Solution of Differential Equations of Dynamics of Inertial Continuously Variable Transmissions

S. Aliukov, L. Shefer, and A. Alyukov

***Abstract*— The inertial continuously variable transmissions are mechanical transmissions that are based on the principle of inertia. These transmissions have a lot of advantages. Usually, the design of the inertial continuously variable transmissions consists of inertia pulsed mechanism with unbalanced inertial elements and two overrunning clutches. Dynamics of the transmissions is described by systems of substantial nonlinear differential equations. In general, precise methods of solution for such equations do not exist. Therefore, in practice, approximate analytical and numerical methods must be employed. The main analytical methods employ successive approximation, a small parameter, or power series expansion. Each approach has its advantages and disadvantages. Therefore, we need to compare them in order to select the best method for dynamic study of such kind of transmissions. In this paper a comparative analysis of approximate methods of solving of differential equations for the inertial continuously variable transmissions is done. The object of the investigation is structural dynamics of the continuously variable automatic inertial mechanical transmissions. Approximate methods of solving the nonlinear differential equations of motion of inertial transmissions based on a pulsed mechanism are compared. These methods take account of the no uniform driveshaft rotation and the dynamic characteristics of the motor. Analysis of the solutions reveals the best method for dynamic study of the given transmissions. The comparative analysis showed that the best method of approximate solution is the method of a small parameter.**

***Index Terms*— Continuously variable transmissions, differential equations, dynamics, methods of solution.**

I. Introduction

THE inertial continuously variable transmission (CVT) is based on the principle of inertia [1-3]. This transmission has a lot of advantages [4,5], namely: compactness, minimum friction losses and high efficiency as a result of the relatively small number of rotating components, a wide range of transformation of the torque. It does not need any conventional friction clutches. This transmission protects the engine from overload when output shaft is braked. This drive guarantees optimum conditions of work for the engine regardless of the changing of load, and smoothly changes output speed according to the load.

In spite of relative simplicity of design of the inertial transmissions, their dynamics is described by complex systems of substantial nonlinear differential equations [6-8]. To solve the equations there are no any exact methods. It is possible to obtain solutions any with help of approximate analytical and numerical methods. To chose the best method with the smallest error we need to do a comparative analysis of exist methods.

II. The Physical Model of the Continuously Variable Transmission

The general scheme of the inertial continuously variable transmission contains the pulsed mechanism with unbalanced inertial elements, for example, planetary gear with unbalanced satellites, and two overrunning clutches [9-11]. The scheme of such a transmission is shown in Fig.1. Here 1 is the drive shaft of the transmission, which is the input shaft of the pulsed mechanism in the same time, 2 ─ the pulsed mechanism with unbalanced inertial elements, 3 ─ the output shaft (the reactor) of the pulsed mechanism, 4 ─ the body overrunning clutch, 5 ─ the output overrunning clutch, 6 ─ the driven shaft of the transmission.

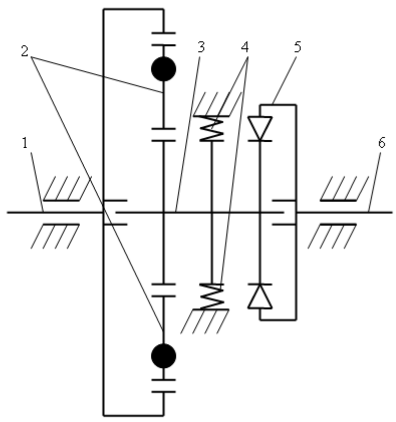

Fig.1. Kinematic scheme of the inertial continuously variable transmission

The main purpose of the pulsed mechanism 2 is to create alternative-sign impulses of torque. One of the overrunning clutches (the output overrunning clutch 5) transmits direct impulses of the torque to the driven shaft 6, the other one (the body overrunning clutch 4) transmits the reverse impulse of the torque on the frame of the transmission [12-15].

The mathematical model of inertia-pulsed transmission may be based on Lagrangian equations of the second kind.

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As the generalized coordinates we take the angle α of rotation of the driving shaft 1, the angle β of rotation of the output shaft 2 of the pulsed mechanism, and the angle γ of the driven shaft 6.

