# Time-Distributed Unknown Input Filtering for a Class of Nonlinear Systems

Chien-Shu Hsieh

*Abstract*—**This paper presents a multi-step delayed input and state estimation (MSDISE) for a class of nonlinear systems by using the newly proposed state-dependent coefficient (SDC)-based system reformation approach, where a time-distribution unknown input filtering (TUIF) technique is used to facilitate the design. It is shown that the addressed nonlinear UIF problem can be easily solved by using the corresponding linear algorithm via this new system reformation approach. A practical illustrative example is given to show the superiority of the proposed method.**

*Index Terms*—**nonlinear state estimation, multi-step delayed input and state estimation, time-distributed input estimation, SDC-based system reformation**

## I. Introduction

THE model-free unknown input filtering (MUIF) problem, where the system dynamics of the unknown inputs is completely unknown, has received many researchers' attention due to its vast applications in different research areas (see [1] and the references therein for details). Most recently, the research work on the MUIF problem has focused on solving more general multi-step delayed input and state estimation (MSDISE) problem [1]-[5]. It is noticed that, the MSDISE problem can be easily solved by making use of some noncausal unknown input reconstruction models, through which yields the delayed unbiased minimum-variance input and state estimation. In this paper, we give a rigorous and detailed derivation of the MSDISE solution, where no unknown input reconstruction models are used.

There are basically two approaches for solving the MSDISE problem. One is the measurement-augmented filtering approach [2], [4], [5] and the other is the time-distributed system reformation filtering approach [1], [3]. Although the measurement-augmented filtering method is straightforward to be applied, its computational complexity may be too complex to implement due to that augmented measurements are used. On the other hand, the system reformation technique does not need to augment the measurements in order to solve the problem. The basic idea behind this method is to transform the original system into one which can reflect all time-delayed estimable inputs associated with the unknown inputs on the measurement equation. Thus, by using a time-distributed unknown input filtering (TUIF) approach the complete input estimation is ready to be achieved via the existing designs [6], [7]. However, it is noted that, this method is effective only if the rank of the feedthrough matrix of the unknown inputs to the output is equivalent to the dimension of the unknown input vector [1]. Unfortunately, the above rank condition may not always be satisfied (see [8] for an illustration). If this happens, suitable modifications are needed to recover the original degraded performance to its optimal one. A heuristic but restricted approach to achieve this can be found in [8]. In this paper, we give a more accurate and general result.

Nonlinear filtering problems arise in many practical applications, e.g., financial estimation, biological and industrial processes, target localization and tracking, robots and robotic manipulators, and traffic state estimation (see [5] and the references therein for details). As is well known, a common approach to solve these problems is generalizing the Kalman filter paradigm for nonlinear systems, e.g., the extended Kalman filter, and representing state uncertainty with a different ensemble set of state vectors [9]-[14]. It should be stressed that all the above methods may yield some approximation errors due to that some nonlinear terms have been truncated. To alleviate this problem, the state dependent coefficient (SDC) factorization method [15]-[17] serves as an effective method to preserve the nonlinear function without any transformation errors. The basic idea embedded this method is to treat nonlinear systems as linear ones. Thus, the dedicated nonlinear filtering problem is recast into the well-known linear UIF problem. This may greatly simplify the nonlinear filter design. Recently, an SDC-based measurement-augmented filtering method has been proposed to successfully solve the MSDISE problem of a nonlinear two-link manipulator system [5]. Nevertheless, it is still unclear how to apply the system reformation filtering approach to solve this SDC-based nonlinear MSDISE problem.

The main aim of this paper is to extend the previous work [1] and continue the research line in investigating the application of the time-distributed system reformation filtering approach for nonlinear systems. In the sequel, a complete and refined MSDISE design, without using a predefined unknown input reconstruction model and with degraded performance recovery, for a class of nonlinear systems by using the newly proposed SDC-based time-distributed system reformation approach is developed.

The paper is organized as follows. In Section II, the statement of the problem is addressed. In Section III, the proposed SDC and input-reconstruction based system reformation method is presented. Specifically, in Section III.A the SDC form of the nonlinear system is derived, in Section III.B the system reformation through the TUIF technique is revisited, and in Section III.C the refined unknown input reconstruction procedure is proposed. Section IV provides the dedicated optimal multi-step delayed input and state estimation design. In Section V, a practical illustrative example is given to show the superiority of the proposed method. Section VI has the conclusions.

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## II. Problem Formulation

In the paper, we consider the following discrete-time nonlinear time-varying system with unknown inputs:

$$x_{k+1} = f_k(x_k) + G_k^x d_k + w_k, \quad (1)$$
$$y_k = g_k(x_k) + H_k^x d_k + v_k, \quad (2)$$

where $x_k \in R^n$, $d_k \in R^m$, $y_k \in R^p$, $w_k \in R^n$, and $v_k \in R^p$ are the state, unknown input, output, process noise, and measurement noise, respectively. Matrices $G_k^x$ and $H_k^x$ are dependant on the system state $x_k$. Noises $w_k$ and $v_k$ are uncorrelated white signals with covariance matrices $Q_k = E[w_k w_k^T] \geq 0$ and $R_k = E[v_k v_k^T] > 0$, respectively. It is assumed that $(f_k, g_k)$ is observable and that $x_0$ is independent of $w_k$ and $v_k$ for all $k$; furthermore, we assume that an unbiased estimate of the initial state $x_0$ is available with covariance matrix $P_0^x$. The problem of interest in the paper is to determine the smoothed state estimate $\hat{x}_{k-s+1|k}$ and unknown input estimate $\hat{d}_{k-s|k}$, where $s$ is a positive integer and named as filter delay, based on the measurements $\{y_0, y_1, \ldots, y_k\}$ and without any information on $d_k$.

