Stochastic Elasticity of Variance, the Global Financial Crisis and Implied Volatility

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***Abstract*—Based on recent financial crises, it is desirable to develop a volatility model appropriate for dynamic markets experiencing sharp crashes. We observe that elasticity of variance of stock or index is randomly fluctuating around a mean level and the mean level itself is time varying contrary to the conventional assumption of constant elasticity of variance. This paper shows how useful the concept called stochastic elasticity of variance is to characterize the global financial crisis. Also, the valuation result for a financial derivative is presented in terms of implied volatility.**

***Index Terms*—financial crisis, implied volatility, option pricing, stochastic elasticity of variance**

I. Introduction

A general form of stochastic differential equation representing the underlying risky asset (stock) price in finance is given by

$$dX_t = rX_t dt + \sigma_t X_t dW_t$$

where r is the risk-free interest rate and σ_t is a volatility process representing standard deviation of the underlying return. It can be given by a function of time, the underlying price and a hidden process as follows:

$$\sigma_t = \sigma(t, X_t, Y_t)$$

The well-known financial models are special cases of this function. For example, there are the renowned Black-Scholes model [1] corresponding to constant σ_t, the Heston stochastic volatility model [2] given by

$$\sigma_t = \sigma_0 \sqrt{Y_t},$$

the constant elasticity of variance (CEV) model [3]-[4] expressed by a power function

$$\sigma_t = \sigma_0 X_t^{\beta},$$

the SABR model [5] given by

$$\sigma_t = \sigma_0 f(Y_t) X_t^{\beta},$$

where f is an exponential function and Y_t is another Brownian motion, and the dynamic elasticity of variance (DEV) model [6] given by

$$\sigma_t = \sigma_0 X_t^{\beta(t)},$$

where $\beta(t)$ is a deterministic function of time.

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In recent decades, the financial crises such as the global financial crisis and Eurozone crisis had a big impact on financial markets which was not negligible in derivative pricing problems. So, it is desirable to develop a volatility model more appropriate for such dynamic markets experiencing sharp crashes.

Our approach starts with introducing the quantity (the log volatility log stock price ratio or LVLSR)

$$\theta_t = 1 + \frac{\log(\frac{\sigma_t}{\sigma_0})}{\log(\frac{X_t}{X_0})}$$

and subsequently define $\eta_t = 2(\theta_t - 1)$ as the 'elasticity of variance' (the skewness parameter) in this paper.

Figure 1 shows the historical elasticity of variance for S&P 500. It demonstrates a fast fluctuating behavior of the LVLSR. So, it appears too idealistic for the LVLSR to be a constant as in the CEV model. Furthermore, it is notable to see that the leverage effect ($\eta_t < 0$), which is commonly observed in equity market, did not hold during the culmination of the global financial crisis (in particular, right after the bankruptcy of the Lehman Brothers) as shown in Figure 1. We also calculated the 50 days moving average of θ_t and found out that the moving average crossed the barrier 1, which is the critical value of the leverage effect, stayed above 1 for a while after the bankruptcy as in Figure 2. It is a sort of phase transition from the leverage effect to the inverse leverage effect. The inverse leverage effect ($\eta_t > 0$) is usually found in commodity market but it happened also in S&P index market during the global financial crisis.

We specify the volatility σ_t in terms of a rapidly fluctuating Ornstein-Uhlenbeck process (cf. [7]) which leads to a volatility model, named as stochastic elasticity of variance (SEV) model, and formulate a derivative pricing problem and derive the implied volatility from it. Some properties of the resultant implied volatility are to be investigated.

This paper is organized as follows. In Section II, the dynamics of the elasticity of variance is formulated in terms of different scales of parameter. Section III presents a pricing formula for European option. In Section IV, the implied volatility formulas are expressed. Section V concludes.

II. Multiscale Formulation

Our choice of the hidden process Y_t is a mean reverting

Ornstein-Uhlenbeck process given by

$$dY_t = \alpha(m - Y_t)dt + \nu dB_t$$

where $\alpha \gg 1$. This is a mixing process which has a normal invariant distribution with variance $\nu^2/2\alpha$. Notation $< \ >$ is going to be used for expectation with respect to this distribution. There is a correlation coefficient ρ between the Brownian motions W_t and B_t.

