Integral on Transcomplex Numbers

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***Abstract*—The usual complex integral is defined in terms of complex numbers in Cartesian form but transcomplex numbers are defined in polar form and almost all transcomplex numbers, with infinite magnitude, have no Cartesian form. However, there are eight infinite, transcomplex numbers which do have a Cartesian form and these can be used to define the transcomplex integral as the limit of sums of these eight numbers. Thus we introduce the transcomplex integral.**

***Index Terms*—transcomplex integral, transcomplex derivative, transcomplex number, transmathematics.**

I. Introduction

THE transreal numbers [2] [10] totalise the real numbers by allowing division by zero in terms of three definite, non-finite numbers: negative infinity, $-\infty = -1/0$; positive infinity, $\infty = 1/0$; and nullity, $\Phi = 0/0$. In earlier work, real elementary functions and real limits were extended to transreal form [1] [6], as were both real differential and integral calculus [5] [7]. This extends real analysis to transreal analysis. Further to this work, a new transreal integral is being developed, but so far we have it only for the extended-real numbers [3].

We are now in the process of extending complex analysis to transcomplex analysis. Starting with the transcomplex numbers [4] the transcomplex topology, elementary functions and limits were developed [8] [9]. In the present paper we develop the transcomplex integral and just as much of the transcomplex derivative as we need. This leaves a totalisation of the transcomplex derivative for future work, which will then extend complex analysis to transcomplex analysis. Thus the present paper can be seen as the penultimate step in extending complex analysis.

In order to understand this present paper, we advise the reader to review the transreal integral [7] and to review transcomplex numbers, their arithmetic, how their topology works, and how the elementary functions are defined on them [9].

The natural numbers have two different definitions, either including or excluding zero. The former definition is popular in Computer Science, the latter in Mathematics. Here we follow the mathematical convention $\mathbb{N} = \{1, 2, 3, \dots\}$.

II. Initial Considerations

In the complex domain, the integral along a curve is defined as follows. If $f : [a, b] \to \mathbb{C}$ is a function then, taking $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ such that $f = u + iv$, f is integrable in $[a, b]$ if and only if u and v are integrable in $[a, b]$ and the integral of f in $[a, b]$ is defined as $\int_a^b f(t)\ \mathrm{d}t := \int_a^b u(t)\ \mathrm{d}t + i \int_a^b v(t)\ \mathrm{d}t$. A smooth path is a differentiable function $\gamma : [a, b] \to \mathbb{C}$ such that γ' is continuous. Given a smooth path $\gamma : [a, b] \to \mathbb{C}$ and $f : \gamma([a, b]) \to \mathbb{C}$, f is integrable on γ if and only if $(f \circ \gamma)\gamma'$ is integrable in $[a, b]$ and the integral of f on γ is defined as $\int_\gamma f(z)\ \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t$.

Notice that the definition of the complex integral is closely linked to the Cartesian form of complex numbers, $a + ib$ where $a, b \in \mathbb{R}$ and i is the imaginary unit. So we have a big problem to define the integral in the transcomplex domain. Since almost all infinite transcomplex number cannot be written as $a + ib$ with $a, b \in \mathbb{R}^T$, not all transcomplex functions can be represented by $u + iv$ with u and v being transreal functions.

Observe that only eight infinite transcomplex numbers can be written as $a + ib$ with $a, b \in \mathbb{R}^T$, namely, $\frac{1}{0}$, $\frac{-1}{0}$, $\frac{i}{0}$, $\frac{-i}{0}$, $\frac{1+i}{0}$, $\frac{-1+i}{0}$, $\frac{-1-i}{0}$ and $\frac{1-i}{0}$, which are: $\infty + i0$, $-\infty + i0$, $0 + i\infty$, $0 + i(-\infty)$, $\infty + i\infty$, $-\infty + i\infty$, $-\infty + i(-\infty)$ and $\infty + i(-\infty)$, respectively. Adding these eight numbers we can get an infinite number of infinite transcomplex numbers which, although they do not have Cartesian form, they are a sum of numbers which have Cartesian form. Note also these eight numbers are, in exponential form: ∞e^{i0}, $\infty e^{i\pi}$, $\infty e^{i\frac{\pi}{2}}$, $\infty e^{-i\frac{\pi}{2}}$, $\infty e^{i\frac{\pi}{4}}$, $\infty e^{i\frac{3\pi}{4}}$, $\infty e^{-i\frac{3\pi}{4}}$, $\infty e^{-i\frac{\pi}{4}}$, respectively. Summing numbers from these eight, we get numbers of the form $\infty e^{i\frac{l\pi}{2^n}}$ with $l, n \in \{0\} \cup \mathbb{N}$. Now notice that $\frac{l}{2^n}$, called dyadic rational numbers, are dense in $\mathbb{R}$. Therefore few infinite transcomplex numbers have Cartesian form but every infinite transcomplex number is the limit of a sequence of numbers which are sums of numbers which have Cartesian form.