A straightforward method to solve the above simultaneous input and nonlinear state estimation problem is through a measurement-augmented unknown input filtering approach [5], where one intends to use the following aggregated measurement equation to obtain the optimal state and input estimates of system (1), (2):

$$\begin{aligned} T_k^s \begin{bmatrix} y_k \\ \vdots \\ y_{k+\alpha} \end{bmatrix} &= T_k^s \begin{bmatrix} g_k(x_k) + H_k^x d_k \\ \vdots \\ g_{k+\alpha}(x_{k+\alpha}) + H_{k+\alpha}^x d_{k+\alpha} \end{bmatrix} \\ &\quad + T_k^s \begin{bmatrix} v_k^T & \cdots & v_{k+\alpha}^T \end{bmatrix}^T, \\ &= \breve{g}_k^s(x_k) + \breve{H}_k^s d_k + \breve{v}_k^s, \end{aligned} \quad (3)$$

where $\alpha = s - 1$, $T_k^s$ is a specific matrix to delete the unknown inputs $d_{k+1}, \cdots, d_{k+\alpha}$ from the augmented measurements and to make $d_k$ be estimable, and $\breve{g}_k^s$, $\breve{H}_k^s$ and $\breve{v}_k^s$ are suitable matrices of appropriate dimensions. Then, the estimation problem is recast to find the optimal estimates $\hat{x}_{k-s+1|k}$ and $\hat{d}_{k-s|k}$ for system (1), (3). It is noticed that the above measurement-augmented filtering approach may not be practical to implement due to the need of finding a suitable matrix $T_k^s$ and the forbidden computational complexity involved in the calculations of $\breve{g}_k^s$ and $\breve{v}_k^s$. Furthermore, the obtained augmented measurement noise $\breve{v}_k^s$ may correlate with the process noise $w_k$. All these considerations may complicate the estimation algorithm design.

In this paper, a nonlinear UIF method which has no need of augmenting the measurement equation in solving the addressed problem is proposed. This new approach serves as an extension of the TUIF in [1] for nonlinear systems.

## III. SDC and Input-Reconstruction Based System Reformation

The main aim of this section is to reform the nonlinear system (1), (2) into a linear-like system in which the unknown inputs can be directly estimated by using the existing estimation methods, e.g. [6], [7].

### A. System Reformation Using SDC Form

We note that a nonlinear function $f(x)$ can be represented as the following specific form: $f(x) = A^x x$, where $A^x$ is the SDC matrix and its general solution is given as follows:

$$A^x = f(x)x^+ + Z(I - xx^+), \quad (4)$$

in which $Z$ is an arbitrary matrix of appropriate dimension. Thus, based on (4) the nonlinear system dynamics (1) and the measurement (2) can be rewritten in the following SDC forms:

$$x_{k+1} = A_k^x x_k + G_k^x d_k + w_k, \quad (5)$$
$$y_k = C_k^x x_k + H_k^x d_k + v_k, \quad (6)$$

where

$$A_k^x = f_k(x_k)x_k^+ + Z_k^A(I - x_k x_k^+), \quad (7)$$
$$C_k^x = g_k(x_k)x_k^+ + Z_k^C(I - x_k x_k^+). \quad (8)$$

### B. TUIF-Based System Reformation Revisited

First, we define the online estimable unknown input vector $\tilde{d}_k^0$, associated with $d_k$, to satisfy the following relationship:

$$H_k^x \tilde{d}_k^0 = H_k^x d_k. \quad (9)$$

Solving (9) for $\tilde{d}_k^0$ yields

$$\tilde{d}_k^0 = \left((\Omega_k^0)^+ \Omega_k^0\right) d_k, \quad \Omega_k^0 = H_k^x. \quad (10)$$

Using (9)-(10), the system (5), (6) is rewritten as follows:

$$x_{k+1} = A_k^x x_k + G_k^x \tilde{d}_k^0 + G_k^x \Pi_k^1 d_k + w_k, \quad (11)$$
$$y_k = C_k^x x_k + \Omega_k^0 \tilde{d}_k^0 + v_k, \quad (12)$$

where

$$\Pi_k^1 = I - (\Omega_k^0)^+ \Omega_k^0. \quad (13)$$

Second, using (11) in (12) yields:

$$\begin{aligned} y_k &= C_k^x (A_{k-1}^x x_{k-1} + G_{k-1}^x \tilde{d}_{k-1}^0 + w_{k-1}) \\ &\quad + C_k^x G_{k-1}^x \Pi_{k-1}^1 d_{k-1} + \Omega_k^0 \tilde{d}_k^0 + v_k. \end{aligned} \quad (14)$$

