Non-Conservative Stability Analysis of Hauger Types of Columns with Different Boundary Conditions

S.A. Fazelzadeh, M. Tashakorian, E. Ghavanloo, M. Amoozgar, M.I. Friswell

Abstract— **Fully intrinsic equations are used to obtain a model compliant with Euler-Bernoulli assumptions for a beam under a linearly distributed follower force known as a Hauger column. The advantage of the intrinsic formulation in modeling problems with non-conservative forces is discussed here. Only intrinsic parameters which are independent of the choice of coordinate system has been used and four different boundary conditions were implemented. Also, a comparison between the present study and similar studies with the classical formulation has been developed. The Generalized Differential Quadrature Method is used to numerically analyze the critical load of the beam. It is well understood that there is a remarkable advantage in terms of convergence using intrinsic equations in comparison with the classical formulation.**

Index Terms— **Differential Quadrature Method, Hauger Column, Intrinsic Equations, Non-Conservative Stability**

I. Introduction

Fully intrinsic equations may be applied to a wide variety of applications of beams because of its proper modeling of forces and moments as motion-dependent quantities. Although mechanical elements such as beams have been the center of interest in recent years and has been analyzed in many different configurations, there are some cases that have not been discussed properly as yet. One of these is the non-conservative stability of beams with linear distributed follower loads which is discussed in this paper using intrinsic equations.

The procedure for deriving the fully intrinsic equations was proposed by Hodges [1]. The fully intrinsic equations include a set of first-order partial differential equations of motion. The displacements and rotations are not presented in these formulations and intrinsic parameters exist in this formulation are independent of the choice of coordinate system.

Hodges [2] presented a systematic derivation of a geometrically exact generalized Timoshenko theory for initially curved and twisted anisotropic beams. Models for elementary beam vibration and stability with various boundary conditions were addressed by Chang and Hodges [3]-[4] and Sotoudeh and Hodges [5]. The fully intrinsic formulation was used by Patil and Hodges [6] to study flight dynamics and aeroelasticity of highly flexible flying wings. This work on flying wings spawned several related works for HALE aircraft based on the fully intrinsic formulation, e.g., Chang and Hodges [7], Chang et al. [8], and Sotoudeh et al. [9]. Fully intrinsic equations were also used to predict the aeroelastic behavior of joined-wing aircraft by Sotoudeh and Hodges [10]. The fully intrinsic formulation was applied to model multi-flexible body dynamics problems by Sotoudeh and Hodges [11]-[12]. This work has led to a series of studies on the effect of engine placement on aeroelastic behavior, passive morphing and body-freedom flutter of flying wings by Mardanpour et al. [13]-[15]. A Galerkin approach was presented for approximate solutions of the nonlinear fully intrinsic equations by Patil and Althoff [16]. Palacios [17] used the intrinsic formulation to obtain the nonlinear normal modes of beams. Hesse and Palacios [18, 19] and Wang et al. [20]-[21] applied the intrinsic formulation for model order reduction, consistent linearization, and modeling of nonlinear aeroservoelasticity for flexible aircraft. Using fully intrinsic equations, Khaneh Masjedi and Ovesy [22]-[23] investigated the static, large deflection of beams under both conservative and nonconservative loads. They used the Chebyshev collocation method to numerically solve the differential equations. In another study, Amoozgar and Shahverdi [24] studied a similar problem by using the generalized differential quadrature (GDQ) method. Recently, dynamic instability of beams under tip follower forces was studied by the same group [25] within the framework of geometrically exact, fully intrinsic equations. More recently, the geometrically exact, fully intrinsic nonlinear beam theory has been utilized to model the dynamic stability of initially twisted beams subjected to distributed follower forces [26].

