# Semi-analytical Solution for Postbuckling Behavior of Highly Deformable Nanobeams

M.A. Maneshi, E.Ghavanloo, S.A.Fazelzadeh, M.I. Friswell, S. Adhikari

**In this paper, exact analytical solutions are developed to describe the size-dependent nonlinear bending behavior of cantilever nano-beams subjected to an end force. Geometric and equilibrium equations of the deformed element are used in conjunction with a nonlocal differential constitutive relation to obtain large deformation of the nano-beam. Here, the nano-beam is considered to be inextensible and the Euler-Bernoulli hypotheses are adopted. Applicability and accuracy of the present formulations are confirmed by comparing the predicted results with those reported in the literature. Furthermore, by using the exact solution presented in this investigation, the deformed configurations of the nano-beams are determined for different loading conditions. Our results reveal that the nano-beam exhibits a softening behavior when nonlocality is increasing.**

***Index Terms*— Postbuckling, Nonlocal elasticity, One-dimensional nanoscopic structures.**

## I. Introduction

Postbuckling behavior of elastic beams is one of the basic problems in different engineering fields. Therefore, it is of substantial practical interest and has been widely studied by many researchers. Nowadays, one-dimensional nanoscopic structures including nanowires, nanorods, nanotubes, nanofibers and nanoribbons, have paved a new way for various advances in future applications [1]. Experimental observations have shown that one-dimensional nanoscopic structures may undergo postbuckling [2]. The understanding of the postbuckling behaviour of these nanoscopic structures is crucial for the design of new nanodevices. Hence, in the last few years, the postbuckling analysis of one-dimensional nanoscopic structures has attracted extensive attention in the nanomechanics community [3-6].

To the authors' knowledge, no closed-form solutions for the postbuckling configurations of nano-beams have been presented so far. It is well-known that the exact solution can serve as reference results for verifying numerical solutions, and so a closed-form expression is highly desirable. In this paper, an attempt is made to propose an explicit closed-form solution for the postbuckling behaviour of a cantilever nano-beam subjected to compressive load at its free end. In this connection, Eringen's nonlocal elasticity theory [7] is used to incorporate the small-scale effect. This theory has been successfully used to solve problems involving the mechanics of nanoscopic structures [8]. The nonlinear governing equations of the problem are presented. Then, the equilibrium shapes of the nano-beam for different conditions are calculated by solving the nonlinear governing equations. The accuracy of the model is examined by the comparison between the present results and those reported in the literature.

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## II. Formulation

Consider an inextensible nano-beam of length $L$ and flexural rigidity $EI$ subjected to a tip axial force $P$ (Fig. 1). As shown in Fig. 1, a Cartesian coordinate $X$-$Y$ is chosen and the tangential angle between the nano-beam axis and the $X$ direction is $\psi$. Using trigonometrical relations applied to a differential element $dS$ (Fig. 2), the following geometrical relations are obtained:

