Proceedings of the World Congress on Engineering 2019
WCE 2019, July 3-5, 2019, London, U.K.

On Isotropy Groups of CNF Formulas

Stefan Porschen*

Abstract—**We study the isotropy groups of CNF formulas regarding the flipping operation. Specifically we present an algorithm for computing the isotropy groups of fibre-formulas. Further, we investigate the lifting process to the general case, and consider the behaviour of several subclasses of CNF with respect to the flipping operation.**

Keywords: satisfiability, isotropy-group, orbit, CNF

1 Introduction

The genuine and one of the most important NP-complete problems in mathematics is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [4]. Specifically one is interested in classes for which SAT can be solved in polynomial time. There are known several classes, for which SAT can be tested efficiently, such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested and co-nested formulas etc. [1, 2, 3, 6, 8, 9, 10, 7, 17, 19]. On basis of the flipping operation on CNF formulas, here we investigate the isotropy groups of formulas. The motivation behind this research is the fact that formulas with large isotropy groups have small orbits. On the other hand the generator sets of isotropy groups are of polynomial size. This might enable one to compute class invariants more efficiently, specifically those that are connected to the satisfiability of formulas like the monotonicity index. The hope here is to identify new subclasses of CNF which behave efficient regarding SAT-decision as well as to gain new structural insight into CNF-SAT in general. After discussing some basic concepts and results we investigate the isotropy group of fibre-formulas and present an algorithm for its determination. Further we discuss to some extent the lifting process to the total case and investigate the isotropy group for members of several explicit subclasses of CNF such as the linear formulas [14, 16]. Methodically the fibre view on clause sets [11] is exploited again for this study.

2 Notation and Preliminaries

A Boolean or propositional variable x taking values from $\{0, 1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$. To *flip* or *complement* a literal always means to negate the underlying variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. If all literals in c are complemented one gets c^γ. A clause containing no negative literal is called *positive*. A clause containing only negated variables is called *negative*. A *unit* clause contains exactly one literal. A conjunctive normal form formula C, for short formula, is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. C^γ is the formula obtained from C by transfering $c \to c^\gamma$ for all $c \in C$. A formula can also be empty which is denoted as $\varnothing$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Let CNF_+ (CNF_-) denote that part of CNF containing only positive (negative) clauses. A formula $C \in \mathrm{CNF}$ is called *linear* if each pair $c_i, c_j \in C$, $i \neq j$, satisfies $|V(c_i) \cap V(c_j)| \leq 1$. By LCNF the class of linear formulas is denoted. Given $C \in \mathrm{CNF}$, let $A(C) := \{c \in C : c^\gamma \notin C\}$ and $S(C) := \{c \in C : c^\gamma \in C\}$ defining the classes $\mathcal{A} := \{C \in \mathrm{CNF} : C = A(C)\}$ of *anti-symmetric* and $\mathcal{S} := \{C \in \mathrm{CNF} : C = C^\gamma\}$ of *symmetric* formulas [15]. Note that $\mathcal{A} \cap \mathcal{S} = \{\varnothing\}$, and that for every non-empty $C \in \mathrm{CNF}$ one has $C = A(C) \cup S(C)$ as disjoint union. However, clearly $\mathcal{S} \cup \mathcal{A}$ is a proper subset of CNF. Let $\mathcal{S}_\pm \subseteq \mathcal{S}$ contain all formulas $C = C \cup C^\gamma$, where $\varnothing \neq C \in \mathrm{CNF}_+$. For a finite set M, let 2^M denote its powerset. As usual for a positive integer n, let $[n] := \{1, \ldots, n\}$, and for convenience we set $[0] := \emptyset$. Throughout log means the logarithm function with respect to base 2; and groups always are assumed to be finite. Given a group G, recall that the order of any subgroup of G is a divisor of its cardinality $|G|$ according to a central theorem of Lagrange. Let $\mathrm{Gn}(G)$ denote a set of generators of G. Let $< g > \leq G$ denote the cyclic subgroup generated by $g \in G$. Further recall that every abelian group can be written as a direct product of cyclic subgroups. Given $C \in \mathrm{CNF}$, SAT asks whether there is a truth assignment $t : V(C) \to \{0, 1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of C. Let $\mathrm{SAT} \subseteq \mathrm{CNF}$ denote the collection of all formulas for which there is a model. Clauses containing a complemented pair of literals are always satisfied. Hence, it is assumed throughout that clauses only contain literals over different variables. Also unit clauses must be satisfied therefore it is no loss of generality to assume for convenience that unit

*Mathematics Group, Department 4, HTW Berlin, D-10313 Berlin, Germany, Email: porschen@htw-berlin.de.