Non-conservative forces occur in a vast range of applications in mechanical and civil engineering as well as aeronautics engineering and aeroelasticity. Most researchers assume a uniformly distributed follower force which is a proper model for fluid friction and similar problems. A detailed review of follower force research can be found in Mardanpour et al. [26] and Fazelzadeh et al. [27]-[28]. Because of difficulties of applying a variable follower load on a deformable beam, the linearly distributed follower

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problem known as the Hauger problem has received less interest. The only research that considered such a problem was published by Marzani et al. [28] in which the stability of the Hauger column was studied using the classical formulation and the GDQ method.

All of the research in follower force problems and the Hauger column has considered only one boundary condition. Using intrinsic equations arbitrary follower forces can be applied and so the Hauger column can be modeled much more easily than in the classical formulation. In this paper, the implementation of a linearly distributed follower load on the beam will be discussed. Also the effects of different boundary conditions will be considered.

II. Theoretical Foundations

A. *Fully Intrinsic Equations*

A set of equations has been proposed by Hodges and developed by him and his colleagues during past decade, named "fully intrinsic" which are independent of the choice of coordinate systems and by considering special constitutive rules will provide a geometrically exact analysis of beams with any arbitrary shape of cross section and curvatures. Fig. 1 illustrates the configuration of the beam and reference frames which are used in the intrinsic formulation. Here ℓ is the undeformed reference line and $\mathcal{L}$ is the deformed reference line of the beam. Also, at every point of both the deformed and undeformed reference lines there is a reference frame for which one of the undeformed reference lines is time-independent and denoted by $\mathbf{b}_i (i = 1,2,3)$, and is called the undeformed reference frame. In a similar manner, a deformed reference frame is considered at every point of the deformed reference line and denoted by $\mathbf{B}_i (i = 1,2,3)$. All intrinsic parameters are measured in the deformed reference frame which makes it easy to apply non-conservative forces.

According to the formulation proposed by Hodges [1], the three dimensional differential equations of motion in intrinsic parameters can be written as:

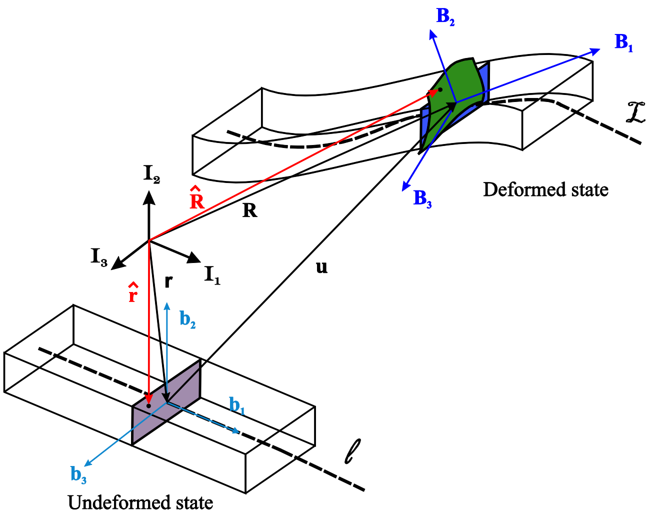

Fig. 1. General configuration of the beam, the reference lines and the reference coordinate systems.

$$\begin{aligned} &\mathbf{F}' + \tilde{\mathbf{K}}\mathbf{F} + \mathbf{f} = \dot{\mathbf{P}} + \tilde{\mathbf{\Omega}}\mathbf{P} \\ &\mathbf{M}' + \tilde{\mathbf{K}}\mathbf{M} + (\tilde{\mathbf{e}}_1 + \tilde{\boldsymbol{\gamma}})\mathbf{F} + \mathbf{m} = \dot{\mathbf{H}} + \tilde{\mathbf{\Omega}}\mathbf{H} + \tilde{\mathbf{V}}\mathbf{P} \end{aligned} \tag{1}$$

