Mechanical Modelling of AM Processes for Centrifugal Deposition of Viscoelastic Material Layers on a Cylindrical Mold

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***Abstract*—Additive technological processes of layer-by-layer deposition of material to the inner surface of an axisymmetric cylindrical mold rotating around its axis at an arbitrary time-varying angular velocity are investigated. The deposed material exhibits properties of linear creep and aging. The flexibility of the mold is not taken into account. A mechanical model of the studied processes based on the general approaches of the mathematical theory of accreted solids developed by the authors is proposed. The corresponding nonclassical boundary value problem for the velocity characteristics of the deformation process of the formed material layer under the action of centrifugal forces is stated. The closed analytical solution of the stated problem in quadratures is obtained. By means of it the evolution of the stress state of the layer under consideration in the process of its additive formation during any number of stages of the material continuous deposition with arbitrary pauses between these stages and after the formation completing is built. The found technological stresses in the having been formed layer, caused by the action of centrifugal forces in the process of its manufacturing, depend on the nature of the process in determining wise. The distributions of these stresses essentially differ from the classical stresses distributions in a similar rotating material layer that did not experience an impact of deformation factors during the manufacturing process. This difference is explained by the fundamental mechanical features of the accreting process itself and causes the inevitable occurrence of residual stresses in the having been formed layer after stopping its rotation and, if the simulated technological process implies it, the subsequent detachment of the completed layer from the mold. The distributions of these residual stresses can be found by means of the dependences constructed in the paper.**

***Index Terms*—accreted solid, additive manufacturing, centrifugal deposition, technological stresses, viscoelastic material.**

I. Introduction

MANY technological processes are accompanied by an increase in the size and, possibly, by a change in the shape of the solids involved due to attaching the additional material to them, that is, building-up, or growing these solids. It is obvious that when studying this kind of processes one should take into account the kinematic and force features of the new substance gradual inflow to the surface of the accreted solid under loads taking place in the technological process that act to this solid simultaneously. However, such accounting is impossible to implement correctly in principle in the framework of the classical mechanics of deformable solids, even if we consider the traditional equations and boundary conditions in a domain variable in time. This is due to the fact that the classical formulation of the mechanical problems always implies the existence in the entire solid in general such a configuration in which there are no stresses. Such a configuration is commonly called natural. It is this configuration that the objective measures of deformation of the considered body, compared with its stress state in one way or another by means of defining relations, are measured from.

Meanwhile, the accreted solid may not have a natural configuration in general case (unlike, for example, the solid exposed to the removal of the material). Indeed, while some of the material elements have already deformed with the solid, others have not yet been incorporated into it. As a consequence of this fundamental fact, the problems of mechanical behavior of accreted solids should possess a number of specific features and constitute a special class of problems of deformable solid mechanics. The development of common approaches to mathematical formulation of such nonclassical problems and developing methods of their solution construction and analysis, as well as studying various growth processes on the basis of these problems present works [1–25]. One can find there specific examples of solutions of such problems and discussing a variety of new mechanical effects discovered thanks to the obtained solutions. Note that among a great number of papers on AM technologies written by technologists, chemists, and physicists (see, e.g., [26–48]) a rare one is devoted to mechanical aspects and analyses of AM fabricated parts and similar problematics.

Mass forces often act as a mechanical loading in the studied processes. These include, in particular, the forces of inertia caused by the movement of the solid in space as a rigid whole. First of all, these are centrifugal forces that must be taken into account in the case of accreting the rotating bodies, in particular in the analysis of manufacturing processes or strengthening a certain type of products, as well as deposing coating layers on products. These are that very processes that the processes studied in this paper are referred to.