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

clauses do not occur in formulas. As usual iff means if and only if. The hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ assigned to a formula $C \in \text{CNF}$ is defined as $B(C) := \{V(c) : c \in C\} \in \text{CNF}_+$. As introduced in [11] the collection of all clauses c such that $V(c) = b$, for a fixed $b \in B(C)$, is the fibre formula C_b of C over b yielding the fibre-decomposition $C = \bigcup_{b \in B(C)} C_b$ of C. Conversely, a hypergraph $\mathcal{H} = (V, B)$ can be regarded as a base hypergraph if its vertex set V is a non-empty finite set of Boolean variables such that for every $x \in V$ there is some $b \in B$ containing x. By $W_b := \{c : V(c) = b\}$ denote the collection of all possible clauses over a fixed $b \in B$. The *intersection graph* $\mathcal{I}(\mathcal{H})$ of $\mathcal{H} = (V, B)$ gets a vertex for each $b \in B$ and there is exactly one edge joining a pair of vertices $b \neq b'$ iff $b \cap b' \neq \emptyset$. A hypergraph $\mathcal{H}$ is called *connected* iff $\mathcal{I}(\mathcal{H})$ is connected in the usual sense. The set of all clauses over $\mathcal{H}$ is $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A $\mathcal{H}$*-based* formula is a subset $C \subseteq K_{\mathcal{H}}$ such that $C_b := C \cap W_b \neq \emptyset$, for every $b \in B$. For a $\mathcal{H}$-based $C \subseteq K_{\mathcal{H}}$, let $\bar{C} := K_{\mathcal{H}} \setminus C$ be its complement formula. According to [11], a fibre-transversal of $K_{\mathcal{H}}$ is a $\mathcal{H}$-based formula $F \subset K_{\mathcal{H}}$ such that $|F \cap W_b| = 1$, for every $b \in B$. By $\mathcal{F}(K_{\mathcal{H}})$ denote the set of all fibre-transversals of $K_{\mathcal{H}}$. For a base-hypergraph $\mathcal{H}$ and a class $\mathcal{C} \subseteq \text{CNF}$ let $\mathcal{C}(\mathcal{H}) := \{C \in \mathcal{C} : \mathcal{H}(C) = \mathcal{H}\}$, denote the $\mathcal{H}$-based fraction of $\mathcal{C}$.

3 Basic Concepts and Results

For a fixed finite and non-empty set of propositional variables V, let $B_t = 2^V$ and $\mathcal{H}_t = (V, B_t)$. Denote by $\text{CNF} := 2^{K_{\mathcal{H}_t}}$ the set of all CNF formulas with $V(C) \subseteq V$, $B(C) =: B \subseteq B_t$. Let c^X be the clause obtained from $c \in K_{\mathcal{H}_t}$ by complementing all variables in $X \cap V(c)$, where X is an arbitrary subset of V, for short we set $c^\gamma := c^{V(c)}$, and further $c^\emptyset := c$. This *flipping operation* $\varphi(c, X) := c^X$ acting on $K_{\mathcal{H}_t}$ induces an action on CNF by observing that $\{c\} \in \text{CNF}$: For $C = \{c_1, \ldots, c_m\} \in \text{CNF}$ and $X \in 2^V$ let $\varphi : \text{CNF} \times 2^V \to \text{CNF}$, such that $\varphi(C, X) := \{c_1^X, \ldots, c_m^X\} =: C^X \in \text{CNF}$. Again set $C^\gamma := C^{V(C)}$ in case that all variables in C are flipped, and $C^\emptyset := C$. Thus formally we obtain the G_V-*action* of the abelian group $G_V := (2^V, \oplus)$ with neutral element $\emptyset$ on CNF. Indeed, first flipping C by $X \in G_V$ then by $Y \in G_V$ obviously yields $(C^X)^Y = C^{X \oplus Y}$, where $\varnothing^X := \varnothing \in \text{CNF}$, for every $X \in G_V$. In case $V(C) \subsetneq V$, the relevant subgroup of G_V is $G_{V(C)} = (2^{V(C)}, \oplus)$. We shall use the abbreviation $E := \{\emptyset\} \leq G_V$ for the trivial group. By $\mathcal{O}(C) := \{C^X : X \in G_{V(C)}\} = \{C^X : X \in G_V\}$ denote the *$(G_V$-$)$orbit* of C in CNF yielding the classes of an equivalence relation on CNF. This quotient space CNF/G_V therefore usually is called the *orbit space*. Recall that a group acts transitively on its orbits. Let $G_{V(C)}(C) := \{X \in G_{V(C)} : C^X = C\}$ denote the *isotropy group* also called *stabilizer* of $C \in \text{CNF}$. For a fibre-subformula $C_b \subseteq C$, there are different kinds of isotropy groups, namely, $G_{V(C)}(C_b) := \{X \in G_{V(C)} : C_b^X = C_b\}$ and $G_b(C_b) := \{X \in G_b : C_b^X = C_b\}$, where $G_b := (2^b, \oplus)$, $V(C_b) = b$; thus $G_b(C_b) \leq G_{V(C)}(C_b)$.

Lemma 1 *Given $\mathcal{H} = (V, B)$, $G \leq G_V$ and $H \leq G_b$ then $|\text{Gn}(G)| \leq |V|$ and $|\text{Gn}(H)| \leq |b|$, $b \in B$.*

PROOF. Clearly $\text{Gn}(G_V) = V$ thus $|\text{Gn}(G)| \leq |\text{Gn}(G_V)| = |V|$. Also $\text{Gn}(G_b) = b$ hence $|\text{Gn}(H)| \leq |b|$, $b \in B$. □

Lemma 2 *Let $C \in \text{CNF}$ with $\mathcal{H}(C) =: (V, B)$.*
(i) $X \in G_V(C)$ is equivalent with $c \in C \Leftrightarrow c^X \in C$.
(ii) $X \in G_V(C)$ iff $X \in G_V(C_b)$, for all $b \in B$.