Then, we can define the online estimable unknown input vector $\tilde{d}_{k-1}^1$, associated with $d_{k-1}$, to satisfy the following relationship:

$$C_k^x G_{k-1}^x \Pi_{k-1}^1 \tilde{d}_{k-1}^1 = C_k^x G_{k-1}^x \Pi_{k-1}^1 d_{k-1}. \quad (15)$$

Solving (15) for $\tilde{d}_{k-1}^1$ yields

$$\tilde{d}_{k-1}^1 = \left((C_k^x G_{k-1}^x \Pi_{k-1}^1)^+ C_k^x G_{k-1}^x \Pi_{k-1}^1\right) d_{k-1}. \quad (16)$$

To promise that unknown input vectors $\tilde{d}_k^0$ and $\tilde{d}_{k-1}^1$ can be accurately estimated, the nonzero columns of matrices $\Omega_k^0$ and $\Omega_k^1$ should be linear independent (see [8] for details). If this is the case, we set $\bar{\Pi}_k^1 = \Pi_k^1$. Otherwise, we modify the input reconstruction matrix $\Pi_k^1$ as follows:

$$\Pi_k^1 = \bar{\Pi}_k^1 + \Delta_k^1, \quad (17)$$

where the nonzero columns of matrices $\Omega_k^0$ and $C_k^x G_{k-1}^x \bar{\Pi}_{k-1}^1$ are linear independent. Thus, we have

$$\tilde{d}_{k-1}^1 = \left((\Omega_k^1)^+ \Omega_k^1\right) d_{k-1}, \quad \Omega_k^1 = C_k^x G_{k-1}^x \bar{\Pi}_{k-1}^1. \quad (18)$$

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and hence system (11), (14) can be rewritten as follows:

$$\begin{aligned} x_{k+1} &= A_k^x x_k + G_k^x \tilde{d}_k^0 + G_k^x \bar{\Pi}_k^1 \tilde{d}_k^1 + G_k^x \Pi_k^2 d_k + w_k, &(19)\\ y_k &= C_k^x (A_{k-1}^x x_{k-1} + G_{k-1}^x \tilde{d}_{k-1}^0 + w_{k-1}) + \Omega_k^1 \tilde{d}_{k-1}^1 \\ &\quad + \Omega_k^0 \tilde{d}_k^0 + C_k^x G_{k-1}^x \Delta_{k-1}^1 d_{k-1} + v_k, &(20) \end{aligned}$$

where

$$\Pi_k^2 = \bar{\Pi}_k^1 \left( I - (\Omega_{k+1}^1)^+ \Omega_{k+1}^1 \right) + \Delta_k^1. \quad (21)$$

Third, following the same procedures to obtain (19) and (20) one can obtain the following alternative system:

$$\begin{aligned} x_{k+1} &= A_k^x x_k + G_k^x \sum_{i=0}^{s} \bar{\Pi}_k^i \tilde{d}_k^i + G_k^x \Pi_k^{s+1} d_k + w_k, &(22)\\ y_k &= C_k^x \Bigg( (A_{k-1}^x)^s x_{k-s} + \sum_{i=1}^{s} (A_{k-1}^x)^{i-1} w_{k-i} \\ &\quad + \sum_{i=1}^{s} \sum_{j=0}^{i-1} (A_{k-1}^x)^{i-1} G_{k-i}^x \bar{\Pi}_{k-i}^j \tilde{d}_{k-i}^j \Bigg) \\ &\quad + \sum_{i=0}^{s} \Omega_k^i \tilde{d}_{k-i}^i + \check{d}_k^* + v_k, &(23) \end{aligned}$$

where

$$\begin{aligned} \tilde{d}_k^i &= \left( (\Omega_{k+i}^i)^+ \Omega_{k+i}^i \right) d_k, &(24)\\ A_k^i &= A_k \times \cdots \times A_{k-i+1}, &(25)\\ \Omega_k^i &= C_k^x (A_{k-1}^x)^{i-1} G_{k-i}^x \bar{\Pi}_{k-i}^i, &(26)\\ \bar{\Pi}_k^i &= \Pi_k^i - \Delta_k^i, \quad \Pi_k^0 = I_m, \quad \Delta_k^0 = 0, &(27)\\ \Pi_k^{i+1} &= \bar{\Pi}_k^i \left( I - (\Omega_{k+i}^i)^+ \Omega_{k+i}^i \right) + \Delta_k^i, &(28)\\ \check{d}_k^* &= \sum_{i=1}^{s-1} C_k^x (A_{k-1}^x)^{i-1} G_{k-i}^x \Delta_{k-i}^i d_{k-i}. &(29) \end{aligned}$$

Note that, from (22) it is clear that the unknown inputs $d_k$ can be completely estimated if the following unknown input reconstruction condition holds:

$$\Pi_k^{s+1} = 0. \quad (30)$$

Now, we are in the position to simplify measurement (23) by defining the following reformed state:

$$\bar{x}_k^s = x_k - \sum_{i=1}^{s} \sum_{j=i}^{s} (A_{k-1}^x)^{i-1} G_{k-i}^x \bar{\Pi}_{k-i}^j \tilde{d}_{k-i}^j. \quad (31)$$