$$\frac{dX}{dS} = \cos(\psi) , \tag{1}$$

$$\frac{dY}{dS} = \sin(\psi) . \tag{2}$$

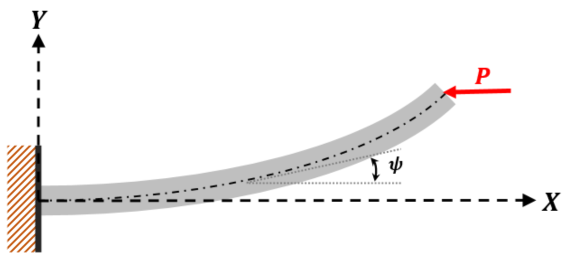


Fig. 1. Cantilever nano-beam subjected to an axial force P

ISBN: 978-988-14048-9-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

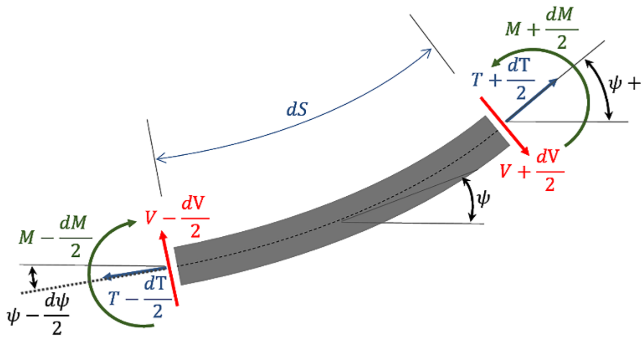


Fig. 2. Free body diagram for a differential element

Furthermore, the static equilibrium equations are derived on the basis of the equilibrium of the deformed element $dS$, i.e.,

$$\frac{dT}{d\psi} = -V \quad , \tag{3}$$

$$\frac{dV}{d\psi} = T \quad , \tag{4}$$

$$\frac{dM}{dS} = V \quad , \tag{5}$$

where $T$, $V$, and $M$ are the normal force, the shear force, and the bending moment. On the other hand, the bending moment-curvature relation for the nonlocal elastic material using Eringen's constitutive relation is given as [9]

$$M - \mu^2 \frac{d^2 M}{dS^2} = EI \frac{d\psi}{dS} \, , \tag{6}$$

where $\mu$ is the nonlocal scale coefficient which incorporates the small scale effect in the constitutive relation. To simplify Eqs. (1)-(6), the following dimensionless parameters are introduced:

$$\begin{gathered} x = \frac{X}{L}, \qquad y = \frac{Y}{L}, \qquad s = \frac{S}{L}, \qquad p = \frac{PL^2}{EI} \\ m = \frac{ML}{EI}, \qquad v = \frac{VL^2}{EI}, \qquad t = \frac{TL^2}{EI}, \qquad e = \frac{\mu}{L} \end{gathered} \tag{7}$$

Substituting Eq. (7) into Eqs. (1)-(6), the nonlinear governing equations are derived as,

$$\frac{dt}{d\psi} = -v \, , \tag{8}$$

$$\frac{dv}{d\psi} = t \, , \tag{9}$$

$$\frac{dm}{ds} = v \, , \tag{10}$$

$$m - e^2 \frac{d^2 m}{ds^2} = \frac{d\psi}{ds} \, , \tag{11}$$

$$\frac{dx}{ds} = \cos(\psi) \, , \tag{12}$$

$$\frac{dy}{ds} = \sin(\psi) \, , \tag{13}$$

Furthermore, the boundary and load conditions are

$$\begin{gathered} x(s=0) = 0 \, , \qquad y(s=0) = 0 \, , \\ m(s=1) = 0 \, , \qquad t(s=1) = -p\cos(\psi_B) \, , \\ \psi(s=0) = 0 \, , \qquad v(s=1) = -p\sin(\psi_B) \, . \\ \psi(s=1) = \psi_B \, , \end{gathered} \tag{14}$$

Using Eqs. (8) and (9), the dimensionless normal and shear forces are found as functions of $\psi$:

$$t = a\cos(\psi) + b\sin(\psi) \tag{15}$$

$$v = a\sin(\psi) - b\cos(\psi) \tag{16}$$

where $a$ and $b$ are constants determined by the loading conditions at the tip. Using Eqs. (15) and (16) at $s$ = 1 and comparing the resultant relations with the load conditions, we obtain

$$a = -p \, , \qquad b = 0 \, . \tag{17}$$

Equations (10)-(11) and (15) are combined and the curvature relation is obtained as follows:

$$m = \left[1 + e^2\left(-p\cos(\psi)\right)\right]\frac{d\psi}{ds} \, . \tag{18}$$

Multiplying each term of Eq. (18) by $dm/ds$, using Eqs. (10) and (16), integrating and after making some simplifications, we have

$$m^2 = -\frac{1}{e^2}(1 - e^2 p\cos(\psi))^2 + c \tag{19}$$

Since the bending moment at the tip is zero, the constant is calculated as follows:

$$c = \frac{1}{e^2}\left(1 + e^2\left(-p\cos(\psi_B)\right)\right)^2 \tag{20}$$

To find the closed-form expression for geometrically nonlinear large deformation of nano-beams, it is necessary to obtain $x$ and $y$ as functions of $\psi$. Using Eqs. (12), (13), (18), (19) and (20), it can be shown that

