Proceedings of the World Congress on Engineering 2019
WCE 2019, July 3-5, 2019, London, U.K.

Alternative Pricing Methods for Shout Call Options

Joanna Goard *

Abstract—**Shout call options are exotic options that give the investor the ability to 'shout' during the life of the option, thus locking in a profit and resetting the strike price to the prevailing spot price. We look at two approaches to value such options. The first approach makes use of canonical variables of the classical heat equation and results in a series solution. In the second approach an integral formulation is used, which can be more amenable to pricing when there is more than one 'shout' allowed.**

Keywords: shout options, strike reset options

1 Introduction

In general, a shout option allows the holder of the option to 'shout' at one or more times during the life of the option and so adjust certain aspects of the option such as the strike price or time to maturity. In their original form however (and possibly still the more well-known form, see e.g.[2]), shout contracts permit the holder to 'shout', in which case the strike price is reset to the then prevailing spot price and a payment at expiry to the holder is locked-in of $\max(S_t - X, 0)$, where S_t is the current asset price and X the strike price. This is further to the usual payoff from the option using the new strike price. Some shout options offer the holder to shout more than once, and the general rule is that if an n shout option is exercised early at time t, the holder receives $\max(S_t - X, 0)$ for a call (at expiry) along with a new at-the-money $(n-1)$ shout option.

The shouting right of the holder necessarily gives rise mathematically to a free boundary problem. Dai *et al* [1] provide an exact representation for the price of a shout floor (which is a special case of the strike reset put (see Section 2) in which the initial strike price is set at zero), and derived a (double) integral representation for a shouting premium for a reset put option. However much of the work done to date is numerical. For example, finite difference methods have successfully been applied to shouts by Windcliff *et al* ([5]). In this paper we look at two approaches to formally derive valuations for shout options and their optimal shout boundaries (OSB). In the first approach the governing PDE is reduced to the classical heat equation and then we make use of the result that for any linear partial differential equation (PDE) with a Lie point symmetry, separation of variables is possible in terms of canonical symmetry coordinates (see [3]). This approach leads to a series solution for the option value. In the second approach we use an integral equation formulation. Both approaches have their advantages. While the first method avoids the need to integrate and finds the coefficients for the option value and OSB simultaneously, the second approach involves the decoupling of the option valuation problem from finding the free boundary and would be able to handle multiple free boundaries corresponding to multiple shouts. Both methods result in the same series form for the OSB.

*School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA. joanna@uow.edu.au

2 Approach 1: Series Solution for Option Value

To price shout call options we begin by focussing on the mathematical model for the related strike reset put option. A *strike reset put* option allows the holder of the put option to 'shout' during the life of the option, upon which the strike of the option is reset to the stock price at the time of the shout. Hence, if the holder does not shout during the life of the option, the payoff at expiry time T from the option is $\max(X - S_T, 0)$, where X is the original exercise or strike price; whereas if a shout is made at time t, then the payoff at expiry time T is given by $\max(S_t - S_T, 0)$. Hence the holder will only shout if $S_t > X$ in order to increase the payoff value. We show in Section 2.3 that the price of a shout call option can be derived from the price of a strike reset put option so we begin by pricing the strike reset put option.

We assume that the stock price S $(= S_t)$ follows the usual risk-neutral lognormal process i.e

$$dS/S = (r - q)dt + \sigma dZ \tag{1}$$

where r, q and σ are the constant risk-free rate, dividend yield and volatility respectively and Z is a Wiener process under a risk-neutral measure. Upon expiry time T, the holder of the strike reset put option receives $\max(X - S_T, 0)$ so that the holder will shout at time $t < T$ only if $S_t > X$. Upon shouting, the option becomes an at-the-money put option, so that its value above the critical

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

shouting boundary, $S_f(t)$ becomes $P(S,t) = P_e(S,t;S)$, where $P_e(S,t;X)$ is the Black-Scholes formula for the European put option with strike X. In the continuation region, $0 \leq S \leq S_f(t)$ the value $P(S,t)$ of the strike reset put option can be shown to satisfy the Black-Scholes (BS) equation

$$\frac{\partial P}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial^2 S} + (r-q)S\frac{\partial P}{\partial S} - rP = 0 \qquad (2)$$

where q is the constant, continuous dividend yield. The option price must necessarily satisfy the smooth-pasting conditions so that across the optimal shouting boundary, the value of the option and its derivative are continuous. Equation (2) then needs to be solved subject to

$$P(S,T) = \max(X-S,0), \qquad (3a)$$

$$P(0,t) = Xe^{-r(T-t)} \qquad (3b)$$

and at $S_f(t) > X$

$$P(S_f(t),t) = S_f(t)[e^{-r(T-t)}N(-d_2) - e^{-q(T-t)}N(-d_1)] \qquad (3c)$$

$$P_S(S_f(t),t) = e^{-r(T-t)}N(-d_2) - e^{-q(T-t)}N(-d_1) \qquad (3d)$$

where T is the expiry date, $N(\cdot)$ is the cumulative distribution function for the standard normal distribution, $d_1 = \frac{(r-q+\frac{\sigma^2}{2})\sqrt{(T-t)}}{\sigma}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

In the following subsection, an exact formal solution will be given for the strike reset put option based on solving (2) subject to (3a)-(3d). The following result will be used, which was proven by Dai *et al* [1].