In showing this, we have the following relationship due to (22) and (30):

$$\begin{aligned} x_k &= (A_{k-1}^x)^s x_{k-s} + \sum_{i=1}^{s} \sum_{j=0}^{s} (A_{k-1}^x)^{i-1} G_{k-i}^x \bar{\Pi}_{k-i}^j \tilde{d}_{k-i}^j \\ &\quad + \sum_{i=1}^{s} (A_{k-1}^x)^{i-1} w_{k-i}. &(32) \end{aligned}$$

Using (32) in (31) yields

$$\begin{aligned} \bar{x}_k^s &= (A_{k-1}^x)^s x_{k-s} + \sum_{i=1}^{s} \sum_{j=0}^{i-1} (A_{k-1}^x)^{i-1} G_{k-i}^x \bar{\Pi}_{k-i}^j \tilde{d}_{k-i}^j \\ &\quad + \sum_{i=1}^{s} (A_{k-1}^x)^{i-1} w_{k-i}, &(33) \end{aligned}$$

by which measurement (23) is rewritten as follows:

$$y_k = C_k^x \bar{x}_k^s + \check{H}_k^x \check{d}_k^s + \check{d}_k^* + v_k, \quad (34)$$

where

$$\begin{aligned} \check{H}_k^x &= \begin{bmatrix} \Omega_k^0 & \Omega_k^1 & \cdots & \Omega_k^s \end{bmatrix}, &(35)\\ \check{d}_k^s &= \begin{bmatrix} (\tilde{d}_k^0)^T & (\tilde{d}_{k-1}^1)^T & \cdots & (\tilde{d}_{k-s}^s)^T \end{bmatrix}^T. &(36) \end{aligned}$$

Note that the effect of the above unknown signal $\check{d}_k^*$ will corrupt the estimation of $\check{d}_k^s$, and hence the measurement (34) can be rewritten as follows:

$$y_k = C_k^x \bar{x}_k^s + \check{H}_k^x \check{d}_k^{s*} + v_k, \quad (37)$$

where

$$\check{d}_k^{s*} = \check{d}_k^s + (\check{H}_k^x)^+ \check{d}_k^*, \quad (38)$$

due to that the nonzero columns of matrix $\check{H}_k^x$ are independent.

Finally, the problem remains to simplify the system dynamics (22). Using (31) in (22) yields:

$$\bar{x}_{k+1}^s = A_k^x \bar{x}_k^s + \check{G}_k^x \check{d}_k^s + w_k, \quad (39)$$

where

$$\check{G}_k^x = \begin{bmatrix} G_k^x & \Xi_k^1 & \cdots & \Xi_k^s \end{bmatrix}, \ \Xi_k^i = (A_k^x)^i G_{k-i}^x \bar{\Pi}_{k-i}^i. \quad (40)$$

Note that, owing to the contamination of vector $\check{d}_k^s$ in the measurement equation (37) the above system dynamics can be equivalently expressed as follows in the sense of filtering:

$$\bar{x}_{k+1}^s = A_k^x \bar{x}_k^s + \check{G}_k^x \check{d}_k^{s*} + w_k. \quad (41)$$

### C. Input Reconstruction

Thanks to (30) and comparing (5) with (22), we can obtain the following unknown input reconstruction model:

$$d_k = \sum_{i=0}^{s} \bar{\Pi}_k^i \tilde{d}_k^i, \quad \bar{\Pi}_k^0 = I_m, \quad (42)$$

which can be expressed more properly as the following $s$-delay form:

$$d_{k-s} = \sum_{i=0}^{s} \bar{\Pi}_{k-s}^i \tilde{d}_{k-s}^i, \quad (43)$$

due to the facts that $\tilde{d}_{k-i}^j$ can only be estimated at time $k-i+j$ and the current measurement is $y_k$. Using (36), the input reconstruction model (43) can be rewritten as follows:

$$d_{k-s} = \sum_{i=0}^{s} \bar{\Pi}_{k-s}^i (\tilde{T}^i \check{d}_{k-s+i}^s), \quad (44)$$

where matrix $\tilde{T}^i$ satisfies the following relationship: $\tilde{d}_k^i = \tilde{T}^i \check{d}_{k+i}^s$. However, for system (37), (41) only the contaminated time-distributed unknown input vector $\check{d}_k^{s*}$ can be estimated. Thus, the problem remains to recover $\check{d}_k^s$ from $\check{d}_k^{s*}$ and $\check{d}_k^*$. This is addressed as below.

Define the following notation:

$$(\tilde{T}^i \check{d}_k^{s*})^* \equiv (\tilde{T}^i \check{d}_k^s) = (\tilde{T}^i \check{d}_k^{s*}) - \tilde{T}^i (\check{H}_k^x)^+ \check{d}_k^*. \quad (45)$$

Thus, the input reconstruction model (44) can be more conveniently represented as follows:

$$\begin{aligned} d_{k-s} &= \sum_{i=0}^{s} \bar{\Pi}_{k-s}^i (\tilde{T}^i \check{d}_{k-s+i}^{s*})^* \\ &= \sum_{i=0}^{s} \bar{\Pi}_{k-s}^i \left( (\tilde{T}^i \check{d}_{k-s+i}^{s*}) - \tilde{T}^i (\check{H}_{k-s+i}^x)^+ \check{d}_{k-s+i}^* \right). &(46) \end{aligned}$$