II. Statement of the Problem

This work is devoted to modeling the processes of gradual deposing the uniform in thickness layers of additional material on the inner surface of a axisymmetric cylindrical

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substrate rotating around its axis at the angular velocity $\Omega(t)$ arbitrarily changing in time t. It is assumed that when deposing the material the velocity of its inflow in the circumferential direction is incomparably higher than the velocity of its inflow in the radial direction. This assumption makes it possible to simulate the considered process of deposing the material as axisymmetric process of deposing the layer being formed simultaneously throughout its whole inner surface. In addition, this deposing process is considered as continuous, in which each infinitely small period of time an infinitely thin additional layer joins the accreting body. Thus, the time dependence of the inner radius of the generated layer $a(t)$ is a continuous function.

It is stated the task to trace the evolution of the stress-strain state of the layer being deposed under the influence of the inertia forces of its rotational motion together with the mold. A sufficiently slow time variation of the mold rotational speed is assumed, $\left|\Omega'(t)\right| \ll \Omega^2(t)$, so that the tangential inertia forces of rotation are negligible if comparing to centrifugal forces. It is supposed that the potentially possible dynamic effects from the attaching the additional material to the surface of the accreted layer are insignificant, and therefore the forces of inertia of its deformation can also be neglected in comparison with the centrifugal forces of inertia and the problem can be considered in a quasi-static statement.

The problem is solved in the approximation of the plane strain state under small strain condition. In view of the latter, it makes no sense to take into account the depending on the time strain component of the inner radius of the formed layer $a(t)$ decreasing due to the inflow of additional material. This dependence can be considered a prescribed program of attaching material, which is implemented in the simulated process. Thus, in the considered problem the dependence $a(t)$ is a given continuous function, strictly decreasing at those time intervals at which the material is deposed, and constant at those time intervals at which the inflow of additional material to the formed layer is temporarily or finally stopped.

It is obvious that if the additional material joins a certain solid that is already in the process of deformation under the exposure of certain impact, then the entire newly joined material is inevitably involved in the process of deformation. One of the objectives of the present study is a refined demonstration of the mechanical effects that arise due to building-up the solid under simultaneous acting on it centrifugal forces regardless of the influence of deformation processes, occurring in the part of the solid under consideration existing before start of accreting (initial). Therefore, in the proposed model the possible compliance of the mold on which the layer is being built is not taken into consideration (although its consideration does not represent a fundamental difficulty), and the mold is considered to be absolutely rigid. The inner radius of the mold let us denote by a_0.

Elements of the additional material to be attached to the solid in the process of accreting, can for some reasons mechanical, physical, chemical undergo pre-deforming when joining. This will cause some initial stresses in them. In this case, stress and strain fields will be formed in the accreted solid even in the absence of an external load. Taking into account the initial stresses in the elements of any accreted solid is an integral part of the formulation of boundary conditions on the surface of its growth. In this paper, these stresses are considered to be zero, that is, when setting the problem of the considered accreted layer deformation, it is considered that the processes and effects accompanying the continuous inclusion of the additional substance in its composition does not lead to the appearance of nonzero stresses in it near the growth surface. It is important to note that this assumption in the studied problem, where the deforming of the accreted solid occurs in the field of mass forces action, from the mathematical point of view, does not simplify the formulation and solution of the problem in comparison with the case of action of some nonzero initial stresses in the attached material. Consideration of zero initial stress in the proposed model only allows to focus on the effect of exlusively centrifugal forces on the development of stress-strain state of the formed layer and convincingly show the fundamental differences of this condition from the state of the layer of similar size and material properties, firstly formed entirely on the surface of a rigid mold without any residual stresses, and only then forced to rotate. The latter state can be obviously determined from the solution of the corresponding classical problem of mechanics, which does not take into account the process of the considered solid formation and involves the applying the load to the solid already in its final composition.

