# DMS Way of Finding the Optimum Number of Iterations for Fixed Point Iteration Method

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***Abstract*— In this paper, the number of iterations is calculated depending on initial approximation ($y_0$), tolerance level ($\epsilon$) and function [$\emptyset(y)$ $in\ y = \emptyset(y)$]. We have shown the relation of these parameters with the number of iterations in the form of an equation. The relation is verified by taking distinct (non-algebraic and algebraic equations) examples in this paper.**

***Index Terms*— Fixed-point iteration method, Initial approximation, Number of iterations, Tolerance Level.**

## I. INTRODUCTION

FROM last few decades, many iteration methods – single and multi-step methods have been developed to solve non-linear algebraic and transcendental equations. In previous years the convergence rate and efficiency index of different methods have been discussed and more iterative methods have been established for higher order convergence rate and higher efficiency index. Noor et al. [1-2] proposed three-step iteration method to solve non-linear (algebraic and transcendental) equations both with cubic convergence. The method was corrected and improved by Chun [11] and Hueso [12] et al. afterwards in 2007 and 2008 respectively. Biazar et al. [3] in 2006 improved the fixed point iteration method to increase the convergence rate and reduce the number of iterations. Noor [4] suggested new decomposition iterative methods to solve nonlinear equations in which performance and efficiency of these methods are illustrated. Two three–step methods were suggested by Cordero et al. [5] with sixth order convergence. The first and second schemes are obtained by Potra-Ptak's and Homeier's methods, respectively. An efficiency index of 1.5651 is obtained from both these methods. An improvement to previous methods is done by Bi et al. [6] in 2009, who propose a three-step method of eighth-order convergence with 1.682 efficiency index. Kang et al. [7] proposed a second order iteration method to solve nonlinear equations with efficiency index of 1.4142. For solving system of non-linear equations, a new iterative method was developed by Huang et al. [8] to get faster convergence. Shah et al. [9] used decomposition technique to develop a multi–step iterative scheme which is advancement to Adomian decomposition technique. Two new iterative schemes of order two and three are given by Saqib et al. [10] and their comparison is done with existing methods. The methods listed in various research papers do not provide the general formula to calculate the minimum number of iterations for any of these or other methods.

In this paper, we are giving the DMS (Shah and Sahni) way to find out optimum number of iterations $(n+1)$ required for fixed-point iteration method while having initial approximation $(y_0)$ known and given tolerance level $(\epsilon)$. Here we use the condition that $0 < \emptyset'(y) < 1$; $\forall y$ in the given interval.

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## II. FIXED POINT ITERATION METHOD

The fixed point iteration method (FPIM) is used to solve both transcendental and non-linear algebraic equation, represented as $f(y) = 0$. The equation can be rewritten, as

$$y = \emptyset(y) \tag{1}$$

where function $\emptyset(y)$ converges to the solution $p$.

The FPIM converges to the solution $p$ under following conditions:

(I) $\exists$ an interval $[c,d]$ such that "$\forall\, y \in [c,d]$; $\emptyset(y) \in [c,d]$",

(II) $\forall\, y \in (c,d)$; $|\emptyset'(y)| < 1$

Considering the following iterative scheme:

$$y_{n+1} = \emptyset(y_n)\,, n = 0,1,2\ldots \tag{2}$$

To start an iterative process an initial approximation $y_0$ is chosen and through sequence of approximations, say $\{y_n\}$, the solution of equation (1) can be found.

The scheme will converge to the root $p$, provided that

(I) The initial approximation $y_0$ is chosen in the interval $[c,d]$,

(II) $\emptyset$ has a continuous derivative on $(c,d)$.

(III) $|\emptyset'(y)| < 1$ for all $y \in [c,d]$

(IV) $c \le \emptyset(y) \le d$ for all $y \in [c,d]$ .

## III. NUMBER OF ITERATIONS

For FPIM, if $e_{n+1}$ is error on $(n+1)^{th}$ iteration and $e_n$ is error on $n^{th}$ iteration, then

$|e_n - e_{n+1}| < \epsilon$, where $e_{n+1} = |p - y_{n+1}|$, $e_n = |p - y_n|$ and $\epsilon$ = tolerance level.