Now, the elasticity of variance η_t is specified as follows:

$$\eta_t = \beta + \delta f(Y_t),$$

where $\delta \ll 1$ and f is any function satisfying the condition required for the existence and uniqueness of solution for X_t. This formulation generalizes the Black-Scholes model which corresponds to $\beta = \delta = 0$ and the CEV model which corresponds to $\delta = 0$.

So, we have three different types of scale involved in the formulation of underlying asset price, i.e., large scale with $\alpha \gg 1$ (fast mean-reverting elasticity), small scale with $\delta \ll 1$ (CEV approximation) and usual O(1) scale. Throughout the paper, we assume that

$$\delta \ll \frac{1}{\sqrt{\alpha}}.$$

III. Option Pricing

Experiencing market crashes often appearing in daily trading activities would affect stock prices or indices as well as financial derivatives greatly. To obtain more accurate option pricing formula appropriate in an unstable market, we present an option pricing result based on the SEV model. The following result is a simplified version of the one in [8].

For a given payoff function h for European option, the option price P is given by

$$P(t,x,y) = E[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y],$$

where expectation is computed by a (market chosen) risk-neutral probability measure. In general, there is no closed form solution for P. So, in this section, we obtain an approximation formula for $\alpha \gg 1$ and $\delta \ll 1$ assuming $\beta = 0$.

Using the multiscale asymptotic analysis of [9], one can obtain the following result for the call option price P:

$$P \approx P_{BS} + \delta(P_1 + \frac{1}{\sqrt{\alpha}}P_2)$$

where P_{BS} is the classical Black-Scholes price and P_1 and P_2 are given by

$$P_1 = K < f > \int_0^{\tilde{\tau}} \int_{-\infty}^{+\infty} \frac{z + \log K}{\sqrt{2\pi s(\tilde{\tau} - s)}} e^{-\frac{(z-\tilde{x})^2}{4(\tilde{\tau}-s)}+z} \phi(d_1) e^*(z - \tilde{x}, s - \tilde{\tau}) dz ds,$$

$$P_2 = \rho\sigma_0 \nu K\sqrt{2} < \psi' > \int_0^{\tilde{\tau}} \int_{-\infty}^{+\infty} \frac{1 + (z + \log K)(1 - \frac{d_1}{\sqrt{2s}})}{\sqrt{2\pi s(\tilde{\tau} - s)}} e^{-\frac{(z-\tilde{x})^2}{4(\tilde{\tau}-s)}+z} \phi(d_1) e^*(z - \tilde{x}, s - \tilde{\tau}) dz ds,$$

$$\tilde{\tau} := \frac{1}{2}\sigma_0^2(T - t),$$

$$d_1(x,t) := \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma_0^2\right)(T - t)}{\sigma_0\sqrt{T - t}},$$

$$e^*(x,t) := e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2 t} \quad (k = \frac{2r}{\sigma_0^2}),$$

respectively. Here, ϕ and ψ are the standard normal cumulative distribution function and the solution of the differential equation

$$L_0\psi = f - < f >$$

with the infinitesimal generator L_0 of the process Y_t, respectively. The market price of elasticity of variance risk has been ignored in the formula above. Also, the $O(\delta^2)$ term is omitted compared to the result in [7] because it appears redundant for the purpose of option pricing in a reasonable time scale to expiration. The Put-Call parity can provide a formula for the put option based on this result.

IV. Implied Volatility

Using the well-known Greeks Vega and Vomma defined by

$$\frac{\partial P_{BS}}{\partial \sigma} = x\phi(d_1)\sqrt{T - t}$$

$$\frac{\partial^2 P_{BS}}{\partial \sigma^2} = x\phi(d_1)\sqrt{T - t}\frac{d_1 d_2}{\sigma_0}$$

respectively, we have the following formula for implied volatility from the option pricing formula in Section III:

$$I \approx \sigma_0 + \frac{\delta P_1 + \frac{\delta}{\sqrt{\alpha}}P_2 - \delta^2 \frac{Vomma}{2(Vega)^2} P_1{}^2}{Vega}$$

We note that the implied formula does not require the knowledge of the original model parameters

$$\sigma_0, \delta, \alpha, m, \nu, \rho$$

and the function f. They are replaced by a number of (reduced) group parameters

$$\sigma_0, \ V_1 := \delta\sigma_0^2 < f >, \ V_2 := \frac{\delta}{\sqrt{\alpha}}\rho\sigma_0\nu < \psi' >$$

Figure 3 shows the implied volatility fitting results of S&P 500 index options with a total of eight expiries. The model fits the market well even in the case of short time-to-maturity.