III. Description of the Used Material

If the elastic solid is formed by deposition, the rate of change of its stress-strain state is obviously determined only by instantaneous characteristics of the processes of its deposing and loading. After deposing is completed and the loading and kinematic constrains are fixed, the state of the solid no longer changes. This is not the case when the deformation response of the material to the mechanical loads applied to it depends on the duration of these loads acting and on the age of the material in which these loads were applied. Extended in time the processes of accreting solids with the use of such materials are quite difficult to simulate as in this case the process of changing the stress-strain state of accreted body is affected at any given time by the entire previous history of deforming its every material element, in particular those that were in the part of the solid existed prior to the increase. However, the study of this very kind of processes is relevant from the point of view of various engineering applications since many materials used in practice exhibit conspicuous rheological properties and their mechanical characteristics are often significantly changed with age regardless of acting loads.

In the model proposed in this paper, we consider a linear viscoelastic uniformly aging isotropic material with the same constant (independent of either the material age or the time elapsed since the application of loads to it) Poisson's ratio $\nu = \text{const}$ for instantaneous elastic strain and creep strain developing over time. For the given material the relation between the stress tensor $\mathbf{T}$ and the small strain tensor $\mathbf{E}$ at each point of the solid $\mathbf{r}$ at any time moment t, calculated from the time of manufacture of the material, has the form [4], [49]:

$$\mathcal{H}_{\tau_0(\mathbf{r})}\mathbf{T}(\mathbf{r},t) = 2\mathbf{E}(\mathbf{r},t) + (\varkappa - 1)\mathbf{1}\operatorname{tr}\mathbf{E}(\mathbf{r},t). \quad (1)$$

Here $\mathbf{1}$ is the unit tensor of the second rank, $\varkappa = 1/(1-2\nu)$, $\mathcal{H}_s$ is the linear viscoelasticity integral operator acting under the rule

$$\mathcal{H}_s f(t) = \frac{f(t)}{G(t)} - \int_s^t \frac{f(\tau)}{G(\tau)} K(t,\tau)\, d\tau, \qquad (2)$$

where $G(t)$ is the elastic shear modulus of the material at its age t, and $K(t,\tau)$ is the creep kernel. The latter can be expressed through various characteristics of the material, describing its behavior in this or that elementary stress state. For example, using the characteristics for the pure shear state it will be

$$K(t,\tau) = G(\tau)\,\frac{\partial\Delta(t,\tau)}{\partial\tau}, \quad \Delta(t,\tau) = \frac{1}{G(\tau)} + \omega(t,\tau),$$

where $\omega(t,\tau)$ is the creep measure for the pure shear, $\omega(\tau,\tau) \equiv 0$. The $\Delta(t,\tau)$ function describes the evolution over time t of specific (per unit of the acting shear stress) shear strain caused by the constant stress state of pure shear created at the time moment τ.

The parameter s of the operator (2) has the meaning of the time moment of occurrence of the stress state in the neighborhood of the considered point of the solid $\mathbf{r}$. Since the accreted solid is increased with new material elements already during the process of its deformation, the moment of occurrence of stresses at the points $\mathbf{r}$ of such a solid will change from point to point and be set by a certain function $\tau_0(\mathbf{r})$, which is taken into account in the constitutive relation (1).

In the additive process of centrifugal deposition of the material to a rigid cylindrical substrate modeled in this paper, we assume that the elementary layers of the additional material join the inner cylindrical surface of the formed solid in the initially non-stressed state (see Section II) and thus begin to deform only as a part of this solid. In this case, the value

$$\tau_0(\mathbf{r}) \equiv \tau_0(\rho) \qquad (3)$$

should be considered as the moment of joining to the growing solid of the material layer with the radius $\rho < a_0$.

It is clear that for all values of ρ that make sense for the solid in question, it is true

$$t = \tau_0(\rho) \implies \rho = a(t), \qquad (4)$$

moreover, at any time interval of continuous growth of the solid due to the strict monotony of the $a(t)$ function the inverse implication will be just as well. From (4) it follows the identity $a\big(\tau_0(\rho)\big) \equiv \rho$, when differentiating which, we get the notation

$$\tau_0'(\rho) = 1/\dot{a}(t), \quad t = \tau_0(\rho), \qquad (5)$$

which we will need in the future. Here and everywhere hereinafter, by stroke the derivative with respect to the coordinate ρ is denoted, and a dot on the top denotes the derivative of the function of one variable t.

