Advection-Diffusion and Isothermal Laminar Flow*. New York: Wiley., 2000.
[2] R. Ingram, “A new linearly extrapolated Crank-Nicolson time-stepping scheme for the Navier-Stokes equations,” *Math. Comp.*, vol. 82, pp. 953–1973, 2013.
[3] C. Xu, Y. Zhang, and J. Zhou, “Convergence of a linearly extrapolated BDF2 finite element scheme for viscoelastic fluid flow,” *Bound. Value Probl.*, vol. 140, 2017.
[4] M. Akbas, S. Kaya, and L. G. Rebholz, “On the stability at all times of linearly extrapolated BDF2 time-stepping for multiphysics incompressible flow problems,” *Numer. Methods Partial Differential Equations*, vol. 33, pp. 999–1017, 2016.
[5] Y. B. Yang and Y. L. Jiang, “Numerical analysis and computation of a type of imex method for the time-dependent natural convection problem,” *Comput. Methods Appl. Math.*, vol. 16(2), p. 321344, 2016.
[6] N. Jiang, M. Mohebujjaman, L. G. Rebholz, and C. Trenchea, “An

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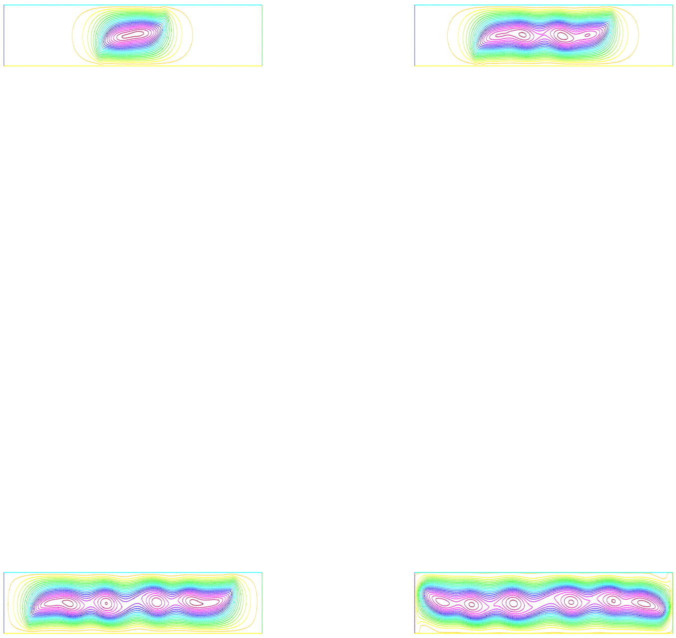

Fig. 2. Streamlines at $t = 2, 4, 6, 8$, respectively

optimally accurate discrete regularization for second order time stepping methods for Navier-Stokes equations," *Comput. Methods Appl. Mech. Engrg.*, vol. 310, pp. 388–405, 2016.

[7] F. Hecht, "New development in freefem++." *J. Numer. Math.*, vol. 20, pp. 251–265, 2012.

[8] J.-G. Liu, C. Wang, and H. Johnston, "A fourth order scheme for incompressible Boussinesq Equations." *J. Sci. Comput.*, vol. 18, pp. 253–285, 2003.

[9] L. G. R. M. A. Belenli, S. Kaya, "An explicitly decoupled variational multiscale method for incompressible, non-isothermal flows," *Comput. Meth.Appl. Math.*, vol. 15, pp. 1–20, 2015.

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