Result 2.1 The optimal shouting boundary for the strike reset put option takes on the value X at expiry i.e $S_f(T) = X$.

2.1 Solution for the Strike Reset Put Option

Consider the value $V(S,t)$ of a strike reset put option $P(S,t)$ plus a forward contract $f(S,t) = Se^{-q(T-t)} - Xe^{-r(T-t)}$. Because an investor will shout only if $S_f(t) > X$, it is worthwhile to split the continuation domain into the 2 regions $0 \leq S < X$ and $X \leq S \leq S_f(t)$. Note that using Result 2.1, the region $X \leq S \leq S_f(t)$ reduces to a single point at $t = T$. In the continuation region of the reset put option, $V(S,t)$ satisfies Equation (2) which in

$X \leq S \leq S_f(t)$ needs to be solved subject to

$$V(S_f(t),t) = S_f(t)[e^{-r(T-t)}N(-d_2) + e^{-q(T-t)}N(d_1)] - Xe^{-r(T-t)} \qquad (4a)$$

$$V_S(S_f(t),t) = e^{-r(T-t)}N(-d_2) + e^{-q(T-t)}N(d_1) \qquad (4b)$$

and in $0 \leq S < X,$ subject to $V(S,T) = 0$.

We also impose the smooth pasting conditions i.e continuity of the value of the option and its derivative across the strike price i.e

$$\lim_{S\to X^-} V = \lim_{S\to X^+} V \quad \text{and} \quad \lim_{S\to X^-} V_S = \lim_{S\to X^+} V_S.$$

We make the following substitutions, the first two of which are the standard substitutions that reduce the BS equation to the classical heat equation:

$$(a) \quad S = Xe^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V = e^{-q(T-t)}X\nu(x,\tau), \qquad (5)$$

and let $G(\tau) = \ln\left(\frac{S_f(t)}{X}\right)$ where $G(0) = 0$.

$$(b) \quad u(x,\tau) = e^{Ax+B\tau}\nu(x,\tau) \qquad (6)$$

where $A = \frac{k-1}{2}$, $B = \frac{(k+1)^2}{4}$, resulting in the governing equation $u_t = u_{xx}$.

$$(c) \quad y = \frac{x}{\sqrt{\tau}}, \tau = \tau \qquad (7)$$

which are canonical coordinates of $u_t = u_{xx}$.

The problem then becomes

$$\tau u_\tau = u_{yy} + \frac{y}{2}u_y \qquad (8)$$

to be solved subject to $u(y,0) = 0$ for $y < 0$ and for $0 \leq y \leq \Psi(\tau)$, where $\Psi(\tau) = \frac{G(\tau)}{\sqrt{\tau}}$

$$u(\Psi(\tau),\tau) = -e^{A\sqrt{\tau}\Psi(\tau)+B\tau}e^{-k\tau} + e^{(A+1)\sqrt{\tau}\Psi(\tau)+B\tau}\Big[N(d_1) + e^{-k\tau}N(-d_2)\Big] \qquad (9a)$$

$$u_y(\Psi(\tau),\tau) = -A\sqrt{\tau}e^{A\sqrt{\tau}\Psi(\tau)+B\tau}e^{-k\tau} + \sqrt{\tau}(A+1)e^{(A+1)\sqrt{\tau}\Psi(\tau)+B\tau}[N(d_1)+e^{-k\tau}N(-d_2)] \qquad (9b)$$

$$\lim_{y\to 0^-} u = \lim_{y\to 0^+} u \quad \text{and} \qquad (9c)$$

$$\lim_{y\to 0^-} u_y = \lim_{y\to 0^+} u_y. \qquad (9d)$$

Equation (8) admits separable solutions of the form

$$u(y,\tau) = e^{\frac{-y^2}{4}}\sum_{i=1}^{\infty}\tau^{\frac{i}{2}}\left[C_i M\left(\frac{1+i}{2},\frac{1}{2},\frac{y^2}{4}\right) + D_i U\left(\frac{1+i}{2},\frac{1}{2},\frac{y^2}{4}\right)\right], \qquad (10)$$

where M and U are the Kummer-M and Kummer-U functions respectively. The separation constant used in (10) is $\lambda_i = \frac{i}{2}$ where i is a positive integer, as power series in square root time have been found to be adequate in solving other free boundary problems involving linear diffusion equations (see e.g [4]). The above Equation (10) describes solutions valid for $0 \leq y \leq \Psi(\tau)$ for some non-zero constants C_i, D_i to be determined, while for $y < 0$,

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

consideration of the initial condition implies solutions of the form

$$u(y,\tau) = e^{\frac{-y^2}{4}} \sum_{i=1}^{\infty} \tau^{\frac{i}{2}} F_i U\left(\frac{1+i}{2}, \frac{1}{2}, \frac{y^2}{4}\right) \tag{11}$$

for constants F_i to be determined.

Determining the Solution Coefficients

In order to satisfy the limit conditions at $y = 0$ we require

$$F_i = \frac{\Gamma(1+\frac{i}{2})}{\sqrt{\pi}} C_i + D_i \quad \text{and} \tag{12a}$$

$$F_i = -D_i \tag{12b}$$

so that we set $C_i = \frac{-2\sqrt{\pi}}{\Gamma(1+\frac{i}{2})} D_i$.