Similar to (1) constitutive relations are widely used to describe the mechanical behavior of various natural and artificial stone (in particular, concrete), polymers, soil, ice, wood. Typical experimental curves representing the evolution with time t of the specific longitudinal strain

$$\frac{\varepsilon(t,\tau)}{\sigma_0} = \frac{[\mathcal{H}_\tau \sigma_0](t)}{2(1+\nu)} \Big/ \sigma_0 = \frac{\Delta(t,\tau)}{2(1+\nu)}$$

of such material at its uniaxial tension by constant tension σ_0 applied at the time moment τ can be borrowed, for example, from [50].

Note that the state equation (1) contains as a special case the state equation of the isotropic linearly elastic material. This case is obtained by taking $\omega(t,\tau) \equiv 0$, $G(t) = \mathrm{const}$.

For further convenience we will use the following shorthand notation of the result of the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ acting to an arbitrary function $g(\mathbf{r},t)$ of point of accreted solid $\mathbf{r}$ and time t:

$$g^\circ(\mathbf{r},t) = \mathcal{H}_{\tau_0(\mathbf{r})}\, g(\mathbf{r},t). \qquad (6)$$

We will call the tensor $\mathbf{T}^\circ$ standing on the left side of the constitutive relation (1) the operator stress tensor.

IV. Statement of the Boundary-Value Problem for the Process of Piecewise-Continuous Material Deposition

Let the continuous deposition of the material on the inner surface of the mold rotating around its axis begins at some time moment. This process may be interrupted at arbitrary times moments by arbitrary pauses during which the inflow of additional material to the material layer having already been formed on the mold is temporarily stopped. Such a piecewise continuous process of accreting the formed layer finally comes to the end after a certain number of stages of continuous deposition of additional material on it. After that the finally formed layer can still continue to rotate coupled with the mold rigidly bound with it.

Let us associate with the rotating around its axis axisymmetric cylindrical rigid mold a polar cylindrical coordinate system (ρ,φ,z) with the right orthonormal reference $\{\mathbf{e}_\rho,\mathbf{e}_\varphi,\mathbf{e}_z\}$, here: $\mathbf{e}_z = \mathrm{const}$ is the direction along the axis of rotation, z is the longitudinal coordinate measured in this direction; $\mathbf{e}_\rho = \mathbf{e}_\rho(\varphi)$ and $\mathbf{e}_\varphi = \mathbf{e}_\varphi(\varphi)$ are the radial and circumferential directions in the cross-section of a rotating cylindrical solid formed on the mold, respectively, ρ and φ are the polar radius and the polar angle in the cross-section. In this movable noninertial base for the considered accreted solid the standard equilibrium equation is valid

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \mathbf{0}, \qquad (7)$$

in which the centrifugal forces of inertia

$$\mathbf{f}(\mathbf{r},t) = \mathbf{e}_\rho(\varphi)\,\rho\, c(t), \quad c(t) = \mu\,\Omega^2(t), \qquad (8)$$

play a part of body forces, where μ is the mass density of the material used.