Using these equations, we obtain a mathematical model of an inertia-pulsed drive in the form of fifth-order system of nonlinear equations [8]:

$$\begin{cases} A_1\ddot{\alpha} + A_2\ddot{\beta} + A_4(\dot{\alpha}-\dot{\beta})^2 + A_6\dot{\beta}^2 = M_Д, \\ A_2\ddot{\alpha} + A_3\ddot{\beta} + A_5(\dot{\alpha}-\dot{\beta})^2 - A_6\dot{\alpha}^2 = -M_С, \\ \dot{M}_Д = \frac{1}{T}\left(M_Н - M_Д - \frac{1}{\nu\cdot\omega_X}\cdot(\dot{\alpha}-\omega_Н)\right), \end{cases} \tag{1}$$

where

$$A_1 = J_1 + nma^2 + \left(\frac{a}{k}+q\right)^2 nJ_3 + 2nmah\left(\frac{a}{k}+q\right)\cos\psi,$$
$$A_2 = nmab + \left(\frac{a}{k}+q\right)\left(\frac{b}{k}-q\right)nJ_3 + nmh\left(\frac{2ab}{k}+(b-a)q\right)\cos\psi,$$
$$A_3 = J_2 + nmb^2 + \left(\frac{b}{k}+q\right)^2 nJ_3 + 2nmbh\left(\frac{b}{k}-q\right)\cos\psi,$$
$$\psi = q(\alpha-\beta),$$
$$k = a+b,$$

$$A_4 = -nmah\left(\frac{a}{k}+q\right)q\sin\psi,$$
$$A_5 = nmbh\left(\frac{b}{k}-q\right)q\sin\psi,$$
$$A_6 = nmkhq\sin\psi,$$

Mc is the reduced drag torque on the driven shaft of the pulsed mechanism,
J_1, J_2 are the moments of inertia of the elements; nJ_3 is the total moment of inertia of the unbalanced elements relative to the geometric center;
nm is the total mass of the unbalanced elements;
h is the distance between the geometric center and the center of mass of the unbalanced elements;
a, b, q are parameters of the pulsed mechanism.

To determine the torque $M_Д$ acting on the driveshaft of the pulsed mechanism, we use the dynamics characteristic of an asynchronous electric motor, taking the influence of electromagnetic transient processes into account

$$M_Д = M_Н - T\cdot\dot{M}_Д - \frac{1}{\nu\cdot\omega_X}\cdot(\dot{\alpha}-\omega_Н),$$

where $M_Н$ is the rated moment of motor rotor; $\omega_X, \omega_Н$ are the angular velocity in ideal idling and the rated angular velocity; T is the electromagnetic time constant of the motor; ν is the slope of the static characteristic.

For the sake of comparison, we solve (1) by several approximate analytical methods [16-21].

III. Method of Small Parameter

We rewrite (1) in the form

$$\begin{cases} B_1\ddot{\alpha} + B_2\ddot{\beta} + (b_1\ddot{\alpha} + b_2\ddot{\beta})\cos\psi + a_4(\dot{\alpha}-\dot{\beta})^2\sin\psi + \\ + a_6\dot{\beta}^2\cdot\sin\psi = M_Д, \\ B_2\ddot{\alpha} + B_3\ddot{\beta} + (b_2\ddot{\alpha} + b_3\ddot{\beta})\cos\psi + a_5(\dot{\alpha}-\dot{\beta})^2\sin\psi - \\ - a_6\dot{\alpha}^2\cdot\sin\psi = -M_С, \\ \dot{M}_Д = \frac{1}{T}\left(M_Н - M_Д - \frac{1}{\nu\cdot\omega_X}\cdot(\dot{\alpha}-\omega_Н)\right), \end{cases} \tag{2}$$

where

$$B_1 = J_1 + nma^2 + \left(\frac{a}{k}+q\right)^2 nJ_3;\ \ b_1 = 2nmah\left(\frac{a}{k}+q\right);$$
$$B_2 = nmab + \left(\frac{a}{k}+q\right)\left(\frac{b}{k}-q\right)nJ_3;\ \ b_2 = nmh\left(\frac{2ab}{k}+(b-a)q\right);$$
$$B_3 = J_2 + nmb^2 + \left(\frac{b}{k}+q\right)^2 nJ_3;\ \ b_3 = 2nmbh\left(\frac{b}{k}-q\right),$$
$$a_4 = -nmah\left(\frac{a}{k}+q\right)q,\quad a_5 = nmbh\left(\frac{b}{k}-q\right)q,\quad a_6 = nmkhq.$$

The coefficients B_1, B_2, B_3 contain the moments of inertia J_1, J_2, J_3 of the elements of the inertial-pulsed transmission and are considerably larger than the other coefficients. This permits the introduction of the small parameter μ in (2). The system then takes the form

$$\begin{cases} B_1\ddot{\alpha} + B_2\ddot{\beta} + \mu\cdot(b_1\ddot{\alpha} + b_2\ddot{\beta})\cos\psi + \mu\cdot a_4(\dot{\alpha}-\dot{\beta})^2\sin\psi + \\ + \mu\cdot a_6\dot{\beta}^2\cdot\sin\psi = \mu\cdot M_Д, \\ B_2\ddot{\alpha} + B_3\ddot{\beta} + \mu\cdot(b_2\ddot{\alpha} + b_3\ddot{\beta})\cos\psi + \mu\cdot a_5(\dot{\alpha}-\dot{\beta})^2\sin\psi - \\ - \mu\cdot a_6\dot{\alpha}^2\cdot\sin\psi = -\mu\cdot M_С, \\ \dot{M}_Д = \frac{1}{T}\left(M_Н - M_Д - \frac{1}{\nu\cdot\omega_X}\cdot(\dot{\alpha}-\omega_Н)\right). \end{cases} \tag{3}$$