Note that for continuity at $x = 0$ of the second derivatives, we require

$$\frac{\sqrt{\pi}}{\Gamma(1+\frac{i}{2})} iF_i = iC_i + \frac{\sqrt{\pi}}{\Gamma(1+\frac{i}{2})} iD_i$$

but this follows automatically from (12a). Hence derivatives of all orders are continuous at $x = 0$.

We let $\Psi(\tau) = \sum_{i=0}^{\infty} s_i \tau^{i/2}$. This is motivated by the classic work of Tao (see e.g. [4]) on Stefan problems in general. Now apply (9a) and (9b) to determine the coefficients s_i and D_i, using the following steps:

(i) Using (10) expand both sides of Equation (9a) in a power series of $\tau^{i/2}$. We call this Expansion A.

(ii) Similarly expand both sides of Equation (9b) in a power series of $\tau^{i/2}$ and call this Expansion B.

Both expansions have no constant terms.

(iii) By equating coefficients of $\tau^{1/2}$ from both sides of Expansion A and similarly from both sides of Expansion B, yields 2 equations to solve for the unknowns D_1 and s_0. The solution to this system (to 4 decimal places) is $s_0 = 1.0304$ and $D_1 = -0.3678$.

Then, equating coefficients of powers of τ from Expansions A and B yields 2 equations to solve for D_2 and s_1. In general we can then continue equating coefficients of $\tau^{\frac{i}{2}}$ from Expansions A and B to get as many constants D_i and s_{i-1} as necessary. These coefficients will be in terms of $k\left(= \frac{2(r-q)}{\sigma^2}\right)$. Then constants F_i can be determined from (12b).

Undoing the change of variables (7), (6), (5) gives $V(S,t)$. Then subtracting the value of the forward contract gives the solution for $P(S,t)$. We thus have

Theorem 2.1: The following series formally satisfies the free boundary problem (2)-(3) for the strike reset put option in the continuation region $0 \le S \le S_f(t)$:

$$\begin{aligned} P(S,t) = {} & g(S,t) \sum_{i=1}^{\infty} \left[\frac{\sigma^2}{2}(T-t)\right]^{i/2} D_i \left\{ U\left(\frac{1+i}{2}, \frac{1}{2}, \frac{(\ln(S/X))^2}{2\sigma^2(T-t)}\right) \right. \\ & \left. + \frac{-2\sqrt{\pi}}{\Gamma(1+\frac{i}{2})} M\left(\frac{1+i}{2}, \frac{1}{2}, \frac{(\ln(S/X))^2}{2\sigma^2(T-t)}\right) \right\} \quad (13a) \\ & -Se^{-q(T-t)} + Xe^{-r(T-t)} \qquad \text{for } X \le S \le S_f(t) \\ = {} & -g(S,t) \sum_{i=1}^{\infty} \left[\frac{\sigma^2}{2}(T-t)\right]^{i/2} D_i U\left(\frac{1+i}{2}, \frac{1}{2}, \frac{(\ln(S/X))^2}{2\sigma^2(T-t)}\right) \\ & -Se^{-q(T-t)} + Xe^{-r(T-t)} \qquad \text{for } 0 \le S < X \quad (13b) \end{aligned}$$

where

$$g(S,t) = e^{-q(T-t)} X \left(\frac{S}{X}\right)^{\frac{1-k}{2}} e^{-\frac{(k+1)^2\sigma^2(T-t)}{8}} e^{-\frac{(\ln(\frac{S}{X}))^2}{2\sigma^2(T-t)}}, \tag{14}$$

$k = \frac{2(r-q)}{\sigma^2}$, and the optimal shouting boundary is given by

$$S_f(t) = X \exp\left(\sum_{i=0}^{\infty} s_i \left[\frac{\sigma^2}{2}(T-t)\right]^{\frac{(i+1)}{2}}\right). \tag{15}$$

The coefficients s_i and D_i are determined from the boundary conditions at $S_f(t)$ as described above. The first seven coefficients $s_0 - s_6$ for the optimal boundary and the first seven coefficients for the option value $D_1 - D_7$ are listed in the Appendix.

Plots showing a comparison of option values with different times to expiry using (13a, 13b) with $\sigma = 0.2,\ X = 1,\ r = 0.03,\ q = 0.02$ are shown in Figure 1.

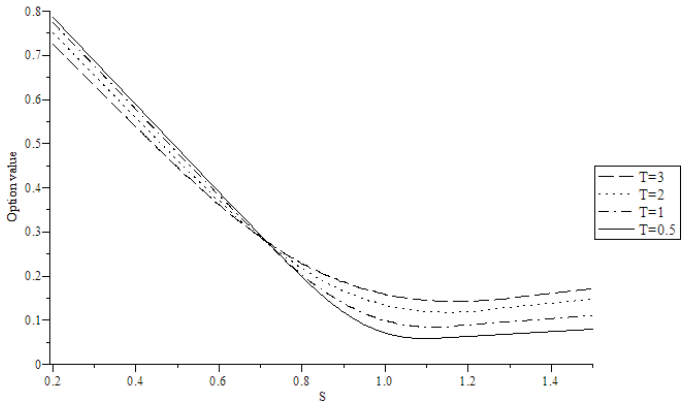

Figure 1: Comparison of strike reset put prices with different times to expiry. Parameters used were $X = 1,\ r = 0.03,\ \sigma = 0.2$ and $q = 0.02$.