As stated in Section I, for an accreted solid it is not possible to introduce deformation measures, traditional for continuous mechanics, due to the absence in such a solid in whole a stress-free configuration that should be taken as a non-deformed one. Meanwhile, it is obvious that in the whole considered accreted solid having been formed by the time moment t the sufficiently smooth velocity vector field $\mathbf{v}(\mathbf{r},t)$ of the deformation motion of its particles is determined. In the process of deposition simulated in this paper, we are talking about a cylindrical solid $a(t) < \rho < a_0$ and the velocities of motion of its particles in the introduced rotating coordinate system. Due to the axial and mirror symmetry of

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the process in this coordinate system in the considered case of plane strain the velocity field will be:

$$\mathbf{v}(\mathbf{r},t) = \mathbf{e}_\rho(\varphi)\, v(\rho,t). \tag{9}$$

For the formulation of mechanical problem in velocities it is necessary to enter into consideration the strain velocity tensor $\mathbf{D} = (\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})/2$. Because of (9) we get

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) = \mathbf{e}_\rho(\varphi)\mathbf{e}_\rho(\varphi)\, D_\rho(\rho,t) + \\ + \mathbf{e}_\varphi(\varphi)\mathbf{e}_\varphi(\varphi)\, D_\varphi(\rho,t),\end{aligned} \tag{10}$$

$$D_\rho(\rho,t) = v'(\rho,t), \quad D_\varphi(\rho,t) = v(\rho,t)/\rho. \tag{11}$$

We can formulate an analogue of the defining relation (1) for the velocity characteristics of the deformation process, i.e. tensor $\mathbf{D}$ and the operator stress velocity tensor $\mathbf{S} = \partial\mathbf{T}^\circ/\partial t$ [3]:

$$\mathbf{S}(\mathbf{r},t) = 2\mathbf{D}(\mathbf{r},t) + (\varkappa-1)\,\mathbf{1}\,\mathrm{tr}\,\mathbf{D}(\mathbf{r},t). \tag{12}$$

From (12) and the representation (10) we get

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) = \mathbf{e}_\rho(\varphi)\mathbf{e}_\rho(\varphi)\, S_\rho(\rho,t) + \\ + \mathbf{e}_\varphi(\varphi)\mathbf{e}_\varphi(\varphi)\, S_\varphi(\rho,t) + \\ + \mathbf{e}_z\mathbf{e}_z\, \nu\big[S_\rho(\rho,t) + S_\varphi(\rho,t)\big],\end{aligned} \tag{13}$$

$$S_{\substack{\rho\\\varphi}}(\rho,t) = (\varkappa+1)\, D_{\substack{\rho\\\varphi}}(\rho,t) + (\varkappa-1)\, D_{\substack{\varphi\\\rho}}(\rho,t). \tag{14}$$

The equation for the tensor $\mathbf{S}$ can be obtained by acting on the equilibrium equation (7) with the linear operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and by differentiating the result by time t. Here it is important to note the fact that in general case the integral operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and the divergence operator $\nabla\cdot$ are not permutable as the lower integration limit of the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ depends on the solid point $\mathbf{r}$. However it can be shown [3] that for the processes considered in this paper these operators commute with respect to the stress tensor $\mathbf{T}$, that is $(\nabla\cdot\mathbf{T})^\circ = \nabla\cdot\mathbf{T}^\circ$ in the entire region occupied at the given moment by the accreted solid, which means that an analogue of the equilibrium equation (7) for operator stresses can be written:

$$\nabla\cdot\mathbf{T}^\circ + \mathbf{f}^\circ = \mathbf{0}, \tag{15}$$

and therefore, for operator stresses velocities:

$$\nabla\cdot\mathbf{S} + \partial\mathbf{f}^\circ/\partial t = \mathbf{0}. \tag{16}$$

For body forces (8) by using integration by parts and the rule of parameter differentiation of an integral we can compute

$$\mathbf{f}^\circ(\mathbf{r},t) = \mathbf{e}_\rho(\varphi)\,\rho c^\circ(\rho,t), \tag{17}$$

$$\partial\mathbf{f}^\circ(\mathbf{r},t)/\partial t = \mathbf{e}_\rho(\varphi)\,\chi(\rho,t), \tag{18}$$