On the basis of the fundamental principle of the method of small parameter, we look for the solution in serial form

$$\begin{cases} \alpha = \alpha_0 + \mu\alpha_1 + \mu^2\alpha_2 + \ldots, \\ \beta = \beta_0 + \mu\beta_1 + \mu^2\beta_2 + \ldots, \\ M_Д = M_{Д0} + \mu M_{Д1} + \mu^2 M_{Д2} + \ldots. \end{cases} \tag{4}$$

Assuming that $\mu = 0$ in (3) and (4), we obtain the generating system

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$$\begin{cases} B_1 \ddot{\alpha}_0 + B_2 \ddot{\beta}_0 = 0, \\ B_2 \ddot{\alpha}_0 + B_3 \ddot{\beta}_0 = 0, \\ \dot{M}_{Д0} + \frac{1}{T} M_{Д0} = \frac{1}{T}\left(M_H - \frac{1}{\nu\omega_X}(\dot{\alpha} - \omega_H)\right). \end{cases}$$

With the initial conditions $t = 0, \alpha_0 = \alpha_{01}, \beta_0 = \beta_{01}, \dot{\alpha}_0 = \dot{\alpha}_{01}, \dot{\beta}_0 = \dot{\beta}_{01}, M_{Д0} = M_{Д01}$ the solution of the generating system is

$$\begin{cases} \alpha_0 = \dot{\alpha}_{01} t + \alpha_{01}, \\ \beta_0 = \dot{\beta}_{01} t + \beta_{01}, \\ M_{Д0} = M_1 \exp\left(\frac{t}{T}\right) + M_2, \end{cases}$$

where

$$M_1 = M_{Д0} - M_H + \frac{1}{\nu\omega_X}(\dot{\alpha}_{01} - \omega_H);$$

$$M_2 = M_H - \frac{1}{\nu\omega_X}(\dot{\alpha}_{01} - \omega_H).$$

The functions $\sin\psi$ and $\cos\psi$ are expanded in the vicinity of the generating solution $\alpha_0(t), \beta_0(t)$

$$\sin\psi = \sin q(\alpha_0 - \beta_0) + \cos q(\alpha_0 - \beta_0)\cdot(\mu q(\alpha_1 - \beta_1) + \\ + \mu^2 q(\alpha_2 - \beta_2) + \ldots)),$$
$$\cos\psi = \cos q(\alpha_0 - \beta_0) + \sin q(\alpha_0 - \beta_0)\cdot(\mu q(\alpha_1 - \beta_1) + \\ + \mu^2 q(\alpha_2 - \beta_2) + \ldots)).$$

Retaining only term where μ is of first order and taking account of the series expansion of the trigonometric functions, we obtain a system of equations for $\alpha_1(t), \beta_1(t), M_{Д1}(t)$

$$\begin{cases} B_1 \ddot{\alpha}_1 + B_2 \ddot{\beta}_1 + (a_4(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 + a_6 \dot{\beta}_{01}^2)\sin q(\alpha_0 - \beta_0) = \\ = M_{Д0}, \\ B_2 \ddot{\alpha}_1 + B_3 \ddot{\beta}_1 + (a_5(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - a_6 \dot{\alpha}_{01}^2)\sin q(\alpha_0 - \beta_0) = \\ = -M, \\ \dot{M}_{Д1} + \frac{1}{T} M_{Д1} = -\frac{\dot{\alpha}_1}{T\nu\omega_X}. \end{cases} \quad (5)$$