PROOF. $X \in G_V(C)$ means $C^X = C$ and (i) follows directly. Addressing (ii) observe that for distinct $b, b' \in B$ there is no $X \in G_V$ such that $C_b^X = C_{b'}$. Thus $X \in G_V(C)$ iff $C^X = \bigcup_{b \in B} C_b^X = C = \bigcup_{b \in B} C_b$ iff $C_b^X = C_b$, $b \in B$, iff $X \in G_V(C_b)$, $b \in B$. □

More generally, $G_V(\mathcal{C}) := \{X \in G_V : C \in \mathcal{C} \Rightarrow C^X \in \mathcal{C}\}$ denotes the isotropy group of the class $\mathcal{C} \subseteq \text{CNF}$. Indeed this is a group, as for $X, Y \in G_V(\mathcal{C})$ and $C \in \mathcal{C}$ assume $C^X =: C' \in \mathcal{C}$ then $C^{X \oplus Y^{-1}} = C^{X \oplus Y} = C'^Y \in \mathcal{C}$ hence $X \oplus Y^{-1} \in G_V(\mathcal{C})$. Further note that according to the theorem of Lagrange every subgroup $G \leq G_V$ here is of order $2^{e(G)}$ with the integer $e(G) := \log |G| \geq 0$. A mapping $g : \text{CNF} \to \text{CNF}$ is G_V*-equivariant*, by definition, if $g(C^X) = [g(C)]^X$, for every $X \in G_V$ and every $C \in \text{CNF}$. As shown in [13] one has:

Lemma 3 *$G_{V(C)}(C') = G_{V(C)}(C)$ for all $C' \in \mathcal{O}(C)$.*

As usual a *fixed point* of an operation [18] is the unique member of an 1-point *invariant* (also called *stable*) subspace, so by definition its isotropy group equals the whole group. According to Theorem 4 proven in [12] one has:

Lemma 4 *$\varnothing \neq C \in \text{CNF}$ is a fixed point of the G_V-action iff $C_b = W_b$, for all $b \in B(C)$.*

As a direct consequence of Lemmata 2 and 4 one obtains:

Corollary 1 *Let $C \in \text{CNF}$, $\mathcal{H}(C) = \mathcal{H}(\bar{C}) =: \mathcal{H} = (V, B)$ then $G_V(C) = G_V(\bar{C})$.*

PROOF. If $X \in G_V(C)$ then $K_{\mathcal{H}} = C \cup \bar{C} = K_{\mathcal{H}}^X = C^X \cup \bar{C}^X = C \cup \bar{C}^X$ hence $X \in G_V(\bar{C})$. The reverse inclusion follows by exchanging the roles of $C, \bar{C}$. □

Lemma 5 *For $C, C' \in \text{CNF}$ with $V(C) = V(C') =: V$, assume $G_V(C) = G_V(C')$ then $|\mathcal{O}(C)| = |\mathcal{O}(C')|$.*

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Proof. Let $G := G_V(C)$, then $G \leq G_V$ is a normal subgroup so that the left coset space G_V/G is the group of cosets $YG := \{Y \oplus X : X \in G\}$, for every $Y \in G_V$. Clearly, $|YG| = |G|$ and $C^{Y'} = C^Y$, for all $Y' \in YG$, implying $|\mathcal{O}(C)| = |G_V/G| = |\mathcal{O}(C')|$ by assumption. □

Recall that for $C \in$ CNF the value $\mu(C) := \min\{\min\{|C'_+|, |C'_-|\} : C' \in \mathcal{O}(C)\}$ is the monotonicity index [12] of C. Hence μ is a *class invariant* having the same value for all orbit members. Moreover, as shown in [12] one has $C \in$ SAT iff $\mu(C) = 0$. Recall that fixed-parameter tractability (FPT) w.r.t. parameter k means a worst case upper bound for the computational time complexity of the form $O(p(n,k)g(k))$, for instances of size n, where p is a polynomial and g is an arbitrary function of the parameter k only, cf. e.g. [5].

Theorem 1 *For a constant positive integer k, let $\mathcal{C} := \mathcal{C}(k) \subseteq$ CNF be such that $\log |\mathrm{Gn}(G_{V(C)}(C))| \geq |V(C)| - k$, and such that $\mathrm{Gn}(G_{V(C)}(C))$ can be computed in polynomial time, for every $C \in \mathcal{C}$, then SAT is FPT w.r.t. k, for instances from $\mathcal{C}$.*

Proof. For $C \in \mathcal{C}$, let $G := G_{V(C)}$ and let $H := G_{V(C)}(C)$ be the isotropy group of C. The factor group G/H may be identified with a set of representatives of its cosets. We claim that $\mathrm{Gn}(G/H) = \mathrm{Gn}(G) \setminus \mathrm{Gn}(H)$. Indeed, first assume that there is any $X \in \mathrm{Gn}(H)$ which is also in $\mathrm{Gn}(G/H)$ then $X \in H$ and also $X \in G/H$ implying $X = \emptyset \in G/H$ but $\emptyset \notin \mathrm{Gn}(H)$ providing a contradiction. Next let $Y \in \mathrm{Gn}(G) \setminus \mathrm{Gn}(H)$ then clearly $\emptyset \neq H \in G/H$ therefore $Y \in \mathrm{Gn}(G/H)$. So, by assumption one has $|\mathrm{Gn}(G/H)| = |\mathrm{Gn}(G)| - |\mathrm{Gn}(H)| \leq k$, and according to the proof of Lemma 5, $\mathcal{O}(C) = \{C^{\bigoplus_{Y \in Z} Y} : Z \subseteq \mathrm{Gn}(G/H)\}$. Thus $|\mathcal{O}(C)| \leq 2^k$ meaning that $\mu(C)$ can be computed in FPT time $O(p(|C|, |V(C)|)2^k)$ where p is an appropriate polynomial. □

For $\mathcal{H} = (V, B)$, a subgroup $G \leq G_V$, and $c \in K_{\mathcal{H}}$ let $\mathcal{O}_G(c) := \mathcal{O}_G(\{c\}) = \{c^X : X \in G\}$ denote the G-*orbit* of c. If $G = G_V$ we also write $\mathcal{O}(c)$ instead of $\mathcal{O}_G(c)$. Clearly the orbit of a clause yields a formula whereas the orbit of a formula yields a subclass of CNF.