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Using (29), (44), and the following facts:

$$\Delta_k^j \bar{\Pi}_k^l = 0, \quad 0 \le l \le j, \tag{47}$$

we can express $\check{d}_{k-s+i}^*$ as follows:

$$\check{d}_{k-s+i}^* = \sum_{j=1}^{s-1} \sum_{l=j+1}^{s} \tilde{\Omega}_{k-s+i}^{jl} (\tilde{T}^l \check{d}_{k-s+i-j+l}^{s*}), \tag{48}$$

where

$$\tilde{\Omega}_k^{jl} = C_k^x (A_{k-1}^x)^{j-1} G_{k-j}^x \Delta_{k-j}^j \bar{\Pi}_{k-j}^l. \tag{49}$$

Note that, in (48) we assume the unknown input subvector $(\tilde{T}^l \check{d}_{k-s+i-j+l}^{s*})$, i.e. $\tilde{d}_{k-s+i-j}^l$, is not contaminated.

From (46) and (48), it is clear that if the following conditions hold:

$$\check{d}_{k-s+i-j+l}^{s*} = 0 \quad (i-j+l > s), \tag{50}$$

which means that the correction will not use future measurements $\{y_{k+1}, \dots, y_{k+\alpha}\}$, the recovery of the contaminated estimates in $d_{k-s}$ can be achieved through a delayed elimination technique. In illustrating this, we consider the same numerical example in [8], which has the following system matrices:

$$A_k^x = \begin{bmatrix} 0.5 & 0.4 & 0 & 0 \\ 0.4 & 0.5 & 0.4 & 0 \\ 0 & 0.4 & 0.5 & 0.4 \\ 0 & 0 & 0.4 & 0.5 \end{bmatrix}, \quad G_k^x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$C_k^x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_k^x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Applying the input reconstruction procedures to the above system, we have the following design parameter matrices associated with the input reconstruction model:

$$\begin{aligned} \bar{\Pi}_k^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\Pi}_k^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \Delta_k^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_k^2 = 0_3, \quad \Pi_k^3 = 0_3. \end{aligned} \tag{51}$$

From (51), it is clear that the unknown inputs can be estimated with two-step delayed input estimation, i.e., $s = 2$. Moreover, from (48) one has

$$\check{d}_k^* = \tilde{\Omega}_k^{12} (\tilde{T}^2 \check{d}_{k+1}^{s*}), \quad \tilde{\Omega}_k^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

Using (26), (35), (51), and (52), we can obtain the following relationship:

$$\begin{aligned} & \check{H}_k^x \check{d}_k^s + \check{d}_k^* \\ = & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 \end{bmatrix} \check{d}_k^s \\ & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\tilde{T}^2 \check{d}_{k+1}^{s*}), \\ = & \check{H}_k^x (\check{d}_k^s + (\check{H}_k^x)^+ \tilde{\Omega}_k^{12} (\tilde{T}^2 \check{d}_{k+1}^{s*})) = \check{H}_k^x \check{d}_k^{s*}, \end{aligned} \tag{53}$$

which verifies (37) and (38). Examining (53), one can easily verify that only the first element of $\tilde{T}^0 \check{d}_k^s (= \tilde{d}_k^0)$ is corrupted by the third element of $\tilde{T}^2 \check{d}_{k+1}^s (= \tilde{d}_{k-1}^2)$ as follows:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\tilde{T}^0 \check{d}_k^{s*}) \\ = & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\tilde{T}^0 \check{d}_k^s) + \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix} (\tilde{T}^2 \check{d}_{k+1}^s) \\ = & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\tilde{T}^0 \check{d}_k^s) + \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix} (\tilde{T}^2 \check{d}_{k+1}^{s*}), \end{aligned} \tag{54}$$

where the last equality holds because the vector $\tilde{d}_{k-1}^2$ is not contaminated. Using (54) in (46) yields the following true input reconstruction model:

$$\begin{aligned} d_{k-2} = & \bar{\Pi}_{k-2}^0 (\tilde{T}^0 \check{d}_{k-2}^{s*}) - \breve{\Omega}_{k-2}^0 (\tilde{T}^2 \check{d}_{k-1}^{s*}) \\ & + \sum_{i=1}^{2} \left( \bar{\Pi}_{k-2}^i (\tilde{T}^i \check{d}_{k-2+i}^{s*}) - \breve{\Omega}_{k-2}^i (\tilde{T}^i \check{d}_{k-2+i}^{s*}) \right). \end{aligned}$$

where

$$\breve{\Omega}_{k-2}^0 = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \breve{\Omega}_{k-2}^1 = \breve{\Omega}_{k-2}^2 = 0_3.$$

The effectiveness of the above recovery process can be found in [8].