Solving (5) with null initial conditions, we obtain expressions for $\alpha_1(t)$ and $\beta_1(t)$

$$\begin{cases} \alpha_1 = \frac{1}{\Delta}\left(\begin{array}{l} (B_3 M_2 + B_2 M)\frac{t^2}{2} + B_3 T^2 M_1 \exp\left(-\frac{t}{T}\right) - \\ -\frac{D_1}{q^2(\dot{\alpha}_{01} - \dot{\beta}_{01})^2}\sin\psi_0 \end{array}\right) + C_1 t + C_3, \\ \beta_1 = -\frac{1}{\Delta}\left(\begin{array}{l} (B_2 M_2 + B_1 M)\frac{t^2}{2} + B_2 T^2 M_1 \exp\left(-\frac{t}{T}\right) + \\ +\frac{D_2}{q^2(\dot{\alpha}_{01} - \dot{\beta}_{01})^2}\sin\psi_0 \end{array}\right) + C_2 t + C_4, \end{cases}$$

where

$$\Delta = B_1 B_3 - B_2^2,$$

$$D_1 = (B_2 a_5 - B_3 a_4)\cdot(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - (B_2 \dot{\alpha}_{01}^2 + B_3 \dot{\beta}_{01}^2)a_6,$$

$$D_2 = (B_2 a_4 - B_1 a_5)\cdot(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 + (B_1 \dot{\alpha}_{01}^2 - B_3 \dot{\beta}_{01}^2)a_6,$$

$$C_1 = \frac{1}{\Delta}\left(B_3 T M_1 + \frac{D_1 \cos q(\alpha_{01} - \beta_{01})}{q(\dot{\alpha}_{01} - \dot{\beta}_{01})}\right),$$

$$C_2 = \frac{1}{\Delta}\left(-B_2 T M_1 + \frac{D_2 \cos q(\alpha_{01} - \beta_{01})}{q(\dot{\alpha}_{01} - \dot{\beta}_{01})}\right),$$

$$C_3 = \frac{1}{\Delta}\left(-B_3 T^2 M_1 + \frac{D_1 \sin q(\alpha_{01} - \beta_{01})}{q^2(\dot{\alpha}_{01} - \dot{\beta}_{01})^2}\right),$$

$$C_4 = \frac{1}{\Delta}\left(B_2 T^2 M_1 + \frac{D_2 \sin q(\alpha_{01} - \beta_{01})}{q^2(\dot{\alpha}_{01} - \dot{\beta}_{01})^2}\right),$$

$$\psi_0 = q(\alpha_0 - \beta_0) = q((\dot{\alpha}_{01} - \dot{\beta}_{01})t + \alpha_{01} - \beta_{01}).$$

The solution for $M_{Д1}(t)$ is

$$M_{Д1} = P_1 + P_2 t + (C_5 + P_3 t)\exp\left(-\frac{t}{T}\right) + P_4 \sin\psi_0 + P_5 \cos\psi_0,$$

where

$$P_1 = \left(C_1 + \frac{B_3 M_2 + B_2 M}{\nu\omega_X}\right)T, \; P_2 = -\frac{B_3 M_2 + B_2 M}{\nu\omega_X}, \; P_3 = \frac{B_3 M_1}{\Delta\nu\omega_X},$$

$$P_4 = \frac{D_1 T}{(1 + T^2 q^2 (\dot{\alpha}_{01} - \dot{\beta}_{01})^2)\Delta\nu\omega_X},$$

$$P_5 = \frac{D_1}{(1 + T^2 q^2 (\dot{\alpha}_{01} - \dot{\beta}_{01})^2)\Delta\nu\omega_X q(\dot{\alpha}_{01} - \dot{\beta}_{01})},$$

$$C_5 = -P_1 - P_4 \sin q(\alpha_{01} - \beta_{01}) - P_5 \cos q(\alpha_{01} - \beta_{01}).$$

Retaining only the first two terms in (4) and assuming that $\mu = 1$, we write the final solution of (1) by the method of small parameter

$$\begin{cases} \alpha = \alpha_{01} + \dot{\alpha}_{01} + \alpha_1, \\ \beta = \beta_{01} + \dot{\beta}_{01} + \beta_1, \\ M_Д = M_1 \exp\left(-\frac{t}{T}\right) + M_2 + M_{Д1}. \end{cases}$$

IV. Successive-Approximation Method

The first approximation is obtained on the basis of the initial conditions

$$\begin{cases} \alpha_1 = \alpha_{01} + \dot{\alpha}_{01} t, \\ \beta_1 = \beta_{01} + \dot{\beta}_{01} t, \\ M_{Д1} = M_{01}. \end{cases}$$

In particular, taking account of the first approximation and the conditions $\ddot{\alpha}_1 = \ddot{\beta}_1 = 0$, we obtain the system of differential equations for the second approximation

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$$\begin{cases} B_1\ddot{\alpha}_2 + B_2\ddot{\beta}_2 = M_{Д2} - (a_4(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 + a_6\dot{\beta}_{01}^2)\sin\psi_0, \\ B_2\ddot{\alpha}_2 + B_3\ddot{\beta}_2 = -M - (a_5(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - a_6\dot{\alpha}_{01}^2)\sin\psi_0, \\ \dot{M}_{Д2} = \dfrac{1}{T}\left(M_H - M_{Д1} - \dfrac{\dot{\alpha}_1 - \omega_H}{\nu\omega_X}\right). \end{cases} \quad (6)$$