$$\begin{aligned}c^\circ(\rho,t) = c\big(\tau_0(\rho)\big)\,\Delta\big(t,\tau_0(\rho)\big) + \\ + \int_{\tau_0(\rho)}^{t} \dot{c}(\tau)\,\Delta(t,\tau)\,d\tau,\end{aligned} \tag{19}$$

$$\begin{aligned}\chi(\rho,t) = \rho\,\partial c^\circ(\rho,t)/\partial t = \\ = \rho\bigg[\frac{\dot{c}(t)}{G(t)} + c\big(\tau_0(\rho)\big)\,\frac{\partial\omega\big(t,\tau_0(\rho)\big)}{\partial t} + \\ + \int_{\tau_0(\rho)}^{t} \dot{c}(\tau)\,\frac{\partial\omega(t,\tau)}{\partial t}\,d\tau\bigg].\end{aligned} \tag{20}$$

The functions $c^\circ(\rho,t)$ and $\chi(\rho,t)$ are known and determined by the concrete programs of deposition and rotation of the formed layer as well as the mass density and viscoelastic properties of the material used, implemented in the simulated additive process. In the special case of rotation of the mold with a constant angular velocity $\Omega = \mathrm{const}$ we have

$$\begin{aligned}c^\circ(\rho,t) = \mu\,\Omega^2\Delta\big(t,\tau_0(\rho)\big),\\ \chi(\rho,t) = \mu\,\Omega^2\rho\,\partial\omega\big(t,\tau_0(\rho)\big)/\partial t.\end{aligned}$$

Taking into account (13) and (18) the component notion of the equation (16) has the form

$$S'_\rho(\rho,t) + \big[S_\rho(\rho,t) - S_\varphi(\rho,t)\big]/\rho + \chi(\rho,t) = 0. \tag{21}$$

As one of the edge conditions that must be set for this equation is the condition of immobility of the outer boundary of the layer being formed in the considered rotating coordinate system:

$$v(a_0,t) = 0. \tag{22}$$

In view of the transition in the process of mathematical formulation of the problem from the operator stresses $\mathbf{T}^\circ$ to their time derivatives $\mathbf{S}$ for the closure of the statement it is required to set some initial conditions on the operator stresses. This can be done taking into account that the simulated process there are absent initial stress in each newly attached elementary material layer (see Section II), that is

$$\mathbf{T}\big(\mathbf{r},\tau_0(\mathbf{r})\big) = \mathbf{0}. \tag{23}$$

Since at the moment $t = \tau_0(\mathbf{r})$ of stresses appearing at the point $\mathbf{r}$ of the growing solid the operator stresses $\mathbf{T}^\circ\big(\mathbf{r},\tau_0(\mathbf{r})\big)$ on the strength of (6) and (2) coincide with the true stresses related to the shear modulus $\mathbf{T}\big(\mathbf{r},\tau_0(\mathbf{r})\big)/G\big(\tau_0(\mathbf{r})\big)$ then the homogeneous initial condition (23) for the true stresses is equivalent to the homogeneous initial condition for operator stresses

$$\mathbf{T}^\circ\big(\mathbf{r},\tau_0(\mathbf{r})\big) = \mathbf{0}. \tag{24}$$

A remarkable fact is that the condition (24) implies the following simple condition on the radial component of the operator stress velocity tensor $\mathbf{S}$ on the internal moving due to the inflow of additional material boundary $\rho = a(t)$ of the growing layer at any time interval of its continuous growth:

$$\begin{aligned}S_\rho\big(a(t),t\big) = -q(t),\\ q(t) = -c(t)\,a(t)\,\dot{a}(t)/G(t) \geqslant 0.\end{aligned} \tag{25}$$

Indeed, given the initial condition (24) we have

$$\mathbf{T}^\circ(\mathbf{r},t) = \int_{\tau_0(\mathbf{r})}^{t} \mathbf{S}(\mathbf{r},\tau)\,d\tau. \tag{26}$$