As with vanilla put options, the longer-lived options are more valuable except for small values of S. Each of the price curves in Figure 1 touch tangentially the value of the

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

at-the-money put option at the stock price corresponding to the optimal shout boundary. A comparison of the optimal shout boundaries for different q values and with $X = 1$ and $r = 0.03,\ \sigma = 0.2$ is given in Figure 2.

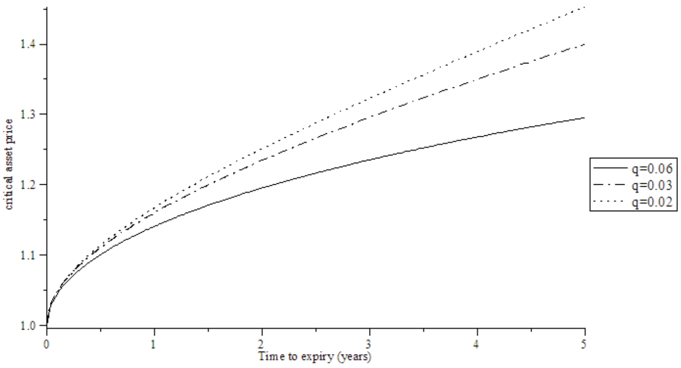

Figure 2: Optimal shout boundaries with parameters $X = 1,\ r = 0.03$ and $\sigma = 0.2$.

From this it can be seen that the larger the dividend yield, then the smaller the optimal shout boundary. This is to be expected as dividend yields have the effect of lowering the growth rate in the stock price. Notably also, when $r < q$ the optimal shout boundary has a lower slope and a greater curvature.

2.2 Examples and Comparisons

We compared results from the series truncated at seven terms, with accurate solutions for a 5 year reset put option, obtained by using 50000 time steps in the Binomial Model, as given by Dai [1] with parameter values $T - t = 5,\ S = 100,\ r = 0.06$ and $0.03,\ q = 0.03$ and $0.06,\ \sigma = 0.2$ and 0.3. It was found that better accuracy was achieved with fewer terms in our formal series solutions when σ is larger and when $r < q$. However in all cases, at last 4 significant figure accuracy was achieved using seven terms.

In order to understand how many terms the series generally requires in order to attain good accuracy, we tried a number of examples with $T - t = 0.5, 1, 2, 3, q = 0.06$, and $0.02,\ r = 0.03,\ X = 1$ and found the number of terms s_n needed in Equation (15) and the number of terms D_n needed in Equations (13a) or (13b) to achieve 4 decimal place accuracy in $S_f(t)$ and $P(S,t)$ respectively. In each case the 'exact' solution was taken as the value where the *relative* differences between successive values using n and $n + 1$ terms was less than 10^{-4}. In every case, the practical criterion for accuracy was satisfied with the number of terms s_n required ranging from 3 to 5 and the number of terms D_n ranging from 2 to 5. In general, the larger the time to expiry, the more terms were needed and when $r > q$ then more s_i and D_i terms were sometimes needed for the same accuracy in S_f and P. Hence for all cases considered, the new series solutions provided fast and accurate answers for times to expiry up to three years.

We now use the results from this section to value shout call options.

2.3 Solution for the Shout Call Option

Theorem 2.2:

An exact formal solution for the shout call option $V(S,t)$ in the continuation region $0 \le S \le S_f(t)$ is

$$\begin{aligned} V(S,t) &= g(S,t)\sum_{i=1}^{\infty}\left[\frac{\sigma^2}{2}(T-t)\right]^{i/2}\left\{D_i U\left(\frac{1+i}{2},\frac{1}{2},\frac{\ln(S/X)^2}{2\sigma^2(T-t)}\right)\right. \\ &\quad \left. + \frac{-2\sqrt{\pi}D_i}{\Gamma(1+\frac{i}{2})}M\left(\frac{1+i}{2},\frac{1}{2},\frac{\ln(S/X)^2}{2\sigma^2(T-t)}\right)\right\} \qquad (16a) \\ &\qquad\qquad \text{for } X \le S \le S_f(t) \\ &= -g(S,t)\sum_{i=1}^{\infty}\left[\frac{\sigma^2}{2}(T-t)\right]^{i/2} D_i U\left(\frac{1+i}{2},\frac{1}{2},\frac{\ln(S/X)^2}{2\sigma^2(T-t)}\right) \\ &\qquad \text{for } 0 \le S < X \qquad (16b) \end{aligned}$$

where $g(S,t)$ is given in (14), $k = \frac{2(r-q)}{\sigma^2}$ and the optimal shout boundary is given by (15). The constant coefficients D_i and s_i are again determined as described in Section 2.1.

Note that in the region $S_f(t) > X$, the value of the shout call option is simply $C_e(S,t;S) + (S - X)e^{-r(T-t)}$ where $C_e(S,t;X)$ is the Black-Scholes value of a European call with strike price X.