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ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$s(\psi)=\int_0^{\psi}\frac{(1-e^2-p\cos(\varphi))d\varphi}{\sqrt{c-\frac{1}{e^2}\left(1-e^2-p\cos(\varphi)\right)^2}}, \tag{21}$$

$$x(\psi)=\int_0^{\psi}\frac{(1-e^2 p\cos(\varphi))\cos(\varphi)d\varphi}{\sqrt{c-\frac{1}{e^2}\left(1-e^2 p\cos(\varphi)\right)^2}}, \tag{22}$$

$$y(\psi)=\int_0^{\psi}\frac{(1-e^2 p\cos(\varphi))\sin(\varphi)d\varphi}{\sqrt{c-\frac{1}{e^2}\left(1-e^2 p\cos(\varphi)\right)^2}}. \tag{23}$$

Equations (21) and (22) are closed-form expressions for geometrically nonlinear large deformation of nano-beams as a function of single unknown parameter $\psi_B$. According to the end condition $\psi(s=1)=\psi_B$, the unknown parameter can be found as follows:

$$\begin{cases} 1-\int_0^{\psi_B}\dfrac{d\phi}{\sqrt{2(p\cos(\phi)-p\cos(\psi_B)}}=0 & e=0, \\ 1-\displaystyle\int_0^{\psi_B}\dfrac{e(1+e^2(q\sin(\phi)-p\cos(\phi)))d\phi}{\sqrt{(1-e^2 p\cos(\psi_B))^2-(1-e^2 p\cos(\phi))^2}}=0 & e>0 \end{cases} \tag{24}$$

Although Eq. (24) is a single nonlinear equation, it is an improper integral of the second kind, and hence better solved by using a numerical method such as the false position method.

## III. Numerical Results

To demonstrate the accuracy of the present closed-form solution, the values of the tip-angle, $\psi_B$, of the nano-beam subjected to pure compressive load are obtained and compared with the results determined earlier by Wang et al [9]. The results are presented in Table 1 and show that, the results obtained from the present model are in good agreement with those reported by Wang et al. [9] for small nonlocal scale coefficients. However, with increasing the nonlocal scale coefficient, a small discrepancy is observed between the results from the proposed closed-form solution and the numerical results reported by Wang et al. [9] on the basis of a shooting method. This discrepancy is attributed to the difference in the nonlinear models and numerical methods. After verifying the accuracy and applicability of the present formulations, they are now applied to various nano-beams.

Table 1. Comparison of the predicted tip angle (degrees)

| $e$ | $p$ = 2.5054 | | $p$ = 2.8417 | |
|---|---|---|---|---|
| | Present study | Wang et al. [9] | Present study | Wang et al. [9] |
| 0.00 | 20.00 | 20.00 | 60.00 | 60.00 |
| 0.02 | 20.64 | 20.68 | 60.20 | 60.45 |
| 0.04 | 22.42 | 22.62 | 60.80 | 61.89 |
| 0.06 | 25.11 | 25.61 | 61.77 | 64.67 |
| 0.08 | 28.42 | 29.48 | 63.10 | - |
| 0.10 | 32.13 | 34.13 | 64.75 | - |

Figure 3 shows the postbuckling shapes of the nano-beam subjected to pure compressive load. In this figure, the dimensionless nonlocal scale coefficient, $e$, is fixed at 0.2, and the nano-beam shapes are shown for eight compressive forces $p$ = 2.25, 2.30, 2.50, 3.00, 5.00, 10.00, 15.00 and 24.99.