V. Conclusion

We have shown that the stochastic elasticity of variance model characterizes the global financial crisis, which has not been reported for stochastic volatility models such as the Heston model, and provides improvement in geometric structure of implied volatility curves over the constant elasticity of variance model. However, from the perspective of computing the correction terms P_1 and P_2, it takes less time to use the finite difference method for the corresponding partial differential equation than to compute the double integral in the formula. So, it is desirable to obtain a better analytic (hopefully, closed form) formula than the integral formula, which is ongoing work [10] to be published later.

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Figure 1. Global Financial Crisis and Elasticity of Variance

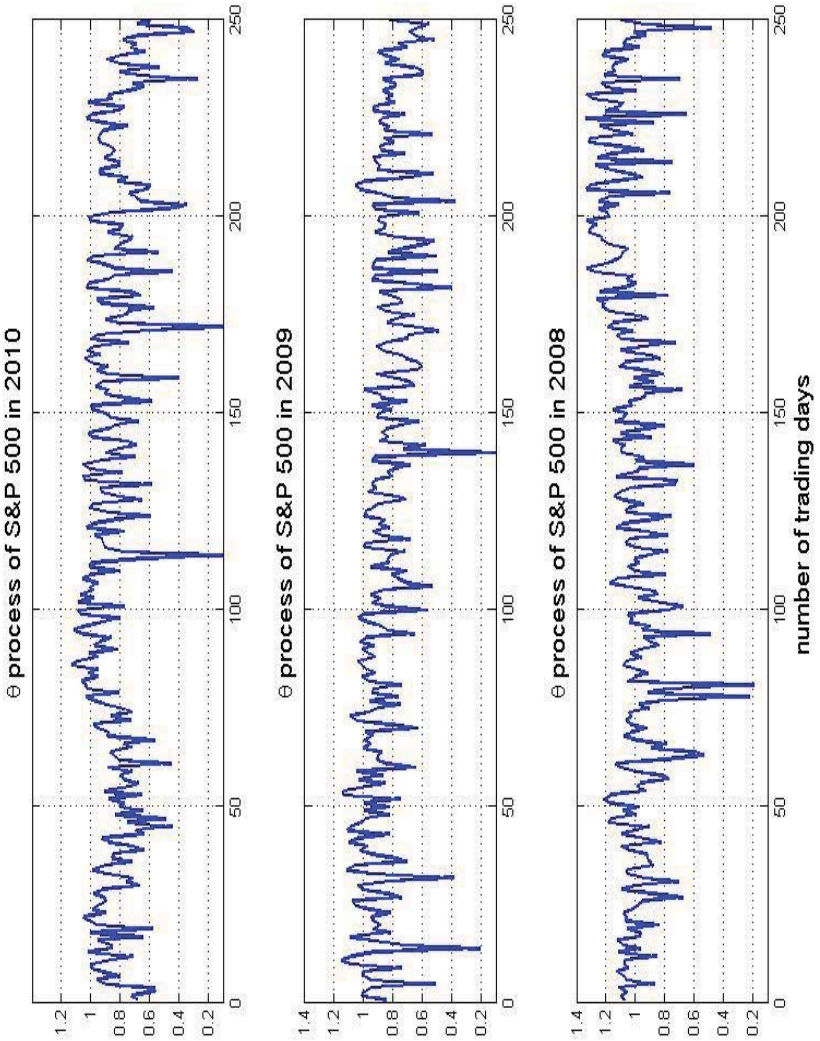

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Figure 2. Moving Average (50 days)