Proof: The holder of a shout call option can 'shout' if $S_t > X$, which resets the strike price to S_t and locks in a payment they will receive at expiry of $S_t - X$. Thus, the payoff from the shout call option is $C_{sh}(S,T) = \max(S_T - X, S_t - X, 0)$. As $\max(S_T - X, S_t - X, 0) = \max(S_t - S_T, 0) + (S_T - X)$, then the shout call option can be replicated using a strike reset put option and a long position in a forward contract with the same strike. Hence the value of the shout call option is simply $V(S,t)$ from Section 2.1 - i.e. the sum of the value of the strike reset put option and a forward contract ie.

$$C_{sh}(S,t) = V(S,t) = P(S,t) + Se^{-q(T-t)} - Xe^{-r(T-t)}$$

where $P(S,t)$ is given in (13a, 13b). The optimal shout boundary is then the same as for the strike reset put option which is given in (15).##

Plots showing a comparison of shout call option values with different times to expiry are given in Figure 3 with parameter values $X = 1,\ r = 0.03,\ q = 0.02,\ \sigma = 0.2$. As with European call options, the values increase with time to expiry.

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

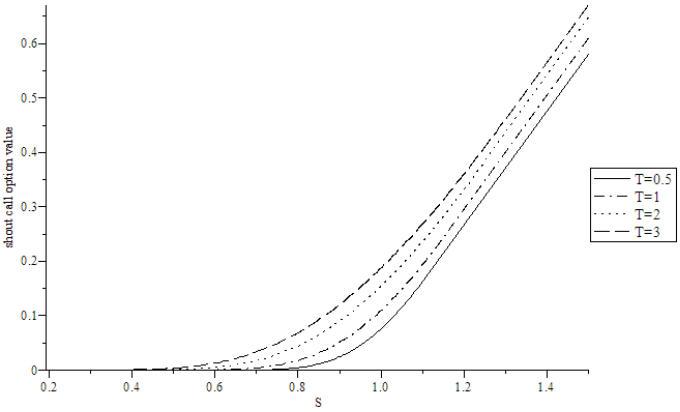

Figure 3: Comparison of shout call option prices with different times to expiry. Parameters used were $X = 1,\ r = 0.03,\ q = 0.02,\ \sigma = 0.2$.

3 Approach 2: An Integral Equation Formulation

With $\tilde{\tau} = T - t$, the BS equation can be written as

$$\mathcal{L}V = \left[\frac{\partial}{\partial\tilde{\tau}} - \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} - (r-q)\,S\frac{\partial}{\partial S} + r\right]V = 0 \quad (17)$$

where we have defined the operator $\mathcal{L}$ for later use.

For American-style options with early exercise features, it follows from an application of Green's theorem that if such an option obeys equation (17) where it is optimal to hold the option, while that from immediate exercise is $P(S,\tilde{\tau})$, then we can write the value of the option as the sum of the value of the corresponding European option $V^{(e)}(S,\tilde{\tau})$ together with another term representing the premium from early exercise,

$$V(S,\tilde{\tau}) = V^{(e)}(S,\tilde{\tau}) + \int_0^{\tilde{\tau}}\int_0^{\infty}\mathcal{F}(Z,\zeta)G(S,Z,\tilde{\tau}-\zeta)dZd\zeta. \quad (18)$$

where G is the Green's function,

$$G(S,Z,\tilde{\tau}) = \frac{e^{-r\tilde{\tau}}}{Z\sigma\sqrt{2\pi\tilde{\tau}}}\exp\left(-\frac{\left[\ln(S/Z) + r_2\tilde{\tau}\right]^2}{2\sigma^2\tilde{\tau}}\right) \quad (19)$$

and we have introduced the shorthand $r_2 = r - q - \sigma^2/2$. In this equation, $\mathcal{F}(S,\tilde{\tau})$ is equal to 0 where it is optimal to hold the option while where exercise is optimal $\mathcal{F}(S,\tilde{\tau})$ is the result of substituting the early exercise payoff $P(S,\tilde{\tau})$ into BS partial differential equation,

$$\mathcal{F}(S,\tilde{\tau}) = \mathcal{L}P, \quad (20)$$

where the operator $\mathcal{L}$ was defined in (17). Similarly, shout options satisfy $\mathcal{L}V = \mathcal{F}(S,\tilde{\tau})$, where $\mathcal{F} = 0$ when it is not optimal to shout and otherwise $\mathcal{F} = \mathcal{L}P$ where P is the payoff from shouting. Hence for shout options, we can use the formulae (18,20) recursively. We shall use the notation $V^{(n)}(S,\tilde{\tau})$ for the value of a shout option with n shouting opportunities and $X^{(n)}$ for the strike price of an n shout option. If held until expiry, an n shout call will pay $\max(S - X^{(n)}, 0)$ while a put will pay $\max(X^{(n)} - S, 0)$. At the first shout, which will occur at the free boundary $S_f^{(n)}(\tilde{\tau})$, we exchange this n shout option for a lock-in payment at expiry of the difference between the current stock price S and the strike price $X^{(n)}$ together with a new at-the-money $(n-1)$ shout option $V^{(n-1)}(S,\tilde{\tau})\mid_{X^{(n-1)}=S}$. At-the-money in this context means that the strike price $X^{(n-1)}$ of this new $(n-1)$ shout option is set equal to the stock price S at the time of exercise. It follows that the payoff from exercise for an n shout call is

$$P^{(n)}(S,\tilde{\tau}) = (S - X^{(n)})e^{-r\tilde{\tau}} + V^{(n-1)}(S,\tilde{\tau})\mid_{X^{(n-1)}=S} \quad (21)$$