Following [16] we substitute this representation into the analogue of the equilibrium equation (15), calculating beforehand the divergence of the tensor (26) according to the rule of parameter differentiation of the integral by using the equation (16) for the tensor $\mathbf{S}$, the identity (3) and the representations

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(5) and (13):

$$\nabla\cdot\mathbf{T}^\circ(\mathbf{r},t)=\nabla\cdot\int_{\tau_0(\mathbf{r})}^{t}\mathbf{S}(\mathbf{r},\tau)\,d\tau=$$
$$=\int_{\tau_0(\mathbf{r})}^{t}\nabla\cdot\mathbf{S}(\mathbf{r},\tau)\,d\tau-\nabla\tau_0(\mathbf{r})\cdot\mathbf{S}\big(\mathbf{r},\tau_0(\mathbf{r})\big)=$$
$$=-\int_{\tau_0(\rho)}^{t}\frac{\partial\mathbf{f}^\circ(\mathbf{r},\tau)}{\partial\tau}\,d\tau-\tau_0'(\rho)\,\mathbf{e}_\rho(\varphi)\cdot\mathbf{S}\big(\mathbf{r},\tau_0(\rho)\big)=$$
$$=\mathbf{f}^\circ\big(\mathbf{r},\tau_0(\rho)\big)-\mathbf{f}^\circ(\mathbf{r},t)-\mathbf{e}_\rho(\varphi)\,\frac{S_\rho\big(\rho,\tau_0(\rho)\big)}{\dot a\big(\tau_0(\rho)\big)},$$

$$(15)\iff\mathbf{e}_\rho(\varphi)\,S_\rho\big(\rho,\tau_0(\rho)\big)=\mathbf{f}^\circ\big(\mathbf{r},\tau_0(\rho)\big)\,\dot a\big(\tau_0(\rho)\big).$$

Taking into account (4), (17), and the representation (19) we come from here to the condition (25).

At those time intervals t when the inflow of additional material to the additively formed rotating layer is temporarily or finally terminated, at each point $\mathbf{r}$ of its fixed inner cylindrical surface $\rho=a(t)=\mathrm{const}$ the condition of this surface unloading is set

$$\mathbf{e}_\rho(\varphi)\cdot\mathbf{T}(\mathbf{r},t)=\mathbf{0}\qquad(27)$$

starting from the moment $t=\tau_0(\mathbf{r})$ of inclusion of this point in the solid composition. From the condition (27) it obviously follows that $S_\rho\big(a(t),t\big)=0$ in the absence of an inflow of new material to the formed solid, that is, at any time interval of constancy of the function $a(t)$. So, since $\dot a(t)\equiv 0$ on these intervals, the boundary condition (25) derived above for the time intervals of continuous growth of the solid formally remains valid even outside of these intervals when the growth of the solid temporarily or permanently terminated.

Thus, for the entire process of piecewise continuous accreting the viscoelastic solid under consideration, including arbitrary prolonged period of time after the end of additive formation of the solid, the boundary value problem (21), (20), (14), (11), (22), (25) will be fair.

V. Solution of the Stated Problem and Constructing the True Stresses Fields Evolution

The exact analytical solution of the boundary value problem formulated in Section IV has the form:

$$v(\rho,t)=\frac{\rho}{2}\Bigg[-\Lambda_1^{(-)}(\rho,t)+\frac{\lambda_{1/\varkappa}^{(+)}\big(a(t),\rho\big)}{\lambda_{1/\varkappa}^{(+)}\big(a(t),a_0\big)}\,\Lambda_1^{(-)}(a_0,t)-\frac{\lambda_1^{(-)}(a_0,\rho)}{\lambda_\varkappa^{(+)}\big(a_0,a(t)\big)}\,q(t)\Bigg],$$