## IV. Optimal Multi-Step-Delay Input and State Estimation Design

### A. SDC-Based RSF Filter Design

Now, we are in the position to present the optimal estimation results of the reformed system (37), (41); this is achieved by using the recently developed RTSKF (robust two-stage Kalman filter) in [1]. For easy reference, the obtained reformed state filter (RSF) is given as follows:

$$\hat{\bar{x}}_{k|k}^s = \hat{\bar{x}}_{k|k} + V_k^{\bar{x}} \check{d}_{k|k}^{s*}, \tag{55}$$
$$P_{k|k}^{\bar{x}^s} = P_{k|k}^{\bar{x}} + V_k^{\bar{x}} P_{k|k}^{\check{d}^{s*}} (V_k^{\bar{x}})^T, \tag{56}$$

where the input-free filter $\hat{\bar{x}}_{k|k}$ is given by:

$$\hat{\bar{x}}_{k|k-1} = A_{k-1}^x \hat{\bar{x}}_{k-1|k-1} + U_{k-1} \check{d}_{k-1|k-1}^{s*}, \tag{57}$$
$$\hat{\bar{x}}_{k|k} = \hat{\bar{x}}_{k|k-1} + K_k^{\bar{x}} (y_k - C_k^x \hat{\bar{x}}_{k|k-1}), \tag{58}$$
$$\begin{aligned} P_{k|k-1}^{\bar{x}} = & A_{k-1}^x P_{k-1|k-1}^{\bar{x}} (A_{k-1}^x)^T \\ & + U_{k-1} P_{k-1|k-1}^{\check{d}^{s*}} U_{k-1}^T + Q_{k-1}, \end{aligned} \tag{59}$$
$$K_k^{\bar{x}} = P_{k|k-1}^{\bar{x}} (C_k^x)^T \tilde{R}_k^{-1}, \tag{60}$$
$$P_{k|k}^{\bar{x}} = (I - K_k^{\bar{x}} C_k^x) P_{k|k-1}^{\bar{x}}, \tag{61}$$

where

$$\tilde{R}_k = C_k^x P_{k|k-1}^{\bar{x}} (C_k^x)^T + R_k, \tag{62}$$

the unknown input filter $\check{d}^{s*}_{k|k}$ is given by:

$$\check{d}^{s*}_{k|k} = K^{\check{d}^{s*}}_k (y_k - C^x_k \hat{\bar{x}}_{k|k-1}), \quad (63)$$
$$K^{\check{d}^{s*}}_k = P^{\check{d}^{s*}}_{k|k} (\check{H}^x_k)^T \tilde{R}^{-1}_k, \quad (64)$$
$$P^{\check{d}^{s*}}_{k|k} = \left((\check{H}^x_k)^T \tilde{R}^{-1}_k \check{H}^x_k\right)^+, \quad (65)$$

and the blending matrices $V^{\bar{x}}_k$ and $U_k$ are given by:

$$V^{\bar{x}}_k = -K^{\bar{x}}_k \check{H}^x_k, \quad U_k = A^x_k V^{\bar{x}}_k + \check{G}^x_k. \quad (66)$$

However, owing to that the SDC matrix $A^x_k$ is implemented as $A^x_k = A^x_k(\hat{\bar{x}}^s_{k|k})$, one can more conveniently modify (57) as follows:

$$\hat{\bar{x}}_{k|k-1} = A^x_{k-1} \hat{\bar{x}}^s_{k-1|k-1} + \check{G}^x_{k-1} \check{d}^{s*}_{k-1|k-1}. \quad (67)$$

Note that the existence condition of the above RSF is given as follows:

$$rank[\check{H}^x_k] = m. \quad (68)$$

### B. Optimal Input and State Estimates Construction

In this subsection, we show how to optimally reconstruct the input and state estimates of the original system (5), (6) via the globally optimal RSF designed in Section IV.A.

First, based on the input reconstruction model (44) and the time-distribution unknown input vector recovery (46), we can obtain the optimal multi-step delayed input estimate and its error covariance matrix at time $t$, where $t = k-s$, as follows:

$$\hat{d}_{t|k} = \sum_{i=0}^{s} \bar{\Pi}^i_t (\tilde{T}^i \check{d}^{s*}_{t+i|t+i})^*, \quad (69)$$
$$P^d_{t|k} = \sum_{i=0}^{s}\sum_{j=0}^{s} \bar{\Pi}^j_t \tilde{T}^j \{\bullet\}^{j,i}_t (\bar{\Pi}^i_t \tilde{T}^i)^T, \quad (70)$$

where $\{\bullet\}^{j,i}_t$ is defined as follows:

$$\{\bullet\}^{j,i}_t = \begin{cases} \left(\{\bullet\}^{i,j}_t\right)^T, & j < i \\ P^{\check{d}^{s*}}_{t+i|t+i}, & j = i \\ E\left[\tilde{d}^{s*}_{t+j} (\tilde{d}^{s*}_{t+j})^T\right], & j > i \end{cases}, \quad (71)$$

in which $\tilde{d}^{s*}_k = \check{d}^{s*}_k - \check{d}^{s*}_{k|k}$. Following the derivations in [1], we can obtain the estimation error covariance matrix $\{\bullet\}^{j,i}_t$, where $j > i$, as follows:

$$\{\bullet\}^{j,i}_t = -K^{\check{d}^{s*}}_{t+j} C^x_{t+j} \{\star\}^{j,i}_t, \quad (72)$$

where $\{\star\}^{j,i}_t$ is given as follows:

$$\{\star\}^{j,i}_t = (A^x_{t+l} - A^x_{t+l} K^{\bar{x}}_{t+l} C^x_{t+l} - U_{t+l} K^{\check{d}^{s*}}_{t+l} C^x_{t+l}) \{\star\}^{l,i}_t,$$

in which $l = j-1$, with the initial condition: $\{\star\}^{i,i}_t = U_{t+i} P^{\check{d}^{s*}}_{t+i|t+i}$.