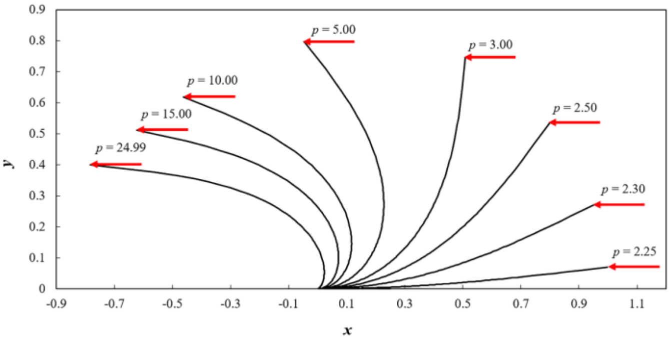


Fig. 3. Postbuckled shapes of the nano-beam with prescribed values of $p$.

To illustrate the effect of the nonlocal scale coefficient on the postbuckling behavior of the nano-beam, the postbuckling shapes of the nano-beam subjected to pure compressive load is displayed in Fig. 4 for $p$ = 2.47 and different values of the dimensionless nonlocal scale coefficient. From the numerical results, it is concluded that there is no continuous solution when $e$ > 0.6352.

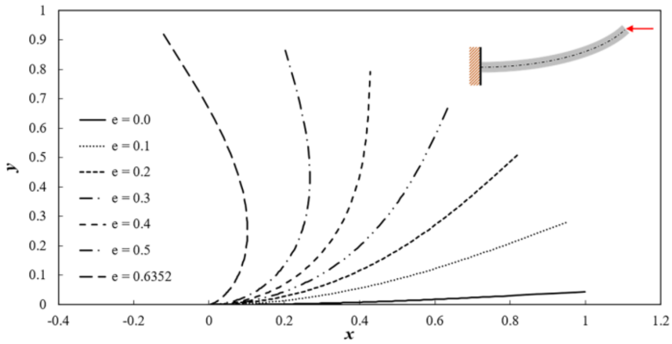


Fig. 4. The effect of the nonlocal scale coefficient on the postbuckled shapes of the nano-beam for $p$ = 2.47.

The postbuckling load-deflection curves of the nano-beams for different values of nonlocal scale coefficient are shown in Fig. 5. It can be seen that, as the postbuckling load increases, the maximum transverse displacement, $y_{end}$, goes up sharply. After each curve takes its peak point at a postbuckling applied load, the maximum deflection decreases by increasing of the load.

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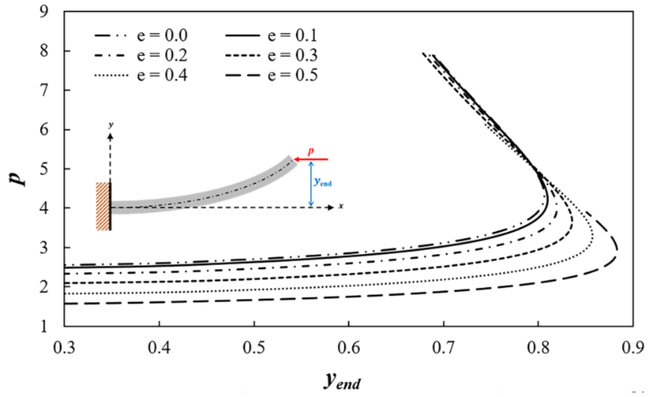


Fig. 5. Postbuckling load-deflection curves for different values of nonlocal scale coefficient.

## IV. Conclusion

An exact analysis of nonlinear large deformation of the cantilever nano-beams subjected to an axial end-tip load has been presented. In this investigation, nonlocal elasticity theory has been used for a nonlinear elastica problem. The present problem is a complex nonlinear boundary value problem including the small scale effect. Despite some achievement in analyzing large deflections of nano-beams, to the authors' knowledge, no closed-form solution for determining the large deformed shape of nano-beams has been presented so far. In this paper, the nonlinear governing differential equations were solved using elliptic integral approach. By using the proposed formulations, we obtained the deformed shapes of cantilever nano-beams under different loading conditions. Furthermore, the load-deflection curves have been plotted by using the closed-form solutions. In addition, the results were compared with existing numerical results in the literature.

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