Lemma 6 *For $\mathcal{H} = (V, B)$, $b \in B$, and $c \in W_b$ one has $W_b = \mathcal{O}(c)$. Specifically, W_b is bijective to G_b, $b \in B$.*

Proof. As $W_b = \{c \in K_{\mathcal{H}} : V(c) = b\}$ one has for given $c \in W_b$ and any $X \in G_V$ that $V(c^X) = V(c^{X \cap b}) = V(c)$ hence $\mathcal{O}(c) \subseteq W_b$ and as $|\mathcal{O}(c)| = |G_V \cap 2^b| = |2^b| = |W_b|$ we have $\mathcal{O}(c) = W_b$ and $|W_b| = |G_b|$. □

Definition 1 *For $\mathcal{H} = (V, B)$, $b \in B$, $G \leq G_V$, and $H \leq G_b$, let $\mathrm{R}_b(G) := \{X \cap b : X \in G\}$ be the (b)-*restriction *of G, and let $\mathrm{L}_b(H) := \{X \in G_V : X \cap b \in H\}$ be the (G_V)-*lift *of H.*

One clearly has $\mathrm{L}_b(G_b) = G_{V(C)}(C_b)$ and $\mathrm{R}_b(E) = E$, for every $b \in B$. More generally:

Lemma 7 *For $\mathcal{H} = (V, B)$, $b \in B$, and any subgroups $G \leq G_V$, $H \leq G_b$, one has (i) $\mathrm{R}_b(G) \leq G_b$, $e(\mathrm{R}_b(G)) \leq \min\{e(G), b\}$, $H \leq \mathrm{L}_b(H) \leq G_V$, and (ii) $G \leq \mathrm{L}_b(\mathrm{R}_b(G))$, $H = \mathrm{R}_b(\mathrm{L}_b(H))$.*

Proof. Recall that $|G| = 2^e$, where $e := e(G) \geq 0$, as a subgroup of G_V. As $b \in G_V$, one has $\{X \cap b : X \in G\} \subseteq G_V \cap G_b$ hence $|\mathrm{R}_b(G)| \leq G_b$, and also $|\mathrm{R}_b(G)| \leq 2^e$. Moreover, $\mathrm{R}_b(G)$ is a subgroup of G_b. Indeed for any $X, Y^{-1} = Y \in G$, let $X_b := X \cap b$, $Y_b := Y \cap b \in \mathrm{R}_b(G)$ then one has

$$\begin{aligned} X_b \oplus Y_b &= (X_b \cup Y_b) \setminus (X_b \cap Y_b) \\ &= (X \cup Y) \cap b \setminus (X \cap Y) \cap b \\ &= (X \oplus Y) \cap b \end{aligned}$$

which can be verified easily. Thus $X_b \oplus Y_b \in \mathrm{R}_b(G)$ being a subgroup. Hence there is $e_b := e(\mathrm{R}_b(G)) \geq 0$ such that $e_b \leq \min\{b, e\}$ and $|\mathrm{R}_b(G)| = 2^{e_b}$. Next, choose arbitrary $X, Y \in G_V$ such that as previously defined $X_b, Y_b \in H$. Then reversing the sequence of equations above one directly obtains $(X \oplus Y) \cap b = X_b \oplus Y_b \in H$ implying $X \oplus Y \in \mathrm{L}_b(H)$ thus being a subgroup of G_V. Further, $H \subseteq G_b \subseteq G_V$ therefore $H \subseteq \mathrm{L}_b(H)$ thus $H \leq \mathrm{L}_b(H)$, hence (i) is verified. Finally, both assertions in (ii) are obvious. □

4 Isotropy Groups of Fibre Formulas

Throughout this section, let C be a non-empty fibre-formula meaning $C \subseteq W_b$ where $b := V(C)$, and let $E \neq H \leq G_b$ be a proper subgroup of the flipping group. Further let $G := G_b(C)$ be the isotropy group of C on the fibre level. Given $c, c' \in W_b$ then due to Lemma 6 there exists, by transitivity, a unique *transition member* $Y(c, c') := V(c \oplus c') \in G_b$ with $c' = c^{Y(c,c')}$, where $c \oplus c'$ is regarded as a set of literals.

Lemma 8 *(1) For two distinct H-orbits, $\mathcal{O} := \mathcal{O}_H(c), \mathcal{O}' := \mathcal{O}_H(c')$, $c, c' \in W_b$, there is exactly one $X \in G_b \setminus H$ which is composed of generators in $\mathrm{Gn}(G_b) \setminus \mathrm{Gn}(H)$ only, such that $\mathcal{O}^X = \mathcal{O}'$. This unique X is called the* primitive (orbit) transition element. *(2) Let $C = \bigcup_{i \in [s]} \mathcal{O}_i$ be the union of $s \geq 1$ disjoint H-orbits: $\mathcal{O}_i := \mathcal{O}_H(c_i)$, where $c_i \in C \subseteq W_b$. Then every $X \in G \setminus H$ provides a non-trivial 2-regular permutation π_X of $[s]$ such that $\mathcal{O}_i^X = \mathcal{O}_{\pi_X(i)}$, $i \in [s]$.*

Proof. Let $c \in \mathcal{O}$, $c' \in \mathcal{O}'$, where the orbits are assumed to be distinct. Then $Y(c, c') \in G_b \setminus H$. By transitivity for every $c_i \in \mathcal{O}$ there is a unique $X_i \in H$ such that $c^{X_i} = c_i$ yielding the unique member $c_i^{Y(c,c')} = c'^{X_i} \in \mathcal{O}'$.