Next, we consider the original state reconstruction. From (31), we can obtain the original state estimate at time $\bar{t}$, where $\bar{t} = k - s + 1$, as follows:

$$\hat{x}_{\bar{t}|k} = \hat{\bar{x}}^s_{\bar{t}|\bar{t}} + \sum_{i=0}^{s-1}\sum_{j=1}^{s-i} (A^x_{\bar{t}-1})^{j-1} G^x_{\bar{t}-j} \bar{\Pi}^{i+j}_{\bar{t}-j} \times (\tilde{T}^{i+j} \check{d}^{s*}_{\bar{t}+i|\bar{t}+i})^*. \quad (73)$$

Finally, we evaluate the error covariance matrix of the state estimation. Defining the following notation:

$$B^i_{\bar{t}} = \sum_{j=1}^{s-i} (A^x_{\bar{t}-1})^{j-1} G^x_{\bar{t}-j} \bar{\Pi}^{i+j}_{\bar{t}-j} \tilde{T}^{i+j}, \quad (74)$$

we can obtain the error covariance matrix at time $\bar{t}$ as follows (see [1] for details):

$$P^x_{\bar{t}|k} = P^{\bar{x}^s}_{\bar{t}|\bar{t}} + \sum_{i=0}^{s-1}\left(B^i_{\bar{t}} P^{\check{d}^{s*}\bar{x}^s}_{\bar{t}+i,\bar{t}} + (B^i_{\bar{t}} P^{\check{d}^{s*}\bar{x}^s}_{\bar{t}+i,\bar{t}})^T\right) + \sum_{i=0}^{s-1}\sum_{j=0}^{s-1} B^j_{\bar{t}} \{\bullet\}^{j,i}_{\bar{t}} (B^i_{\bar{t}})^T, \quad (75)$$

where

$$P^{\check{d}^{s*}\bar{x}^s}_{\bar{t},\bar{t}} = P^{\check{d}^{s*}}_{\bar{t},\bar{t}} (V^{\bar{x}}_{\bar{t}})^T, \quad P^{\check{d}^{s*}\bar{x}^s}_{\bar{t}+i,\bar{t}} = -K^{\check{d}^{s*}}_{\bar{t}+i} C^x_{\bar{t}+i} \{*\}^i_{\bar{t}}, \quad (76)$$

in which $\{*\}^i_{\bar{t}}$ is updated as follows:

$$\{*\}^{i+1}_{\bar{t}} = A^x_{\bar{t}+i}(I - K^{\bar{x}}_{\bar{t}+i} C^x_{\bar{t}+i}) \{*\}^i_{\bar{t}} + U_{\bar{t}+i} P^{\check{d}^{s*}\bar{x}^s}_{\bar{t}+i,\bar{t}}, \quad (77)$$

with the initial condition $\{*\}^1_{\bar{t}} = A^x_{\bar{t}} P^{\bar{x}^s}_{\bar{t},\bar{t}} + \check{G}^x_{\bar{t}} P^{\check{d}^{s*}\bar{x}^s}_{\bar{t},\bar{t}}$.

## V. Performance Evaluation

In order to illustrate the superiority of the proposed method, we consider a special case of the discrete-time two-link manipulator model in [5] that ignores the external disturbance. Thus, the considered system is given as follows:

$$x_{k+1} = f_k(x_k) + G^x_k(u_k + d_k) + w_k, \quad (78)$$
$$y_k = C_k x_k + v_k, \quad C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (79)$$

where $u_k$ is the control input, $d_k$ represents the friction forces acting at the joints of the system,

$$f_k(x_k) = \begin{bmatrix} f^1_k & f^2_k & f^3_k & f^4_k \end{bmatrix}^T, \quad (80)$$
$$G^x_k = \begin{bmatrix} 0 & (G^2_k)^T & 0 & (G^4_k)^T \end{bmatrix}^T, \quad (81)$$

in which

$$f^1_k = x^1_k + T_s x^2_k, \quad (82)$$
$$f^2_k = \left(1 - T_s\theta^1_k \frac{m_{12}m_{21}}{2m_{22}} sin(2(x^1_k - x^3_k))x^2_k\right) x^2_k + T_s\theta^1_k k_{11} sin(x^1_k) - T_s\theta^1_k m_{21} sin(x^1_k - x^3_k)(x^4_k)^2 - T_s\theta^1_k \frac{m_{12}k_{12}}{m_{22}} cos(x^1_k - x^3_k) sin(x^3_k), \quad (83)$$
$$f^3_k = x^3_k + T_s x^4_k, \quad (84)$$
$$f^4_k = \left(1 + T_s\theta^2_k \frac{m^2_{21}}{2m_{11}} sin(2(x^1_k - x^3_k))x^4_k\right) x^4_k + T_s\theta^2_k k_{12} sin(x^3_k) - T_s\theta^2_k m_{21} sin(x^1_k - x^3_k)(x^2_k)^2 - T_s\theta^2_k \frac{m_{21}k_{11}}{m_{11}} cos(x^1_k - x^3_k) sin(x^1_k), \quad (85)$$
$$G^2_k = \begin{bmatrix} T_s\theta^1_k & -T_s\theta^1_k \frac{m_{12}}{m_{22}} cos(x^1_k - x^3_k) \end{bmatrix}, \quad (86)$$
$$G^4_k = \begin{bmatrix} T_s\theta^2_k \frac{m_{21}}{m_{11}} cos(x^1_k - x^3_k) & T_s\theta^2_k \end{bmatrix}, \quad (87)$$
$$\theta^1_k = \frac{m_{22}}{m_{11}m_{22} - m_{12}m_{21}\cos^2(x^1_k - x^3_k)}, \quad (88)$$
$$\theta^2_k = \frac{m_{11}}{m_{11}m_{22} - m_{12}m_{21}\cos^2(x^1_k - x^3_k)}. \quad (89)$$