where $\mathbf{F}$ is a vector of cross sectional forces measured in the deformed basis and is unknown. Similarly, $\mathbf{M}$ is the vector of cross sectional moments, $\mathbf{P}$ is the linear momentum and $\mathbf{H}$ is the angular momentum per unit length. There are also kinematical intrinsic parameters, which are: generalized linear velocity $\mathbf{V}$ and angular velocity $\mathbf{\Omega}$, generalized strain $\boldsymbol{\gamma}$ and curvatures $\boldsymbol{\kappa}$ which all is measured in deformed reference frame. Furthermore, $\mathbf{f}$ and $\mathbf{m}$ are the distributed force and moment vectors. In equation (1) there is another parameter which indicates the curvature of the beam in the deformed state and denoted by $\mathbf{K}$, which can be written in the form of undeformed beam curvature $\mathbf{k}$ and generalized curvature $\boldsymbol{\kappa}$ as: $\mathbf{K} = \mathbf{k} + \boldsymbol{\kappa}$.

Here $\mathbf{F}'$ is the space derivative of the cross sectional force with respect to x_1, the path variable of the reference line and $\dot{\mathbf{P}}$ is the time derivative of the linear momentum, and so on. In equation (1), and all of this paper, the tilde notation is used to simplify the cross product of vectors. Given a vector $\mathbf{K}$ with three elements, $\mathbf{K} = [\mathrm{K}_1, \mathrm{K}_2, \mathrm{K}_3]^{\mathrm{T}}$, the tilde notation will give:

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & -\mathrm{K}_3 & \mathrm{K}_2 \\ \mathrm{K}_3 & 0 & -\mathrm{K}_1 \\ -\mathrm{K}_2 & \mathrm{K}_1 & 0 \end{bmatrix} \tag{2}$$

Equation (1) provides equilibrium conditions for a three dimensional beam with intrinsic parameters, but cannot be solved in isolation and needs relations between the kinematical parameters. Here these relations for the kinematical intrinsic parameters are [2]:

$$\begin{aligned} &\mathbf{V}' + \tilde{\mathbf{K}}\mathbf{V} + (\tilde{\mathbf{e}}_1 + \tilde{\boldsymbol{\gamma}})\mathbf{\Omega} = \dot{\boldsymbol{\gamma}} \\ &\mathbf{\Omega}' + \tilde{\mathbf{K}}\mathbf{\Omega} = \dot{\boldsymbol{\kappa}} \end{aligned} \tag{3}$$

Equations (1) and (3) simultaneously provide a set of fully intrinsic equations for the beam.

Now we can simplify the three dimensional equations to make a beam model consistent with Euler-Bernoulli beam theory. In order to achieve this, all of the intrinsic parameters are set to zero except $\mathrm{V}_2, \Omega_3, \gamma_{11}, \gamma_{12}, \mathrm{F}_1, \mathrm{F}_2, \mathrm{M}_3, \mathrm{P}_2$ and κ_3. Equations (1) and (3) can then be written in simplified scalar form as:

$$\begin{aligned} &\mathrm{F}_1' = \kappa_3 \mathrm{F}_2 - \mathrm{P}_2 \Omega_3 - \mathrm{f}_1 \\ &\mathrm{F}_2' = -\kappa_3 \mathrm{F}_1 + \dot{\mathrm{P}}_2 \\ &\mathrm{M}_3' = -\mathrm{F}_2 - \gamma_{11}\mathrm{F}_2 + \dot{\mathrm{H}}_3 \\ &\mathrm{V}_2' = \Omega_3 + \gamma_{11}\Omega_3 + \dot{\gamma}_{12} \\ &\Omega_3' = \dot{\kappa}_3 \end{aligned} \tag{4}$$

where f_1 is the distributed axial follower load acting on the reference line of the beam. Assuming an homogenous, isotropic beam with the mass centroid coincident with the reference line and the principle axis of the cross section, the constitutive relations of the beam can be written as:

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$$\begin{aligned} &P_2 = \mu V_2 \\ &H_3 = i_3 \Omega_3 \\ &\kappa_3 = \frac{M_3}{EI_3} \\ &\gamma_{11} = \frac{F_1}{EA} \\ &\gamma_{12} = \frac{F_2}{GA_2} \end{aligned} \tag{5}$$