[while for an n shout put it is

$$P^{(n)}(S,\tilde{\tau}) = (X^{(n)} - S)e^{-r\tilde{\tau}} + V^{(n-1)}(S,\tilde{\tau})\mid_{X^{(n-1)}=S}]$$

An option with zero shouts remaining is just a vanilla European, $V^{(0)}(S,\tilde{\tau}) = V^{(e)}(S,\tilde{\tau})$. If we exercise a one shout option, we will receive at expiry the difference between the stock price and the initial strike price and we receive immediately a zero shout option which is simply an at-the-money European option. So the payoff from shouting for a one shout call, with $r_1 = r_2 + \sigma^2$, is

$$\begin{aligned} P^{(1)}(S,\tilde{\tau}) &= (S - X^{(1)})e^{-r\tilde{\tau}} + V^{(0)}(S,\tilde{\tau})\mid_{X^{(0)}=S}, \\ &= (S - X^{(1)})e^{-r\tilde{\tau}} \quad (22) \\ &+ \frac{S}{2}\left[e^{-q\tilde{\tau}}\mathrm{erfc}\left(-\frac{r_1\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right) - e^{-r\tilde{\tau}}\mathrm{erfc}\left(-\frac{r_2\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right)\right], \end{aligned}$$

Using (20), the forcing term in the formula (18) for a one shout call is

$$\begin{aligned} &\mathcal{F}^{(1)}(S,\tilde{\tau}) \quad (23) \\ &= -\frac{S}{2}e^{-r\tilde{\tau}}\left[(r-q)\mathrm{erfc}\left(\frac{r_2\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right) - \frac{\sigma}{\sqrt{2\pi\tilde{\tau}}}\exp\left(-\frac{r_2^2\tilde{\tau}}{2\sigma^2}\right)\right] \end{aligned}$$

Applying formula (18) to a shout call where it is optimal to hold if $S < S_f^{(1)}(\tilde{\tau})$ and exercise if $S \geq S_f^{(1)}(\tilde{\tau})$, we find

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

the value of a one shout call,

$$
\begin{aligned}
&V^{(1)}(S,\tilde{\tau}) = V^{(e)}(S,\tilde{\tau}) \\
&+\int_0^{\tilde{\tau}}\int_{S_f^{(1)}(\zeta)}^{\infty}\left[\left(\frac{-rZ}{2}e^{-r\zeta}+\frac{qZ}{2}e^{-r\zeta}\right)\operatorname{erfc}\left(\frac{r_2\zeta}{\sigma\sqrt{2\zeta}}\right)\right. \\
&\left.+\frac{Z\sigma e^{-r\zeta}}{2\sqrt{2\pi\zeta}}\exp\left(-\frac{r_2^2\zeta}{2\sigma^2}\right)\right]G(S,Z,\tilde{\tau}-\zeta)dZd\zeta \\
&=\frac{Se^{-q\tilde{\tau}}}{2}\operatorname{erfc}\left(-\frac{\ln(S/X^{(1)})+r_1\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right) \\
&-\frac{Xe^{-r\tilde{\tau}}}{2}\operatorname{erfc}\left(-\frac{\ln(S/X^{(1)})+r_2\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right) \\
&+\int_0^{\tilde{\tau}}\left[\frac{-(r-q)}{2\sigma\sqrt{2\pi}\sqrt{\tilde{\tau}-\zeta}}e^{-r\tilde{\tau}}\operatorname{erfc}\left(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\right)\right. \\
&\left.+\frac{e^{-r\tilde{\tau}}}{4\pi\sqrt{\zeta}\sqrt{\tilde{\tau}-\zeta}}\exp\left(-\frac{r_2^2\zeta}{2\sigma^2}\right)\right] \\
&\times\frac{\sigma S\sqrt{\pi(\tilde{\tau}-\zeta)}}{\sqrt{2}}e^{(r-q)(\tilde{\tau}-\zeta)} \\
&\times\operatorname{erfc}\left(\frac{\ln(S_f(\zeta))-\ln(S)-r_1(\tilde{\tau}-\zeta)}{\sigma\sqrt{2(\tilde{\tau}-\zeta)}}\right)d\zeta. \qquad (24)
\end{aligned}
$$

3.1 Integral Equations For One Shout Options

To obtain integral equations for the location of the free boundary $S = S_f^{(1)}(\tilde{\tau})$ for a one shout option we substitute the expressions for the one shout call (24) into the conditions at the free boundary. To simplify the analysis, we will write $S_f^{(1)}(\tilde{\tau}) = X^{(1)}\exp\left(x_f^{(1)}(\tilde{\tau})\right)$, noting as in Section 2 that the free boundary starts from $S_f^{(1)}(0) = X^{(1)}$ or equivalently $x_f^{(1)}(0) = 0$ at expiry. The conditions at the free boundary are that the option price and the delta of the option are continuous there, so that $V^{(1)} = P^{(1)}$ and $(\partial V^{(1)}/\partial S) = (\partial P^{(1)}/\partial S)$ at $S = S_f^{(1)}(\tilde{\tau})$. This is equivalent to (4a) and (4b).