Now, we consider the case where $e_0 > 0, e_n > 0, e_{n+1} > 0$ and $0 < \emptyset'(y) < 1$. So, considering the above case we get $e_{n+1} = p - y_{n+1}$, $e_n = p - y_n$, which implies

$$e_n - e_{n+1} < \epsilon\,. \tag{3}$$

Now if the initial approximation $y_0 \in [c,d]$ is given then

there exists first iteration $y_1 = \emptyset(y_0)$ in $[c, d]$ and we have
$\Delta y_0 = y_1 - y_0 = \emptyset(y_0) - y_0$. (4)

After some calculations from (3) and (4) we get the formula for number of iterations, as

$$n > \frac{ln\left(\frac{\varepsilon}{\Delta y_0}\right)}{lnS} \quad (5)$$

where

$$S = \frac{e_{n+1}}{e_n} \cong \frac{1+\emptyset'(y_0)-\sqrt{[1-\emptyset'(y_0)]^2-4\Delta y_0\emptyset''(y_0)}}{2} \quad (6)$$

and $[1 - \emptyset'(y_0)]^2 - 4\Delta y_0\emptyset''(y_0) \geq 0$.

Now, for optimum (minimum) number of iterations we apply lowest integer function to (5) and

$$n = \left[\frac{ln\left(\frac{\epsilon}{\Delta y_0}\right)}{\ln S}\right] \quad \text{and} \quad n+1 = \left[\frac{ln\left(\frac{\epsilon}{\Delta y_0}\right)}{\ln S}\right] + 1 \quad (7)$$

which is minimum number of iterations required to get the solution $p$ from $y_0$ with some tolerance $\epsilon$.

## IV. Numerical Illustration

To show the authenticity of the technique, we apply it to 4 distinct examples.

**Example 1**

Consider the equation $f_1(y) = y^3 - 2y^2 - y - 1 = 0$ which has unique root in [2, 3] from Intermediate Value theorem (IVT), we get$(f_1(2)\ f_1(3) < 0)$.

Rewriting equation $f_1(y) = 0$ as

$$\emptyset(y) = (2y^2 + y + 1)^{1/3} = y \quad (8)$$

and from (8) we get

$$\emptyset'(y) = \frac{(4y+1)}{3(2y^2+y+1)^{2/3}}$$

and

$$\emptyset''(y) = -\frac{2(4y^2+2y-5)}{9(2y^2+y+1)^{5/3}}$$

Let $y_0 = 2$ and $\varepsilon = 10^{-5}$, so we get

$$\emptyset(y_0) = 2.22398, \quad \emptyset'(y_0) = 0.60654,$$
$$\emptyset''(y_0) = -0.06127 \ and\ \Delta y_0 = 0.22398$$

From the equation (6), S is calculated as 0.57430.

Now using equation (5) and (7), $n > 18.06$ and $n + 1 = 20$.

Table I lists the results of $n + 1, y_n, y_{n+1}\ and\ |y_{n+1} - y_n|$ for example 1.

TABLE I
NUMBER OF ITERATIONS FOR EXAMPLE 1

| $n+1$ | $y_n$ | $y_{n+1}$ | $\lvert y_{n+1} - y_n\rvert$ |
|---|---|---|---|
| 1 | 2 | 2.22398 | 0.22398 |
| 2 | 2.22398 | 2.35832 | 0.13434 |
| … | … | … | … |
| 20 | 2.5468 | 2.54681 | 0.00001 |

**Example 2**

Consider the equation $f_2(y) = 2y - log_{10}y - 7 = 0$ which has unique root in $[3, 4]$ from IVT, we get $(f_2(3)\ f_2(4) < 0)$.

Rewriting the function $f_2(y) = 0$ as

$$\emptyset(y) = \frac{log_{10}y+7}{2} = y \quad (9)$$

and from (9) we get $\emptyset'(y) = \frac{log_{10}e}{2y}, \emptyset''(y) = -\frac{log_{10}e}{2y^2}$.

Let $y_0 = 3.6$ and $\epsilon = 10^{-4}$, so we get

$$\emptyset(y_0) = 3.7781, \quad \emptyset'(y_0) = 0.0603,$$
$$\emptyset''(y_0) = -0.0168, \quad \Delta y_0 = 0.1781$$

From the equation (6), $S$ is calculated as 0.0571

Now using equation (5) and (7), $n > 2.61$ and $n + 1 = 4$.