Substituting equation (5) into (4) results in a set of intrinsic equations with respect to only $V_2, \Omega_3, F_1, F_2, M_3$. Thus,

$$\begin{aligned} &F_1' = \frac{M_3}{EI_3} F_2 - \mu V_2 \Omega_3 - f_1 \\ &F_2' = -\frac{M_3}{EI_3} F_1 + \mu \dot{V}_2 \\ &M_3' = -F_2 - \frac{F_1}{EA} F_2 + i_3 \dot{\Omega}_3 \\ &V_2' = \Omega_3 + \frac{F_1}{EA} \Omega_3 + \frac{\dot{F}_2}{GA_2} \\ &\Omega_3' = \frac{\dot{M}_3}{EI_3} \end{aligned} \tag{6}$$

For a linear varying distribution of axial force, known as Hauger column, where the magnitude is p_0 at the root and zero at the free end (Fig. 2), the distributed axial force may be written as:

$$f_1 = \frac{p_0}{L}\left(x - L\right) \tag{7}$$

If the non-linear terms in the first of equations (6) is neglected, then a decoupled ordinary differential equation is created which can be solved analytically for given boundary conditions Thus

$$\begin{aligned} &F_1' = -\frac{p_0}{L}\left(x - L\right) \\ &F_1\left(L\right) = 0 \end{aligned} \tag{8}$$

and the solution is:

$$F_1 = -\frac{p_0}{2L}(x - L)^2 \tag{9}$$

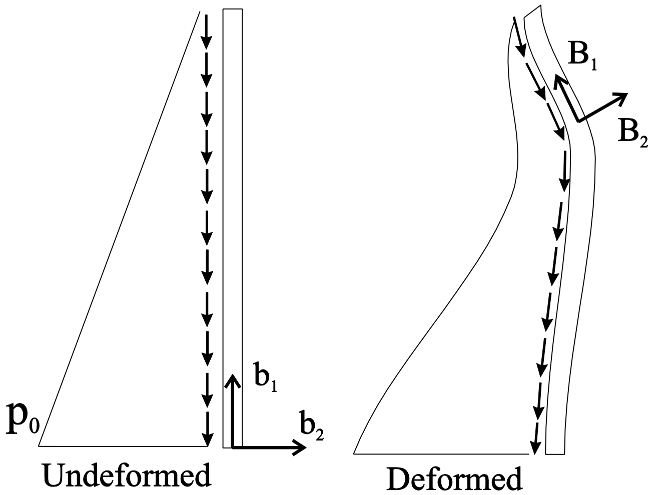

Fig. 2. Reference coordinate system and load model for a Hauger column in the undeformed and deformed states.

Substituting equation (9) into other equations of (6) and ignoring shear effects ($\frac{1}{GA_2} = \frac{1}{EA} = 0$) and rotary inertia ($i_3$=0), results in four coupled first-order differential equations which can be solved for the stability analysis of the problem.

These equations can also be written in non-dimensional form to enable an easier assessment of the numerical performance. Thus

$$\begin{aligned} &\bar{V}_2' = \bar{\Omega}_3 \\ &\bar{\Omega}_3' = \dot{\bar{M}}_3 \\ &\bar{F}_2' = -\frac{p_0}{2}(\xi - 1)^2 + \dot{\bar{V}}_2 \\ &\bar{M}_3' = -\bar{F}_2 \end{aligned} \tag{10}$$

One can also write the non-dimensional parameters as:

$$\begin{aligned} &\bar{F}_2 = \frac{F_2 L^2}{EI_3} \\ &\bar{M}_3 = \frac{M_3 L}{EI_3} \\ &\bar{V}_2 = V_2 \sqrt{\frac{\mu L^2}{EI_3}} \\ &\bar{\Omega}_3 = \Omega_3 \sqrt{\frac{\mu L^4}{EI_3}} \\ &\bar{p}_0 = \frac{p_0 L^3}{EI_3} \\ &\frac{d}{dx_1}(\ ) = \frac{1}{L}\frac{d}{d\xi}(\ ) \\ &\frac{d}{dt}(\ ) = \sqrt{\frac{EI_3}{\mu L^4}}\frac{d}{d\tau}(\ ) \end{aligned} \tag{11}$$