Solving (6) with the specified initial conditions, we obtain the second approximation of the solution (1). For the motor torque, the second approximation takes the form $M_{Д2} = M_3 t + M_{Д01}$, where

$$M_3 = \frac{1}{T}\left(M_H - M_{Д01} - \frac{\dot{\alpha}_{01} - \omega_H}{\nu\omega_X}\right).$$

The second approximation for the angle of drive shaft rotation is

$$\alpha_2 = \frac{1}{\Delta}\left(K_1\frac{t^3}{3} + K_2\frac{t^2}{2} + \frac{K_3}{q(\dot{\alpha}_{01} - \dot{\beta})}\sin\psi_0\right) + C_6 t + C_7,$$

where

$$K_1 = \frac{1}{2}M_3 B_3,\ K_2 = M_{Д01}B_3 + MB_2,$$

$$K_3 = \frac{1}{q}\left(\frac{(B_2\dot{\alpha}_{01}^2 + B_3\dot{\beta}_{01}^2)a_6}{\dot{\alpha}_{01} - \dot{\beta}_{01}} - (B_2 a_5 - B_3 a_4)(\dot{\alpha}_{01} - \dot{\beta}_{01})\right),$$

$$C_6 = \dot{\alpha}_{01} - \frac{K_3}{\Delta}\cos q(\alpha_{01} - \beta_{01}),$$

$$C_7 = \alpha_{01} - \frac{K_3}{\Delta q(\dot{\alpha}_{01} - \dot{\beta}_{01})}\sin q(\alpha_{01} - \beta_{01}).$$

Confining our attention to the second approximation for α and β, we find the third approximation for the motor torque

$$M_{Д3} = \frac{1}{T}\left(\begin{aligned}&\left(M_H + \frac{\omega_H}{\nu\omega_X} - M_{Д01}\right)t - M_3\frac{t^2}{2} - \\ &-\frac{1}{\Delta\nu\omega_X}\left(K_1\frac{t^3}{3} + K_2\frac{t^2}{2} + \frac{K_3\sin\psi_0}{q(\dot{\alpha}_{01} - \dot{\beta}_{01})} - \frac{C_6 t}{\nu\omega_X}\right)\end{aligned}\right) + C_{10},$$

where

$$C_{10} = M_{Д01} + \frac{K_3\sin q(\alpha_{01} - \beta_{01})}{T\Delta\nu\omega_X q(\dot{\alpha}_{01} - \dot{\beta}_{01})}.$$

V. Expansion in Power Series

By this method, the solution of the equation (1) is sought in the form

$$\begin{cases} \alpha = \alpha(0) + \dfrac{\dot{\alpha}(0)}{1!}t + \dfrac{\ddot{\alpha}(0)}{2!}t^2 + \dfrac{\dddot{\alpha}(0)}{3!}t^3 + \ldots, \\ \beta = \beta(0) + \dfrac{\dot{\beta}(0)}{1!}t + \dfrac{\ddot{\beta}(0)}{2!}t^2 + \dfrac{\dddot{\beta}(0)}{3!}t^3 + \ldots, \\ M_Д = M_Д(0) + \dfrac{\dot{M}_Д(0)}{1!}t + \dfrac{\ddot{M}_Д(0)}{2!}t^2 + \dfrac{\dddot{M}_Д(0)}{3!}t^3 + \ldots. \end{cases} \quad (7)$$

We find $\alpha(0), \beta(0), M_Д(0), \dot{\alpha}(0), \dot{\beta}(0)$ from the initial conditions

$$\alpha(0) = \alpha_{01},\ \beta(0) = \beta_{01},\ M_Д(0) = M_{Д01},\ \dot{\alpha}(0) = \dot{\alpha}_{01},\ \dot{\beta}(0) = \dot{\beta}_{01}.$$

To determine the second derivatives of α, and β, and the first derivative of the motor torque when $t = 0$, we solve (1) for higher derivatives

$$\begin{cases} \ddot{\alpha}(0) = \dfrac{M_{Д01}A_{30} + MA_{20} + (A_{20}A_{50} - A_{30}A_{40})(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - -(A_{20}\dot{\alpha}_{01}^2 - A_{30}\dot{\beta}_{01}^2)A_{60}}{A_{10}A_{30} - A_{20}^2}, \\ \ddot{\beta}(0) = \dfrac{-M_{Д01}A_{20} - MA_{10} + (A_{20}A_{40} - A_{10}A_{50})(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 + +(A_{10}\dot{\alpha}_{01}^2 + A_{20}\dot{\beta}_{01}^2)A_{60}}{A_{10}A_{30} - A_{20}^2}, \\ \dot{M}_Д(0) = \dfrac{1}{T}\left(M_H - M_{Д01} - \dfrac{\dot{\alpha}_{01} - \omega_H}{\nu\omega_X}\right), \end{cases}$$