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Here, $x_k^1$ and $x_k^2$ represent the position angle and its velocity, respectively, of joint 1 and $x_k^3$ and $x_k^4$ represent those of joint 2.

To facilitate filter design, it is necessary to obtain the SDC matrix of the nonlinear functions $f_k(x_k)$. In this study, we give an analytical approach to implement the SDC form (7). Thus, the SDC matrix of $f_k(x_k)$ can be obtained as follows:

$$A_k^x = \begin{bmatrix} 1 & T_s & 0 & 0 \\ A_{21,k}^x & A_{22,k}^x & A_{23,k}^x & A_{24,k}^x \\ 0 & 0 & 1 & T_s \\ A_{41,k}^x & A_{42,k}^x & A_{43,k}^x & A_{44,k}^x \end{bmatrix}, \tag{90}$$

where

$$\begin{aligned} A_{21,k}^x &= T_s\theta_k^1 k_{11}\frac{sin(x_k^1)}{x_k^1}, \\ A_{22,k}^x &= \left(1 - T_s\theta_k^1\frac{m_{12}m_{21}}{2m_{22}}sin(2(x_k^1 - x_k^3))x_k^2\right), \\ A_{23,k}^x &= -T_s\theta_k^1\frac{m_{12}k_{12}}{m_{22}}cos(x_k^1 - x_k^3)\frac{sin(x_k^3)}{x_k^3}, \\ A_{24,k}^x &= -T_s\theta_k^1 m_{21}sin(x_k^1 - x_k^3)x_k^4, \\ A_{41,k}^x &= -T_s\theta_k^2\frac{m_{21}k_{11}}{m_{11}}cos(x_k^1 - x_k^3)\frac{sin(x_k^1)}{x_k^1}, \\ A_{42,k}^x &= -T_s\theta_k^2 m_{21}sin(x_k^1 - x_k^3)x_k^2, \\ A_{43,k}^x &= T_s\theta_k^2 k_{12}\frac{sin(x_k^3)}{x_k^3}, \\ A_{44,k}^x &= \left(1 + T_s\theta_k^2\frac{m_{21}^2}{2m_{11}}sin(2(x_k^1 - x_k^3))x_k^4\right). \end{aligned}$$

In the simulation, the system model used in the proposed method is implemented as follows:

$$\bar{x}_{k+1}^s = A_k^x\bar{x}_k^s + G_k^x u_k + \breve{G}_k^x\tilde{d}_{k-2}^2 + w_k, \tag{91}$$
$$y_k = C_k\bar{x}_k^s + \breve{C}_k G_{k-2}^x\tilde{d}_{k-2}^2 + v_k, \tag{92}$$

where

$$\breve{G}_k^x = A_k^x A_{k-1}^x G_{k-2}^x, \tag{93}$$
$$\breve{C}_k = C_k A_{k-1}^x = \begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 0 & 1 & T_s \end{bmatrix}. \tag{94}$$

On the other hand, the system model used in the measurement-augmented filtering approach, e.g. [5], is given as follows:

$$x_{k+1} = A_k^x x_k + G_k^x u_k + G_k^x d_k + w_k, \tag{95}$$
$$\begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} C_k \\ \breve{C}_{k+1} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ C_{k+1} \end{bmatrix} w_k + \begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix}. \tag{96}$$

From the simulation results and comparing (91)-(94) with (95)-(96), we have the following remarks:

(1) The process noise and the augmented measurement noise are correlated, which will complicate the filter design.

(2) There is a need to accumulate the past measurements in the measurement-augmented filtering approach; this does not conform to the usual filter algorithm design. Furthermore, the computational complex of implementing the filter will increase.

(3) Simulation results show that the state estimates of both algorithms are the same; however, the unknown input estimates of the measurement-augmented filtering approach are slightly worse than those of the proposed approach. One possible explanation to this is that we ignored the correlation effect in the former for the sake of easy implementation. Note that the effectiveness of the measurement-augmented filtering approach can be found in the recent work [5].

(4) There are some overhead in implementing the system reformation, i.e. calculating the matrix $\breve{G}_k^x$ in (93), and acquiring the original state estimates as in (73). Thus, how to reduce the computational cost of the above overhead become a practical issue in applying the proposed method, which will be a research topic in the future.

## VI. Conclusion

In the paper, a multi-step delayed input and state estimation for a class of nonlinear systems using SDC factorization technique is developed. Specifically, the recently developed TUIF method is revisited and refined. Moreover, a possible filtering degradation problem existed in the TUIF has also been completely solved. The effectiveness of the proposed method is verified by considering a two-link manipulator system.

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