Thus $Y(c, c')$ provides a bijection from $\mathcal{O}$ to $\mathcal{O}'$ meaning $\mathcal{O}^{Y(c,c')} = \mathcal{O}'$. As $Y(c, c') \subseteq b$ and $\mathrm{Gn}(H) \subseteq 2^b$ we have $X := Y(c, c') \setminus \bigcup_{Y \in \mathrm{Gn}(H)} Y \subseteq b$. Clearly, also $X \in G_b \setminus H$ and so $\mathcal{O}^X = \mathcal{O}'$ as above. Moreover X is unique: Let $\tilde{c} \in \mathcal{O}$, $\tilde{c}' \in \mathcal{O}'$ be another pair of clauses then there are unique $\tilde{Y}, \tilde{Y}' \in H$ with $\tilde{c}^{\tilde{Y}} = c$, $\tilde{c}'^{\tilde{Y}'} = c'$. It follows that $Y(\tilde{c}, \tilde{c}') = \tilde{Y} \oplus Y(c, c') \oplus \tilde{Y}'$ implying $Y(\tilde{c}, \tilde{c}') \setminus \bigcup_{Y \in \mathrm{Gn}(H)} Y$ $= Y(c, c') \setminus \bigcup_{Y \in \mathrm{Gn}(H)} Y = X$. For proving (2), let $X \in G \setminus H$. According to (1), the H-orbits $\mathcal{O}_i^X$, $i \in [s]$, are pairwise different, and their union must yield C as $X \in G$. Thus X induces a bijection π_X on $[s]$ such that $\mathcal{O}_i^X = \mathcal{O}_{\pi_X(i)} \subseteq C$, $i \in [s]$. As $< X >$ is cyclic of order 2, this permutation decomposes into disjoint transpositions, i.e., 2-cycles, namely $\pi_X = (i_1, \pi_X(i_1)) \cdots (i_r, \pi_X(i_r))$, for $r = s/2$, $i_j = \min([s] \setminus \{i_k, \pi(i_k) : k \in [j-1]\})$, for every $j \in [r]$, implying (2). □

Observe that under the assumptions in Lemma 8 one has $H \leq G$, so the previous proof directly implies:

Corollary 2 *Let C be the union of $s \geq 1$ disjoint H-orbits, for an integer s. If there is $X \in G \setminus H$, then s is even.*

Theorem 2 *The isotropy group G of a fibre-formula $C \subseteq W_b$ can be computed in time $O(|b|^2 \cdot |C|^2 \cdot \log^2 |C|)$ as a direct product of cyclic subgroups.*

Proof. If $C = \varnothing$ we have $G = G_b$. Otherwise compute G by iteratively enlarging the number of factors in the current direct product of cyclic groups H, as long as there is a new generator $X \in \mathrm{Gn}(G)$ yielding the next factor $< X >$. Initially setting $H := E$, C can be regarded as the union of $s := |C| \geq 1$ pairwise disjoint H-orbits $\mathcal{O}_i := \{c_i\}$, $i \in [s]$. If $s = 1 \mod 2$ the procedure stops with $G := H$ according to Corollary 2. Otherwise, one has to check in the current iteration whether there is $X \in G \setminus H$. To that end, let c_i be an arbitrary member of the orbit $\mathcal{O}_i$, $i \in [s]$. Considering these clauses as the vertices of a complete graph K_s, we label every edge $c_i - c_j$ by its unique primitive orbit transition member $X_{i,j} := Y(c_i, c_j) \setminus \bigcup_{Y \in \mathrm{Gn}(H)} Y$, $i, j \in [s], i < j$, where $\mathrm{Gn}(H)$ is the generator set in the current iteration. Then due to Lemma 8 our problem is equivalent to identify a perfect matching in K_s such that all its members carry an equal label X which therefore belongs to G, as it provides a bijection of the current set of orbits. One might implement a clever version of a minimum weight perfect matching algorithm, which however for dense graphs is rather slow. On the other hand we do not need a matching, only a suitable label which can be determined faster as follows: By the lexicographic order, based on a pre-ordering of the variables in b, sort all primitive orbit transition elements $X_{i,j}$ in a sequence T. Now equal labels are grouped together. Finally linearly went through T searching for a first consecutive subsequence $t = (X_{i_1,j_1}, \ldots, X_{i_r,j_r})$ of T having the properties: (i) $|t| = s/2 =: r$, (ii) all its elements are equal to an X, and (iii) $\sum_{k=1}^{r}(i_k + j_k) = s(s+1)/2$. Observe that these conditions ensure that the corresponding edges $c_{i_k} - c_{j_k}$, $k \in [r]$, form a perfect matching in K_s of equal label X. The lexicographic sorting of $O(s^2)$ labels can be executed in time $O(|b|^2 \cdot s^2 \cdot \log s)$ dominating the time amount for computing the primitive transition elements relying on Lemma 1, as well as the time amount for the subsequence search. If there is a subsequence as required yielding label X, then set $H \leftarrow H \times < X >$ and join each pair $\mathcal{O}_i$, $\mathcal{O}_{\pi_X(i)}$ of the current orbits to the new H-orbit $\mathcal{O}_i \cup \mathcal{O}_{\pi_X(i)}$ according to Lemma 8 (2). Then $s \leftarrow s/2$ is the new number of orbits. Otherwise, the procedure stops with $G := H$. The joining operation clearly is dominated by the sorting bound as stated above. As every newly added cyclic group factor corresponds to exactly one generator of the isotropy group we have at most $|b|$ such iterations due to Lemma 1. On the other hand the number of iterations is bounded by $\log |C| \leq |b|$ because of the repeated joining process. So the overall upper bound for the time complexity amounts to $O(|b|^2 \cdot |C|^2 \cdot \log^2 |C|)$. □