where

$$\begin{aligned} A_{10} &= B_1 + b_1\cos q(\alpha_{01} - \beta_{01}), \\ A_{20} &= B_2 + b_2\cos q(\alpha_{01} - \beta_{01}), \\ A_{30} &= B_3 + b_3\cos q(\alpha_{01} - \beta_{01}), \\ A_{40} &= a_4\sin q(\alpha_{01} - \beta_{01}), \\ A_{50} &= a_5\sin q(\alpha_{01} - \beta_{01}), \\ A_{60} &= a_6\sin q(\alpha_{01} - \beta_{01}). \end{aligned}$$

By differentiating Eq. (1), we find the third derivatives, when $t = 0$

$$\begin{cases} \dddot{\alpha}(0) = \dfrac{A_{30}S_1 - A_{20}S_2}{A_{10}A_{30} - A_{20}^2}, \\ \dddot{\beta}(0) = \dfrac{A_{10}S_2 - A_{20}S_1}{A_{10}A_{30} - A_{20}^2}, \\ \ddot{M}_Д(0) = -\dfrac{1}{T}\left(\dot{M}_Д(0) + \dfrac{\ddot{\alpha}(0)}{\nu\cdot\omega_X}\right), \end{cases}$$

where

$$\begin{aligned} S_1 = {} & \dot{M}_Д(0) - \dot{A}_1(0)\ddot{\alpha}(0) - \dot{A}_2(0)\ddot{\beta}(0) - \dot{A}_4(0)(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - \\ & - 2A_{40}(0)(\dot{\alpha}_{01} - \dot{\beta}_{01})(\ddot{\alpha}(0) - \ddot{\beta}(0)) - \dot{A}_6(0)\dot{\beta}_{01}^2 - 2A_{60}\dot{\beta}_{01}\ddot{\beta}(0), \\ S_2 = {} & -\dot{A}_2(0)\ddot{\alpha}(0) - \dot{A}_3(0)\ddot{\beta}(0) - \dot{A}_5(0)(\dot{\alpha}_{01} - \dot{\beta}_{01})^2 - \\ & - 2A_{50}(0)(\dot{\alpha}_{01} - \dot{\beta}_{01})(\ddot{\alpha}(0) - \ddot{\beta}(0)) + \dot{A}_6(0)\dot{\alpha}_{01}^2 + 2A_{60}\dot{\alpha}_{01}\ddot{\alpha}(0). \end{aligned}$$

Differentiating the equation for the motor torque again, we obtain the third derivative of this torque when $t = 0$

$$\dddot{M}_Д(0) = -\frac{1}{T}\left(\ddot{M}_Д(0) + \frac{\dddot{\alpha}(0)}{\nu\omega_X}\right).$$

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Substituting the derivatives in (7), we obtain the final solution of (1).

VI. Comparison of the Methods

The analytical solutions obtained are approximate. Therefore, we must consider the accuracy of the approximation. The use of analytical estimates to this end is extremely difficult, since the theory of such estimates is commonly based on majorant series and as a result overestimates the errors (often considerably). In practice, therefore, other options are used to assess the accuracy of the method: for example, the comparison of successive approximations. The comparison of the zero, first, and second approximations will generally give a good idea of the quality of the method. The error of this method may also be estimated by comparing the solution with that obtained by another method or with experimental results. Such comparisons do not completely determine the accuracy of a particular method, but permit sufficient confidence regarding the results in practice.

We now compare the results obtained by the analytical methods and by numerical solution of the nonlinear differential equations. Numerical methods permit the determination of practically accurate solutions, since the error is specified initially and may be as small as is desired. The iterative process is continued until the specified accuracy has been obtained. For purposes of comparison, we adopt the well known fourth order Runge–Kutta method.

In Fig. 2, we plot the solutions of (1) obtained by approximate analytical methods and by the Runge–Kutta method using MathCAD Professional software, with the following parameters of the inertial transmission:
The numerical solution of the system (5) was received by using the Runge-Kutta method in MathCAD software. The transmission parameters were taken as follows:

$$J_1 = 4{,}1\,kg\cdot m^2,\ J_2 = 0{,}53\,kg\cdot m^2,\ nJ_3 = 0{,}052\,kg\cdot m^2,\ nm = 10{,}5\,kg,$$

$$a = 0{,}06\,m,\ b = 0{,}03\,m,\ \mathrm{k} = 0{,}09\ \mathrm{m},\ h = 0{,}05\,m,\ q = \frac{4}{3},\ M_C = 1\,N\cdot m,$$