B. Numerical Discretization

In order to make a numerical stability analysis of a Hauger column, the Generalized Differential Quadrature Method (GDQM) has been used. In this method, the space derivatives of a function is approximated by a series of weighting coefficients multiplied by grid point values of that function, as proposed by Bellman and his colleagues [30]-[31] in the early 1970s. There are several approaches to compute the weighting coefficients values. Here, we will use Shu’s general approach [32], which is an efficient and reliable method and has been used many times recently.

The first derivative in this method, can be written as:

$$\frac{df}{dx}\left(x_i\right) = \sum_{j=1}^{N} A_{ij}^{(1)} f\left(x_j\right) \tag{12}$$

where $A_{ij}^{(1)}$ are the values of the weighting coefficients, $f\left(x_j\right)$ are the values of the function at the grid points and N is the number of grid points. A larger number of grid points gives more accurate results, although this also makes the computation more difficult and the calculation time longer.

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The matrix of weighting coefficients can be computed using Shu's general approach as follows:

$$A_{ij}^{(1)} = \frac{\prod_{k=1,k\neq i}^{N}(x_i - x_k)}{(x_i - x_j)\prod_{k=1,k\neq j}^{N}(x_j - x_k)} \quad i \neq j \tag{13}$$

$$A_{ii}^{(1)} = -\sum_{k=1,k\neq i}^{N} A_{ik}^{(1)}$$

The other parameter that determine the accuracy of the GDQM is the method of choosing the grid points. Here, for faster convergence and higher accuracy, a non-uniform distribution of grid points has been implemented [32], as

$$x_i = \frac{L}{2}\left[1-\mathbf{cos}\left(\frac{i-1}{N-1}\pi\right)\right] \quad i = 1,\ldots, N \tag{14}$$

III. Results and Discussion

Using the Differential Quadrature Method, one can turn equation (10) into a system of first order differential equations with respect to time. Indeed, the equations of motion can be written in standard matrix form as:

$$[\mathbf{A}]\{\dot{\mathbf{q}}\} + [\mathbf{B}]\{\mathbf{q}\} = \{\mathbf{0}\} \tag{15}$$

where **A** and **B** are numerical matrices. In order to obtain the natural frequencies of beam, harmonic motion is assumed and therefore, equation (15) can be rewritten in the form of a standard eigen-value problem as:

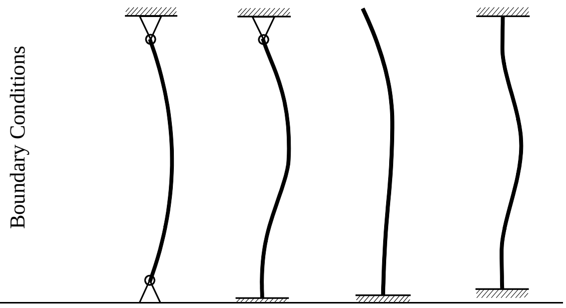
$$([\mathbf{A}]\lambda + [\mathbf{B}])\{\hat{\mathbf{q}}\} = \{\mathbf{0}\} \tag{16}$$

Where λ is the eigen-frequency and $\hat{\mathbf{q}}$ is the eigenvector in generalized coordinates. Using a standard algorithm, the eigen-frequency for different load parameters can be easily computed.