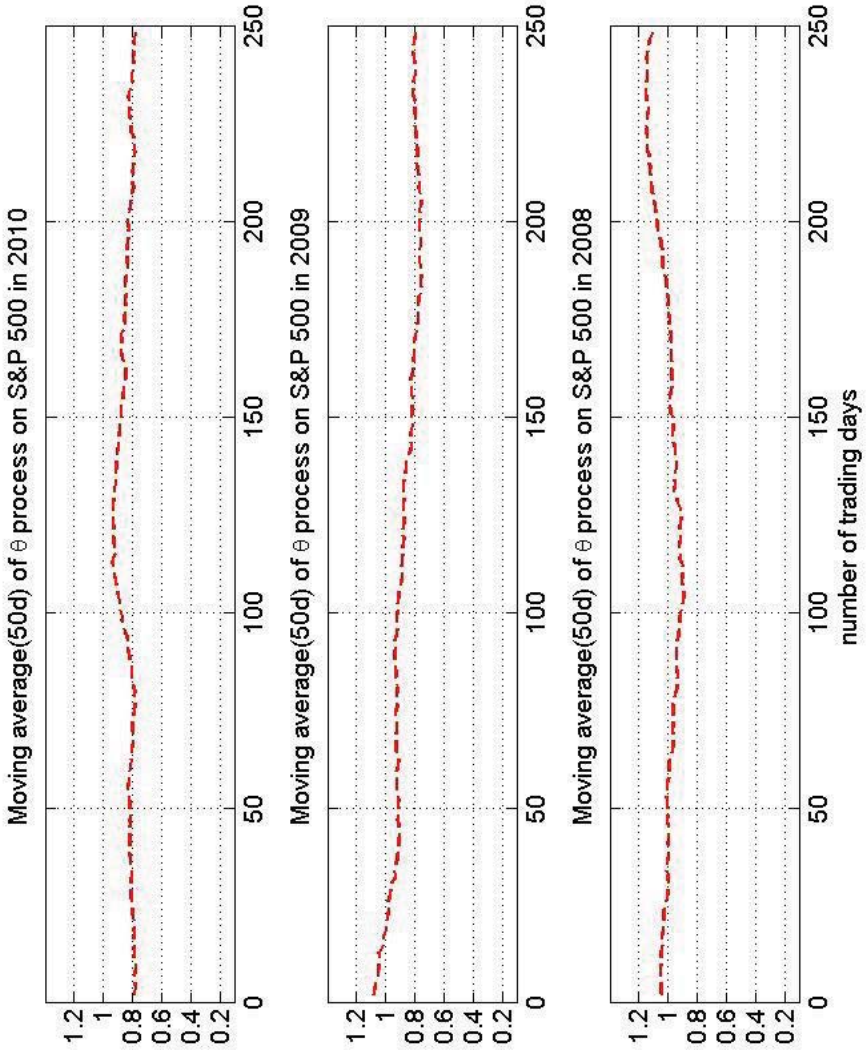

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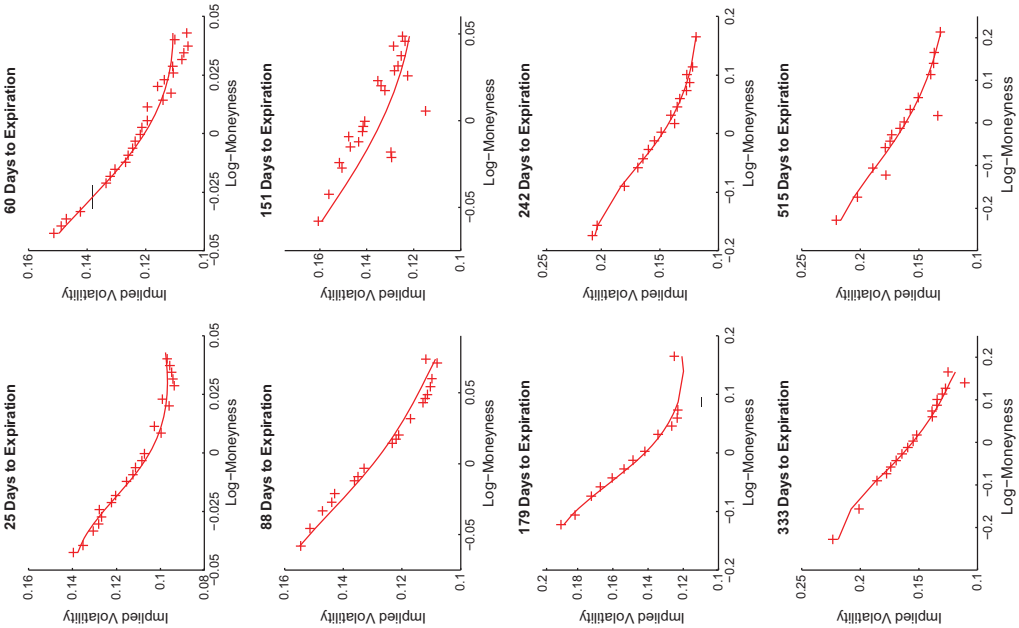
Figure 3. Implied Volatility Fit



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