$$M_H = 10{,}2\ N\cdot m,\ \omega_X = 157\ \frac{\mathrm{rad}}{\mathrm{s}},\ \omega_H = 150\ \frac{\mathrm{rad}}{\mathrm{s}},\ T = 0{,}038\ \mathrm{s},$$

$$\nu = 0{,}0024\,\frac{s^2}{\mathrm{kg}\cdot m^2}.$$

The initial conditions are as follows:

$$t = 0,\ \alpha_{01} = 1{,}2\ rad,\ \beta_{01} = 0,\ \dot{\alpha}_{01} = 150\,\frac{\mathrm{rad}}{\mathrm{s}},\ \dot{\beta}_{01} = 0,\ M_{Д01} = 5\ N\cdot m.$$

The thick continuous curves in Fig. 2 correspond to the Runge-Kutta solution; the thin continuous curves correspond to power series expansion; the dotted curves to the small parameter method; and the dashed curves to successive approximation. In Figs. 2a and 2d, we plot the angle α of driveshaft rotation, after subtraction of the trend component, as a function of *t*. The trend component must be eliminated or otherwise the plots of α for different methods will practically coalesce, since the drive shaft rotates at high speed, with little nonuniformity. In Figs. 2b and 1e, we plot the angle β of driven shaft rotation. In Figs.

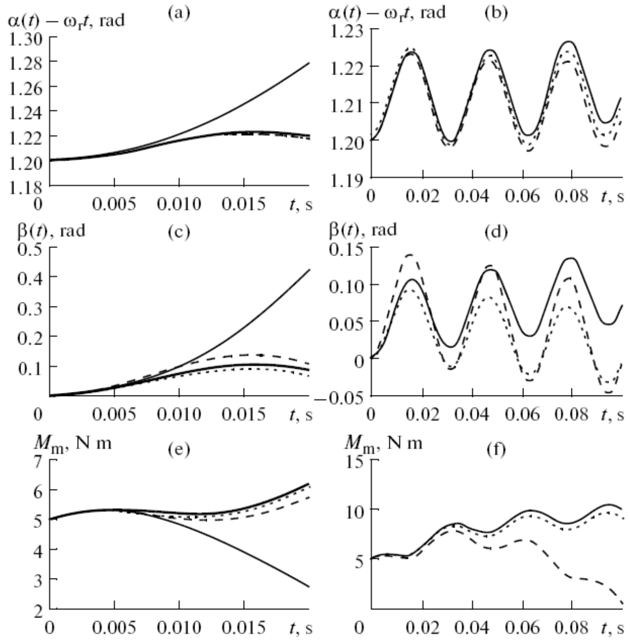

Fig. 2. Dependence of the angles of drive shaft rotation (a, d), and driven shat rotation (b, e), and motor torque (c, f) on the time t

2c and 2f, we plot the motor torque as a function of the time *t*.

The thick continuous curves in Fig. 2 correspond to the Runge-Kutta solution; the thin continuous curves correspond to power series expansion; the dotted curves to the small parameter method; and the dashed curves to successive approximation. In Figs. 2a and 2d, we plot the angle α of driveshaft rotation, after subtraction of the trend component, as a function of *t*. The trend component must be eliminated or otherwise the plots of α for different methods will practically coalesce, since the drive shaft rotates at high speed, with little nonuniformity. In Figs. 2b and 1e, we plot the angle β of driven shaft rotation. In Figs. 2c and 2f, we plot the motor torque as a function of the time *t*.

Analysis shows that power series expansion is of acceptable accuracy only within a short initial interval (Figs. 1a–1c). Given that this method is no simpler than other analytical methods in terms of the structure of the coefficients in solving Eq. (2), we regard this method as the least acceptable for the investigation of mathematical models of inertial transmissions. To improve its accuracy, we could determine additional terms in the expansion, but this entails intolerable complexity.

The solutions obtained by the small parameter method and by successive approximation are similar and are in good agreement with the accurate solution. As follows from Figs. 1a–1c, the small parameter method is somewhat preferable. It is difficult to identify clear differences between these methods in Figs. 1a–1c, on account of the interfering effect of the curves corresponding to power series expansion.

Accordingly, in Figs. 1d–f, we retain only the curves corresponding to the Runge–Kutta method, the small parameter method, and successive approximation. Having excluded the results obtained by power series expansion, we may consider a much larger time interval. It follows from Fig. 1f that, for the motor torque, the results given by

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the small parameter method are much closer to the accurate solution. For the angle α of driveshaft rotation, the difference is not so great, but again favors the small parameter method. For the angle β of driven shaft rotation, the results given by successive approximation are more accurate in some time intervals, but the opposite is true in other intervals. In this situation, it is difficult to clearly establish which method is best, although for a brief initial period the small parameter method is indubitably preferable.