Using (24,22) in the condition that $V^{(1)} = P^{(1)}$ at the free boundary $S = S_f^{(1)}(\tilde{\tau})$ yields the first equation, say Equation X while the condition that $(\partial V^{(1)}/\partial S) = (\partial P^{(1)}/\partial S)$ gives the second which we call Equation Y.

These resultant two equations constitute a pair of integral equations for the location of the free boundary $x_f^{(1)}(\tilde{\tau}) = \ln(S_f^{(1)}(\tilde{\tau})/X^{(1)})$ for a one shout call. To solve the integral equations X and Y we assume as in Section 2, a solution of the form

$$
x_f^{(1)}(\tilde{\tau}) \sim \sum_{n=1}^{\infty} x_n^{(1)}\tilde{\tau}^{n/2}. \qquad (25)
$$

We substitute the series (25) for $x_f^{(1)}(\tilde{\tau})$ into Equations X and Y and expand, collect and equate powers of $\tilde{\tau}$. To evaluate the integrals, we make the change of variable $\zeta = \tilde{\tau}\eta$, which enables us to pull the $\tilde{\tau}$ dependence outside of the integrals when we expand. From Equation X, at $\mathcal{O}\left(\tilde{\tau}^{1/2}\right)$ we find

$$
\begin{aligned}
&\frac{x_1^{(1)}}{2}\operatorname{erfc}\left(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\right)+\frac{\sigma}{\sqrt{2\pi}}\left(1-\exp\left(-\frac{x_1^{(1)2}}{2\sigma^2}\right)\right) \\
&=\frac{\sigma}{4\sqrt{2\pi}}\int_0^1\frac{1}{\sqrt{\eta}}\operatorname{erfc}\left(-\frac{x_1^{(1)}\left(1-\sqrt{\eta}\right)}{\sigma\sqrt{2\left(1-\eta\right)}}\right)d\eta, \qquad (26)
\end{aligned}
$$

while from Equation Y, at $\mathcal{O}\left(\tilde{\tau}^{0}\right)$ we find

$$
\begin{aligned}
&\frac{1}{2}\operatorname{erfc}\left(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\right) \qquad (27) \\
&=\frac{1}{4\pi}\int_0^1\frac{1}{\sqrt{\eta(1-\eta)}}\exp\left(-\frac{x_1^{(1)2}\left(1-\sqrt{\eta}\right)^2}{2\sigma^2\left(1-\eta\right)}\right)d\eta.
\end{aligned}
$$

These two equations (26,27) have a numerical root $x_1^{(1)} = 0.728600109\sigma$, which agrees with the coefficient of $(T-t)^{1/2}$ in (15), namely $\frac{s_0\sigma}{\sqrt{2}}$.

Continuing with our expansion, at the next order, from Equation X at $\mathcal{O}\left(\tilde{\tau}\right)$ and from Equation Y at $\mathcal{O}\left(\tilde{\tau}^{1/2}\right)$ we find the resulting equations have a numerical root of $x_2^{(1)} = 0.5516261057(r-q)+0.04898978883\sigma^2$. Similarly the next term is found to be $x_3^{(1)} = 0.413244516\frac{(r-q)^2}{\sigma} + 0.218773888\sigma(r-q) + 0.003039954446\sigma^3$. These terms agree with those found in Section 2.

3.2 Two Shout Options - Outline

Although a detailed study of multiple shout options is beyond the scope of this study, we will touch on the free boundary for a two shout option. As we noted earlier, for a two shout option the payoff at the free boundary $S_f^{(2)}(\tilde{\tau})$ is the present value of the difference between the stock price and the strike price together with an at-the-money one shout option.

Using the general expression (21),

$$
\begin{aligned}
&P^{(2)}(S,\tilde{\tau}) = (S-X^{(2)})e^{-r\tilde{\tau}} + V^{(1)}(S,\tilde{\tau})\mid_{X^{(1)}=S} \\
&=(S-X^{(2)})e^{-r\tilde{\tau}} \\
&+\frac{Se^{-q\tilde{\tau}}}{2}\operatorname{erfc}\left(-\frac{r_1\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right)-\frac{Se^{-r\tilde{\tau}}}{2}\operatorname{erfc}\left(-\frac{r_2\tilde{\tau}}{\sigma\sqrt{2\tilde{\tau}}}\right) \\
&+\int_0^{\tilde{\tau}}\left[\frac{-(r-q)}{2\sigma\sqrt{2\pi}\sqrt{\tilde{\tau}-\zeta}}e^{-r\tilde{\tau}}\operatorname{erfc}\left(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\right)\right. \qquad (28) \\
&\left.+\frac{e^{-r\tilde{\tau}}}{4\pi\sqrt{\zeta}\sqrt{\tilde{\tau}-\zeta}}\exp\left(-\frac{r_2^2\zeta}{2\sigma^2}\right)\right] \\
&\times\frac{\sigma S\sqrt{\pi(\tilde{\tau}-\zeta)}}{\sqrt{2}}e^{(r-q)(\tilde{\tau}-\zeta)}\operatorname{erfc}\left(\frac{x_f^{(1)}(\zeta)-r_1(\tilde{\tau}-\zeta)}{\sigma\sqrt{2(\tilde{\tau}-\zeta)}}\right)d\zeta.
\end{aligned}
$$