5 Lifting to the Total Case

For $C \in \mathrm{CNF}$ with $\mathcal{H}(C) =: (V, B)$ recall that $G_b(C_b) \leq G_b$ is the isotropy group of C_b over $V(C_b) = b$, $b \in B$.

Lemma 9 *For $\mathcal{H} = (V, B)$, a subgroup $G \leq G_V$, and $b \in B$, let $C = C_b$ be the union of $s > 0$ G-orbits. Then for fixed $c' \in C$ and $M_E := \bigcup_{c \in C_b} Y(c', c)$ one has*

$$G_V(C) = \begin{cases} \mathrm{L}_b(2^{M_E}), & \text{if } \log s = |M_E| - e(\mathrm{R}_b(G)) \geq 0 \\ \mathrm{L}_b(\mathrm{R}_b(G)), & \text{if } s \text{ is odd} \end{cases}$$

Proof. Let $e := e(\mathrm{R}_b(G)) \geq 0$ hence $|\mathcal{O}_G(c)| = |\mathcal{O}_{\mathrm{R}_b(G)}(c)| = 2^e$, for every $c \in C_b$ thus $|C| = s \cdot 2^e$. If s is odd, by contraposing Corollary 2 one has $G_b(C) = \mathrm{R}_b(G)$ directly implying $G_V(C) = \mathrm{L}_b(\mathrm{R}_b(G))$. Next, assume $\log s = |M_E| - e \geq 0$ where $M_E := \bigcup_{c \in C_b} Y(c', c) \in G_b$ and define $G' := \{Y(c', c) : c \in C_b\} \subset G_b$ for any fixed $c' \in C_b$. Then obviously $C_b = \{c'^X : X \in G'\}$. Hence C_b equals exactly one G'-orbit, respectively, one $\mathrm{L}_b(G')$-orbit iff $G' \leq G_b \Leftrightarrow \mathrm{L}_b(G') \leq G_V$, which is claimed to be true. Therefore $\mathrm{L}_b(\mathrm{R}_b(G')) = \mathrm{L}_b(G')$ is the isotropy group of C according to the result previously proven. To establish the claim, observe that all $Y(c', c)$ are pairwise distinct therefore $|G'| = |C_b| = s2^e$ implying $\log |G'| = \log s + e = |M_E|$. Hence $|G'| = |2^{M_E}|$ and, as every member in G' is a subset of M_E, it follows that $G' = 2^{M_E} \leq G_b$. Here one has $G' = \mathrm{R}_b(G)$ if $\log s = 0$ which means an odd s, finishing the proof. □

Given $X \in G_b(C_b)$ and setting $\tau(X) := \{X \cup U : U \in 2^{V \setminus b}\}$, $\Pi_b := \bigcup_{X \in G_b(C_b)} \tau(X)$ one obtains:

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Theorem 3 *$G_V(C) = \bigcap_{b\in B} \mathrm{L}_b(G_b(C_b))$, for a $\mathcal{H}$-based formula C with $\mathcal{H} = (V, B)$. Moreover $\mathrm{L}_b(G_b(C_b)) = \Pi_b \leq G_V$, $b \in B$.*

Proof. Clearly $G_V(C_b) = \{X \in G_V : C_b^X = C_b\} = \{X \in G_V : X \cap b \in G_b(C_b)\} = \mathrm{L}_b(G_b(C_b)) =: \mathrm{L}_b \leq G_V$, for every $b \in B$, according to Lemma 7 (i). Therefore $G_V(C) = \bigcap_{b\in B} \mathrm{L}_b$ immediately follows on the set level relying on Lemma 2 (ii). Further one has $\bigcap_{b\in B} \mathrm{L}_b \leq G_V$ and the first statement is settled. Addressing the last assertion let $Y \in \Pi_b \subseteq 2^V$ then there is a unique $X \in G_b(C_b)$ such that $Y \in \tau(X)$. Hence there is $U \in 2^{V\setminus b}$: $Y = X \cup U$ implying $Y \cap b = X \cap b = X \in G_b(C_b)$ as $U \cap b = \emptyset$. So $Y \in \mathrm{L}_b$. Reversely, let $Y \in \mathrm{L}_b$ then there is $X \in G_b(C_b) : Y \cap b = X$ implying $X \subseteq Y$ and $U := Y \setminus X \in 2^{V\setminus b}$ hence $Y = X \cup U \in \tau(b)$ establishing $\mathrm{L}_b = \Pi_b$, $b \in B$. Hence also $\Pi_b \leq 2^V$ is true. □

One obtains the following sufficient condition for the trivial isotropy group E.