VII. Conclusions

Overall, we may conclude that the small parameter method is best for analytical solution and investigation of the nonlinear equations of motion of inertial pulsed transmissions based on a pulsed mechanism with two degrees of freedom. Successive approximation gives fair results and even outstrips the small parameter method in some circumstances, but overall the small parameter method is unquestionably superior. To obtain the accuracy provided by the other methods, power series expansion proves much more laborious and unwieldy. Thus, in the future, there is no need to use different methods to solve the differential equations of motion of inertial transmissions. Attention may be confined to the small parameter method.

References

[1] Leonov, A.I., "Inertial Automatic Torque Transformers," Moscow: Mashinostroenie, 1978.
[2] Alyukov, S.V. "Improved Models of Inertial Torque Transformers," Russian Engineering Research, 2010, vol. 30, no.6, pp. 535-542.
[3] Alyukov, S. and Gladyshev, S., "Dynamics of an Inertial Continuously Variable Transmission with High Load Ability," SAE Technical Paper 2013-01-2442, 2013, doi:10.4271/2013-01-2442.
[4] Aliukov, S. and Gorshenin, V., "On the Question of External Characteristic of the Inertial Continuously Variable Transmission," SAE Technical Paper 2014-01-1733, 2014, doi:10.4271/2014-01-1733.
[5] Alyukov, S., "Characteristic of an improved continuous inertial transmission," Russian Engineering Research, 2014, vol. 34, no. 7, pp. 436-439.
[6] Poletskii, A.T. and Vasin, G.G., "Integrating the Equations of an Inertial Torque Transformer," Dynamics of Machine, Moscow: Mashinostroenie, 1969, pp. 64─69.
[7] Liu, K., Zhang, H., and Bamba, E., "Dynamic Analysis of an Free-wheel mechanism for the Pulse-Continuously-Variable-Speed Transmission," SAE Technical Paper 980827, 1998, doi:10.4271/980827.
[8] Aliukov, S., "Dynamics of inertial continuously variable transmissions." INFRA-M, Moscow, 2013, (in Russian).
[9] Kremer, J. and Altidis, P., "Roller One-Way Clutch System Resonance," SAE Technical Paper 981093, 1998, doi:10.4271/981093.
[10] Aliukov, S., Keller, A., and Alyukov, A., "Design and Calculating of Relay-Type Overrunning Clutch," SAE Technical Paper 2016-01-1134, 2016, doi:10.4271/2016-01-1134.
[11] Maltsev, V., "Free-Wheel Mechanisms of Roller Type," Moscow: Mashinostroenie, 1968.
[12] Aliukov, S., "Free-Wheel Mechanisms of Relay Type," Tyazholoye Mashinostroeniye, vol. 12, 2010.
[13] Aliukov, S., Keller, A., and Alyukov, A., "Dynamics of Overrunning Clutches of Relay Type," SAE Technical Paper 2015-01-1130, 2015, doi:10.4271/2015-01-1130.
[14] Aliukov, S., Keller, A., and Alyukov, A., "Method of Calculating of Relay Type Free-Wheel Mechanism," SAE Technical Paper 2015-01-2782, 2015, doi:10.4271/2015-01-2782.
[15] Alyukov S., "Relay-type free-wheel mechanism," Russian Engineering Research, 2014, vol. 34, # 9, pp. 549-553.
[16] Keller, A. and Aliukov, S., "Rational Criteria for Power Distribution in All-wheel-drive Trucks," SAE Technical Paper 2015-01-2786, 2015, doi:10.4271/2015-01-2786.
[17] Keller, A. and Aliukov, S., "Analysis of possible ways of power distribution in an all-wheel drive vehicle," Lecture Notes in Engineering and Computer Science, vol. 2218, 2015, pp. 1154-1158.
[18] Keller, A. and Aliukov, S., "Methodology of System Analysis of Power Distribution among Drive Wheels of an All-wheel drive Truck," SAE Technical Paper 2015-01-2788, 2015, doi:10.4271/2015-01-2788.
[19] Alyukov, S.V., "Improved models of inertial torque transformers," Russian Engineering Research, vol. 30, Issue 6, 2010, pp. 535-542.
[20] Velikanov, N.L., Koryagin, S.I., Sharkov, O.V., "Definition of Locked-up Stresses around a Rectilinear Welding Seam," IOP Conference Series: Materials Science and Engineering, vol. 124, Issue 1, 2016.
[21] Sharkov, O.V., Kalinin, A.V., "Kinematic Characteristics of Pulse Speed Regulators," Russian Engineering Research, vol. 29, Issue 6, 2009, pp. 551-554.

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