Furthermore, the boundary condition must be defined. For different boundary conditions, the values of the intrinsic parameters should be determined at the boundaries. Four different traditional boundary conditions are studied in this research and the values of the intrinsic parameters are as follows:

- Simply supported beam (S-S)
  $V_2(0)=0 \qquad V_2(L)=0$
  $M_3(0)=0 \qquad M_3(L)=0$
- Clamped-Simply supported (C-S)
  $V_2(0)=0 \qquad V_2(L)=0$
  $\Omega_3(0)=0 \qquad M_3(L)=0$
- Clamped-Free (C-F)
  $V_2(0)=0 \qquad F_2(L)=0$
  $\Omega_3(0)=0 \qquad M_3(L)=0$
- Clamped (C-C)
  $V_2(0)=0 \qquad V_2(L)=0$
  $\Omega_3(0)=0 \qquad \Omega_3(L)=0$

TABLE I
CRITICAL LOAD FOR DIFFERENT BOUNDARY CONDITIONS USING INTRINSIC AND CLASSICAL FORMULATIONS

	S-S	C-S	C-F	C-C
Boundary Conditions				
Present	61.86	314.42	150.62	374.92
Ref.[29]	61.87	313.50	150.64	375.02
Difference (%)	0.01	0.3	0.01	0.03

The critical load for the four boundary conditions are given in Table I. The Hopf bifurcation method was used to determine the critical condition. Also, in order to compare the intrinsic results and the classical results, the results of Marzani et al. [29] are also presented in Table I. The results are in a good agreement with those reported by Marzani at al.. While the number of grid points in the classical formulation using the DQ method is reported to be 51, but it is reduced to only 17 when using the intrinsic formulation in the same numerical method. This is because of the first order derivatives and intrinsic parameters implemented in the intrinsic equations.

For more design applications, a semi-log plot of critical load versus slenderness ratio is provided. It can be seen that with an increase of the slenderness ratio, the critical load for all boundary conditions will decrease. Furthermore, the most critical boundary condition from the point of critical load is determined to be the simply supported beam. Also, it is easy to see that the least critical boundary condition from this point of view is the clamped beam.

IV. Conclusion

In this paper, the three dimensional fully intrinsic equations of beams have been used to create an intrinsic model conistent with Euler-Bernoulli beam theory in order to analyze a column under a linearly distributed follower force known as a Hauger column. The advantage of the

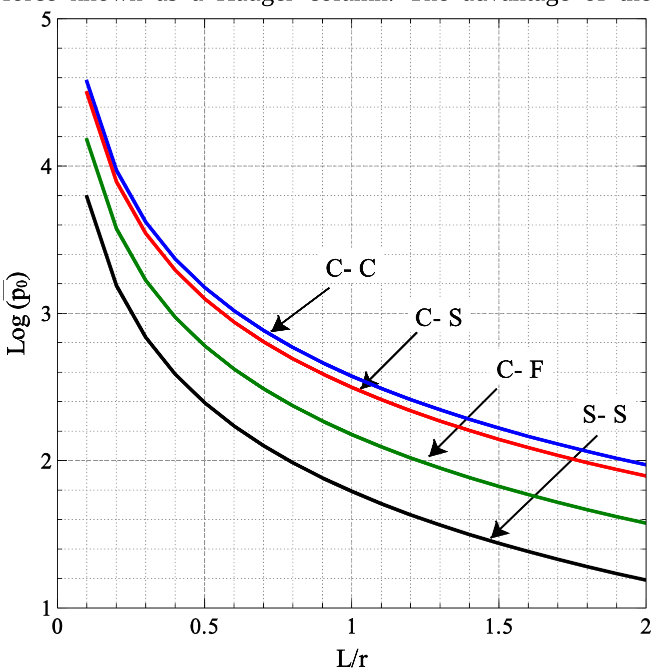


Fig. 3. The dimensionless critical load versus the slenderness ratio, when the number of grid point is 17.

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intrinsic formulation is clearly seen where applying non-conservative loads because of the ease of describing motion-dependent loads in this formulation. Four different boundary conditions have been studied and compared with each other. A comparison between the present formulation and the classical formulation is also presented and the significant advantage of the implementation using the intrinsic formulation was highlighted. For more design applications, a semi-log plot of critical load versus slenderness ratio for four boundary conditions is provided.

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