Corollary 3 *Let $C \in \mathrm{CNF}$ such that $|C_b|$ is odd, for all $b \in B(C)$ then $G_{V(C)}(C) = E$.*

Proof. Set $\mathcal{H}(C) =: (V, B)$. Relying on Lemma 9 the assumption implies $G_V(C_b) = \mathrm{L}_b(E) = 2^{V\setminus b}$, for every $b \in B$, using Theorem 3. Thus $G_V(C) = \bigcap_{b\in B} 2^{V\setminus b} = 2^{V\setminus \bigcup_{b\in B} b} = 2^{\emptyset} = E$. □

Theorem 4 *For every $C \in \mathrm{LCNF} \cup \mathrm{CNF}_+ \cup \mathrm{CNF}_-$ one has $G_{V(C)}(C) = E$.*

Proof. As by assumption members of CNF are considered to be free of unit clauses it follows for any linear or monotone formula C that $|C_b| = 1$ for every $b \in B(C)$. Thus the assertion is implied by Corollary 3. □

In terms of the intersection graph one has for $\mathcal{S}_\pm$:

Lemma 10 *For $C \in \mathcal{S}_\pm$ with $\mathcal{H}(C) =: \mathcal{H} = (V, B)$, let $\{\mathcal{I}_1, \ldots, \mathcal{I}_k\}$ be the set of connected components of the intersection graph $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$. Then $\mathrm{Gn}(G_V(C)) = \{X_i := \bigcup_{b\in V(\mathcal{I}_i)} b : i \in [k]\}$.*

Proof. Assume $C \in \mathcal{S}_\pm$ then $C = B \cup B^\gamma = \bigcup_{b\in B}\{b, b^\gamma\}$, where $B \in \mathrm{CNF}_+$. Hence $\mathcal{O}_{G_b(C_b)}(b) = C_b$ where $G_b(C_b) = \{\emptyset, b\} \leq G_b$ meaning that $G_V(C_b) = \mathrm{L}_b(G_b(C_b))$, for every $b \in B$, according to Lemma 9, because $\mathrm{R}_b(G_b(C_b)) = G_b(C_b)$. Let $M := \{X_i := \bigcup_{b\in V(\mathcal{I}_i)} b : i \in [k]\}$. To verify the assertion, we first show by induction that for any integer $n \geq 1$ and members $X_{i_j} \in M$, $j \in [n]$, one has $\bigoplus_{j=1}^n X_{i_j} \in G_V(C)$. So, given $X_i \in M$ and any $b \in B$ then either $X_i \cap b = \emptyset$ iff $b \notin V(\mathcal{I}_i)$. Or $X_i \cap b = b$ iff $b \in V(\mathcal{I}_i)$ hence $X_i \in \mathrm{L}_b(G_b(C_b))$ for every $b \in B$ meaning $M \subseteq G_V(C)$ according to Theorem 3. Next let $X_{i_j} \in M$, $i_j \in [k]$, $j \in [n]$ and assume the assertion holds true for up to $n-1$ members of M, $n \geq 2$. Hence, there either is $l \in [n-1]$ such that $i_l = i_n$ hence $X_{i_l} = X_{i_n}$ then $\bigoplus_{j\in[n]} X_{i_j} = \bigoplus_{j\in[n]\setminus\{l,n\}} X_{i_j} \in G_V(C)$. Or all X_{i_j}, $j \in [n]$, have pairwise distinct indices, hence are pairwise disjoint by construction meaning $\bigoplus_{j\in[n]} X_{i_j} = \bigcup_{j\in[n]} X_{i_j} = \bigcup_{j\in[n]} \bigcup_{b\in V(\mathcal{I}_{i_j})} b$. Thus given any $b' \in B$ it either follows $\bigoplus_{j\in[n]} X_{i_j} \cap b' = b'$ iff $b' \in \bigcup_{j\in[n]} V(\mathcal{I}_{i_j})$, or this intersection is empty implying $\bigoplus_{j\in[n]} X_{i_j} \in G_V(C)$ according to Theorem 3. So, everything that can be generated by members of M belongs to $G_V(C)$. Reversely, any $X \in G_V(C)$ induces a bipartition $B'(X) \cup B(X) = B = V(\mathcal{I}(\mathcal{H}))$ of the vertex set of $\mathcal{I}(\mathcal{H})$ defined through $X \cap b = \emptyset$, for all $b \in B'(X)$, and $X \cap b = b \neq \emptyset$, for all $b \in B(X)$. Further this bipartition equals an empty cut in $\mathcal{I}(\mathcal{H})$, indeed, otherwise there were $b' \in B'$ and $b \in B$ such that $\emptyset \neq b' \cap b = (b' \cap X) \cap b$ implying $X \cap b' \neq \emptyset$ hence a contradiction. Therefore given $i \in [k]$ one either has $V(\mathcal{I}_i) \subseteq B'(X)$, then set $i \in [k]'(X)$. Or one has $V(\mathcal{I}_i) \subseteq B(X)$, then set $i \in [k](X)$, yielding a bipartition of the index set $[k] =: [k]'(X) \cup [k](X)$ implying $X = \bigoplus_{i\in[k](X)} X_i$. Hence every member of $G_V(C)$ can be generated by elements in M finishing the proof. □

The next result is stated in [13] here it is proven:

Theorem 5 *For $\mathcal{H} = (V, B)$ one has: (i) There is an G_V-equivariant bijection $\sigma : \mathcal{A}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$. (ii) Given $C \in \mathcal{A}(\mathcal{H})$ then $\mathrm{Gn}(G_V(\sigma(C))) = \mathrm{Gn}(G_V(C)) \cup \mathrm{Gn}(G_V(B \cup B^\gamma))$. Moreover for input $\mathrm{Gn}(G_V(C))$, $\mathrm{Gn}(G_V(\sigma(C)))$ can be computed in polynomial time.*