$$\underset{\varphi}{S_\rho}(\rho,t)=-\Lambda_\varkappa^{(\pm)}(\rho,t)+\frac{\lambda_1^{(\mp)}\big(a(t),\rho\big)}{\lambda_{1/\varkappa}^{(+)}\big(a(t),a_0\big)}\,\Lambda_1^{(-)}(a_0,t)-\frac{\lambda_\varkappa^{(\pm)}(a_0,\rho)}{\lambda_\varkappa^{(+)}\big(a_0,a(t)\big)}\,q(t),\qquad(28)$$

$$\lambda_\alpha^{(\pm)}(\xi,\eta)=\alpha\pm\xi^2/\eta^2,$$
$$\Lambda_\alpha^{(\pm)}(\rho,t)=\frac{1}{\varkappa+1}\int_{a(t)}^{\rho}\chi(\xi,t)\,\lambda_\alpha^{(\pm)}(\xi,\rho)\,d\xi.$$

The formulas (13) and (28) (where $\varkappa$ is the positive material constant (see Section III), $a(t)$ is the given law of reducing the inner radius of the formed layer by adding a new material to it, a_0 is the constant radius of its outer surface bound with the rotating rigid mold (see Section II), and the functions $q(t)$ and $\chi(\rho,t)$ are defined by (25) and (20), respectively) give us the evolution of the operator stress velocity tensor $\mathbf{S}(\mathbf{r},t)$ at each point $\mathbf{r}$ of considered piecewise continuously accreted aging viscoelastic solid on a time beam $t>\tau_0(\mathbf{r})$, covering the entire history of deformation of the neighborhood of a given point in the composition of this solid. Therefore the evolution of the tensor of the operator stresses $\mathbf{T}^\circ(\mathbf{r},t)$ at any point of the solid $\mathbf{r}$ at $t\geqslant\tau_0(\mathbf{r})$ can be restored using the integration procedure taking into account the initial condition (24):

$$\mathbf{T}^\circ(\mathbf{r},t)=\int_{\tau_0(\mathbf{r})}^{t}\mathbf{S}(\mathbf{r},\tau)\,d\tau.$$

As soon as at the point $\mathbf{r}$ of the considered solid we know the entire evolution of the tensor $\mathbf{T}^\circ(\mathbf{r},t)$, that is the law of changing this tensor since the moment $t=\tau_0(\mathbf{r})$ of appearing stresses at the given point, we according to (6), (2) can find the complete evolution of the true stresses tensor $\mathbf{T}(\mathbf{r},t)$ at the point $\mathbf{r}$ by means of the integral transform $\mathcal{H}_{\tau_0(\mathbf{r})}^{-1}$ inverse to $\mathcal{H}_{\tau_0(\mathbf{r})}$:

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)}=\mathbf{T}^\circ(\mathbf{r},t)+\int_{\tau_0(\mathbf{r})}^{t}\mathbf{T}^\circ(\mathbf{r},\tau)\,R(t,\tau)\,d\tau,\qquad(29)$$

where $R(t,\tau)$, called the relaxation kernel, is the resolvent of creep kernel $K(t,\tau)$.

When using a specific analytic approximation of the creep kernel $K(t,\tau)$ describing the experimental creep data of the material used the analytic expression for the corresponding relaxation kernel $R(t,\tau)$ may not be known in a closed form or may be too cumbersome. In this case to construct the evolution of the true stress fields it is advisable, instead of using the analytical formula (29), to refer to the procedure of numerical solution of the Volterra integral equation of the 2nd kind (see (6), (2))

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)}-\int_{\tau_0(\mathbf{r})}^{t}\frac{\mathbf{T}(\mathbf{r},\tau)}{G(\tau)}\,K(t,\tau)\,d\tau=\mathbf{T}^\circ(\mathbf{r},t)$$

with the parameter $\mathbf{r}$ relative to the desired time function $\mathbf{T}/G$ with the known right part $\mathbf{T}^\circ$. As a quite simple and efficient numerical method of such equation solution it is possible to suggest a method of quadratures [51].

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