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Using this payoff (28) in the formula (18,20), we can arrive at a set of integral equations which involve $S_f^{(1)}(\tilde{\tau})$ (which in principle is now known), as well as $S_f^{(2)}(\tilde{\tau})$. As with the one shout option, we will solve these equations to find expressions for the location of the free boundary $x_f^{(2)}(\tilde{\tau}) = \ln(S_f^{(2)}(\tilde{\tau})/X^{(2)})$ in the limit $\tilde{\tau} \to 0$. In doing so, we use the series we found $x_f^{(1)}(\tilde{\tau})$ earlier. This again is an example of how the pricing of shout options is a recursive problem: to find the free boundary $S_f^{(n)}(\tilde{\tau})$ for an n shout option, we first need to know $S_f^{(1)}(\tilde{\tau})$, $S_f^{(2)}(\tilde{\tau})$, $\cdots$, $S_f^{(n-1)}(\tilde{\tau})$.

We again assume that $x_f^{(1)}(\tilde{\tau})$ has the form (25), with the coefficients found earlier, and that $x_f^{(2)}(\tilde{\tau})$ has the form

$$x_f^{(2)}(\tilde{\tau}) \sim \sum_{n=1}^{\infty} x_n^{(2)} \tilde{\tau}^{n/2}. \qquad (29)$$

At leading order, we find a pair of equations which have a numerical root $x_1^{(2)} = 0.478602511\sigma$. Continuing with our expansion, at the next order, we find a pair of equations which solve to give $x_2^{(2)} = 0.3691038999(r-q) + 0.04142004125\sigma^2$.

4 Discussion

In this paper we have demonstrated how exact formal solutions to shout call options (and also strike reset put options) can be found. We used both a PDE approach 1) utilising canonical coordinates and resulting in series solutions (16a, 16b) and 2) using an integral equation approach (24). Both methods necessarily resulted in the same series solution for the OSB.

Once the coefficients in the solutions of the OSB (and the option value for the first approach) have been determined in terms of $k\left(=\frac{2(r-q)}{\sigma^2}\right)$, (and this need only be done once) then the solutions can provide fast, accurate valuations for times to expiry that are not impractically large.

In addition to the solution for the one shout calls, we showed how it is possible to use the formulae (18, 20) recursively to price n shout options for which the early exercise payoff is the difference between the current stock price and the strike price (paid at expiry), together with a new at-the-money $(n-1)$ shout option.

These solutions can potentially not only be very useful to practitioners, but they can provide insight and be valuable benchmarks against which numerical schemes can be tested.

5 Appendix

This appendix lists the first seven coefficients s_i for Equation (15) and the first seven coefficients D_i for equations (13a)-(13b) and (16a)-(16b) in terms of $k = \frac{2(r-q)}{\sigma^2}$.

$$\begin{aligned}
s_0 &= 1.030396\\
s_1 &= 0.0979796 + 0.551626k\\
s_2 &= 0.00859713 + 0.309393k + 0.292208k^2\\
s_3 &= 0.00045512 + 0.125178k + 0.357924k^2 + 0.20398k^3\\
s_4 &= -1.7188 \times 10^{-5} + 0.04264k + 0.270601k^2\\
&\quad + 0.38222k^3 + 0.1639k^4\\
s_5 &= -6.53893 \times 10^{-6} + 0.0129311k + 0.157956k^2\\
&\quad + 0.420397k^3 + 0.415477k^4 + 0.140830k^5\\
s_6 &= -5.0169 \times 10^{-7} + 0.0035923k + 0.0775674k^2\\
&\quad + 0.346227k^3 + 0.596232k^4 + 0.453015k^5 + 0.127271k^6\\
D_1 &= -0.367849\\
D_2 &= -0.0586993 - 0.266876k\\
D_3 &= -0.103131 + 0.0201712k - 0.229123k^2\\
D_4 &= -0.0170184 - 0.123276k - 0.0307579k^2\\
&\quad - 0.137031k^3\\
D_5 &= -0.0263029 + 0.0131288k - 0.211362k^2\\
&\quad + 0.00824740k^3 - 0.0929249k^4\\
D_6 &= -0.00437383 - 0.0440872k - 0.0329975k^2\\
&\quad - 0.166962k^3 - 0.0144274k^4 - 0.0524000k^5\\
D_7 &= -0.00660354 + 0.00556156k - 0.120486k^2\\
&\quad + 0.00669872k^3 - 0.181079k^4 - 0.000111471k^5\\
&\quad - 0.0326374k^6
\end{aligned}$$

References

[1] Dai, M., Kwok, Y.K., Wu, L., "Optimal shouting policies of options with strike reset right," *Math. Finance,* V14, N3, pp. 383–401, /04

[2] Hull, J.C., *Options, Futures and Other Derivatives*, 7th Edition, Pearson Prentice Hall, 2009.

[3] Miller, W. Jr., *Symmetry and Separation of Variables*, Addison-Wesley, 1977.

[4] Tao, L.N., "On free boundary problems with arbitrary initial and flux conditions", *Zeitschrift fr angewandte Mathematik und Physik* (ZAMP), V30, N3, pp. 416-426, /79

[5] Windcliff, H., Forsyth, P.A., Vetzal, K.R., "Shout options: a framework for pricing contracts which can be modified by the investor", *J. Comput. Appl. Math.*, V134, pp. 213–241, /01

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)