Proof. Given $C \in \mathcal{A}(\mathcal{H})$ then set $\sigma(C) := C \cup C^\gamma \in \mathcal{S}(\mathcal{H})$ which is uniquely determined by C. Conversely, given $S \in \mathcal{S}(\mathcal{H})$ then there is the unique subformula $A(S) \in \mathcal{A}(\mathcal{H})$ such that $A(S) \cup [A(S)]^\gamma = S = \sigma(A(S))$ hence $A(S) = \sigma^{-1}(S)$. Now let $X \in G_V$, $C \in \mathcal{A}(\mathcal{H})$ then $[\sigma(C)]^X = [C \cup C^\gamma]^X = C^X \cup (C^X)^\gamma = \sigma(C^X)$, and also $\sigma^{-1}(S^X) = [\sigma^{-1}(S)]^X$ hence σ, and σ^{-1} are equivariant implying (i). Regarding (ii) one has $G_V(C) = G_V(C^\gamma)$ because $C^\gamma \in \mathcal{O}(C)$ relying on Lemma 3. Moreover the equivariance of σ directly implies $G_V(C) \leq G_V(\sigma(C))$. Hence, $G_V(\sigma(C)) \setminus G_V(C)$ can only consist of such elements $X \in G_V$ bijectively mapping the clauses in C to the clauses in C^γ. Since $c \in C \Leftrightarrow c^\gamma \in C^\gamma$ these elements are provided by $G_V(B \cup B^\gamma)$, where $B \cup B^\gamma \in \mathcal{S}_\pm(\mathcal{H})$. Finally, the assertion regarding the computational complexity therefore is implied by Lemma 10. □

6 Open Problems

Observe that given $\mathcal{H} = (V, B)$ and any $F \in \mathcal{F}(K_\mathcal{H})$, then Lemma 4 together with Corollary 3 and Corollary 1 imply that the isotropy group jumps from all to trivial, i.e., from $G_{V(C)}$ to E if one switches from $C := K_\mathcal{H}$ to $C' := K_\mathcal{H} \setminus F$, i.e., when exactly one arbitrary clause

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

is removed from W_b, for all $b \in B$. These properties of formulas shall be studied more intensive. To decrease the upper bound for the time complexity in Theorem 2 is a further research task. Also the computation of the liftings of the fibre groups to the total space has to be investigated further. Finally, the FPT-classes have to be identified more concretely.

References

[1] Aspvall, B., Plass, M.R., Tarjan, R.E., "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Information Process. Lett.* pp. 121-123, 8/1979.

[2] Boros, E., Crama, Y., Hammer, P.L., "Polynomial-time inference of all valid implications for Horn and related formulae," *Annals of Math. Artif. Intellig.* pp. 21-32, 1/1990.

[3] Boros, E., Crama, Y., Hammer, P.L., Sun, X., "Recognition of q-Horn formulae in linear time," *Discrete Appl. Math.* pp. 1-13, 55/1994

[4] Cook, S.A., "The Complexity of Theorem Proving Procedures," *3rd ACM Symposium on Theory of Computing* pp. 151-158, 1971.

[5] Downey, R.G., Fellows, M.R., *Parameterized Complexity*, Springer-Verlag, New York, 1999.

[6] Franco, J., VanGelder, A., "A perspective on certain polynomial-time solvable classes of satisfiability," *Discrete Appl. Math.* pp. 177-214, 125/2003.

[7] Minoux, M., "LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation," *Information Process. Lett.* pp. 1-12, 29/1988.

[8] Knuth, D.E., "Nested satisfiability," *Acta Informatica* pp. 1-6, 28/1990.

[9] Kratochvil, J., Krivanek, M., "Satisfiability of co-nested formulas," *Acta Informatica* pp. 397-403, 30/1993.

[10] Lewis, H.R., "Renaming a Set of Clauses as a Horn Set," *J. ACM* pp. 134-135, 25/1978.

[11] Porschen, S., "A CNF Formula Hierarchy over the Hypercube," Proc. AI 2007, *LNAI* pp. 234-243, 4830/2007.

[12] Porschen, S., "Structural Aspects of Propositional SAT," Proc. IMECS 2016, Hong Kong, pp. 126-131.

[13] Porschen, S., "Base Hypergraphs and Orbits of CNF Formulas," Proc. IMECS 2018, Hong Kong, pp. 106-111.

[14] Porschen, S., Schmidt, T., Speckenmeyer, E., Wotzlaw, A., "XSAT and NAE-SAT of linear CNF classes," *Discrete Appl. Math.* pp. 1-14, 167/2014.

[15] Porschen, S., Speckenmeyer, E., "A CNF class generalizing Exact Linear Formulas," Proc. SAT 2008, *LNCS* pp. 231-245, 2008.

[16] Porschen, S., Speckenmeyer, E., Zhao, X., "Linear CNF formulas and satisfiability," *Discrete Appl. Math.* pp. 1046-1068, 157/2009.

[17] Schlipf, J., Annexstein, F.S., Franco, J., Swaminathan, R.P., "On finding solutions for extended Horn formulas," *Information Process. Lett.* pp. 133-137, 54/1995.

[18] Spannier, E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[19] Tovey, C.A., "A Simplified NP-Complete Satisfiability Problem", *Discrete Appl. Math.* pp. 85-89, 8/1984.

ISBN: 978-988-14048-6-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)