Table II lists the results of $n + 1, y_n, y_{n+1}\ and\ |y_{n+1} - y_n|$ for example 2.

TABLE II
NUMBER OF ITERATIONS FOR EXAMPLE 2

| $n+1$ | $y_n$ | $y_{n+1}$ | $\lvert y_{n+1} - y_n\rvert$ |
|---|---|---|---|
| 1 | 3.6 | 3.7781 | 0.1781 |
| 2 | 3.7781 | 3.7886 | 0.0105 |
| 3 | 3.7886 | 3.7892 | 0.0004 |
| 4 | 3.7892 | 3.7892 | 0.0000 |

**Example 3**

Consider the equation $f_3(y) = 7y - 6e^{\frac{y}{5}} = 0$ which has unique root in $[0.2, 1.2]$ from IVT, we get $(f_3(0.2)f_4(1.2) < 0)$.

Rewriting the function $f_3(y) = 0$ as

$$\emptyset(y) = \frac{6e^{\frac{y}{5}}}{7} = y \quad (10)$$

and from (10) we get $\emptyset'(y) = \frac{6e^{\frac{y}{5}}}{35}, \emptyset''(y) = \frac{6e^{\frac{y}{5}}}{175}$.

Let $y_0 = 0.3$ and $\epsilon = 10^{-5}$, so we get

$$\emptyset(y_0) = 0.91014, \quad \emptyset'(y_0) = 0.18203,$$
$$\emptyset''(y_0) = 0.03640, \quad y_0 = 0.61014$$

From the equation (6), $S$ is calculated as 0.21015

Now using equation (5) and (7), $n > 7.063$ and $n + 1 = 9$.

Table III lists the results of $n + 1, y_n, y_{n+1}\ and\ |y_{n+1} - y_n|$ for example 3.

TABLE III
NUMBER OF ITERATIONS FOR EXAMPLE 3

| $n+1$ | $y_n$ | $y_{n+1}$ | $\lvert y_{n+1} - y_n\rvert$ |
|---|---|---|---|
| 1 | 0.3 | 0.91014 | 0.61014 |
| 2 | 0.91014 | 1.02827 | 0.11813 |
| … | … | … | … |
| 9 | 1.05943 | 1.05944 | 0.00001 |

**Example 4**

Consider the equation $f_4(y) = y^3 - y - 1 = 0$ which has unique root in [0.5, 1.5] from IVT, we get $(f_4(0.5)\ f_4(1.5) < 0)$.

Rewriting the function $f_4(y) = 0$ as

$$\emptyset(y) = (y + 1)^{\frac{1}{3}} = y \quad (11)$$

and from (11) we get $\emptyset'(y) = \frac{1}{3(y+1)^{\frac{2}{3}}}, \emptyset''(y) = \frac{-2}{9(y+1)^{\frac{5}{3}}}$.

Let $y_0 = 0.9$ and $\varepsilon = 10^{-5}$, so we get

$$\emptyset(y_0) = 1.23856, \quad \emptyset'(y_0) = 0.21729,$$
$$\emptyset''(y_0) = -0.07624, y_0 = 0.33856$$

From the equation (6), $S$ is calculated as 0.18559

Now using equation (5) and (7), $n > 6.19$ and $n + 1 = 8$.

Table IV lists the results of $n + 1, y_n, y_{n+1}\ and\ |y_{n+1} - y_n|$ for example 4.

ISBN: 978-988-14047-9-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE IV
NUMBER OF ITERATIONS FOR EXAMPLE 4

| $n+1$ | $y_n$ | $y_{n+1}$ | $\lvert y_{n+1} - y_n \rvert$ |
|---|---|---|---|
| 1 | 0.9 | 1.23856 | 0.33856 |
| 2 | 1.23856 | 1.30815 | 0.06959 |
| … | … | … | … |
| 8 | 1.32471 | 1.32472 | 0.00001 |

## V. Conclusion

Till now it was known that the number of iterations is dependent on tolerance $(\epsilon)$, initial guess $(y_0)$ and the function $\emptyset(y)$. Here we have shown the relation in mathematical equation while having the derivative of function $\emptyset'(y) \in (0,1)\ for\ all\ y \in [a,b]$.

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