The transcomplex integral, which we define here, is closely grounded in the above fact. For each transcomplex function f we take $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ and f_n can be written as $\sum_{k=1}^m (u_k + iv_k)$ for some $m \in \mathbb{N}$ where u_k and v_k are transreal functions.

III. The Integral on Transcomplex Numbers

A series of complex numbers is defined as the sequence $(s_n)_{n\in\mathbb{N}}$ where $s_n := \sum_{i=1}^n z_i = z_1 + \cdots + z_n$ and $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$. We define transreal series in the same way [6]. But we need to be careful when defining series of transcomplex numbers because transcomplex addition is not associative. For example, $z_1 + z_2 + z_3$ is not well defined since $(z_1 + z_2) + z_3$ can be different from $z_1 + (z_2 + z_3)$.

Definition 1: Let $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^T$. We define $\sum_{k=1}^1 z_k := z_1$ and, for each $n \geq 2$, $\sum_{k=1}^n z_k := \left(\sum_{k=1}^{n-1} z_k\right) + z_n$. For each $n \in \mathbb{N}$ denote $s_n := \sum_{k=1}^n z_k$. The sequence $(s_n)_{n\in\mathbb{N}}$ is called a *series of transcomplex numbers* and is denoted by $\sum z_n$, each s_n is called a *partial sum of* $\sum z_n$ and z_n is called the *n-th term of* $\sum z_n$. We say that $\sum z_n$ *converges* or *is convergent* if and only if there is the $\lim_{n\to\infty} s_n$.

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Otherwise, $\sum z_n$ *diverges* or *is divergent*. When $\sum z_n$ is convergent we denote $\sum_{k=1}^{\infty} z_k := \lim_{n\to\infty} \sum_{k=1}^{n} z_k$.

Definition 2: We denote

$$\mathcal{A} := \mathbb{C} \cup \{\Phi\} \cup \left\{\infty e^{i\frac{l\pi}{2^n}};\ l,n \in \{0\}\cup\mathbb{N}\right\}$$

and for each $z \in \mathcal{A}$ we define $\sum(a_k + b_k i)$, named *the Cartesian form of* z, in the following way:

I) If $z \in \mathbb{C}$ then take $a, b \in \mathbb{R}$ such that $z = a + bi$ and define

$$\begin{array}{llll} a_1 := a & \text{and} \quad a_k := 0 & \text{for all} \quad k \geq 2 & \text{and} \\ b_1 := b & \text{and} \quad b_k := 0 & \text{for all} \quad k \geq 2. & \end{array}$$

II) If $z = \Phi$ then define

$$\begin{array}{llll} a_1 := \Phi & \text{and} \quad a_k := 0 & \text{for all} \quad k \geq 2 & \text{and} \\ b_k := 0 & \text{for all} \quad k \in \mathbb{N}. & & \end{array}$$

III) If $z = \infty$ then define

$$\begin{array}{llll} a_1 := \infty & \text{and} \quad a_k := 0 & \text{for all} \quad k \geq 2 & \text{and} \\ b_k := 0 & \text{for all} \quad k \in \mathbb{N}. & & \end{array}$$

IV) If $z \in \left\{\infty e^{i\frac{l\pi}{2^n}};\ l,n \in \{0\}\cup\mathbb{N}\right\} \setminus \{\infty\}$ then take $n \in \{0\}\cup\mathbb{N}$ and l odd with $l \in \{1,\ldots,2^{n+1}\}$ such that $z = \infty e^{i\frac{l\pi}{2^n}}$ and define $a_k := a_k^{(n,l)}$ and $b_k := b_k^{(n,l)}$ where $\left(a_k^{(n,l)}\right)_{k\in\mathbb{N}}$ and $\left(b_k^{(n,l)}\right)_{k\in\mathbb{N}}$ are defined inductively in the following way:

For $n = 0$:

$$\begin{array}{llll} a_1^{(0,1)} := -\infty & \text{and} \quad a_k^{(0,1)} := 0 & \text{for all} \quad k \geq 2, \\ b_k^{(0,1)} := 0 & \text{for all} \quad k \in \mathbb{N}. & \end{array}$$

For $n = 1$:

$$\begin{array}{llll} a_1^{(1,1)} := 0 & \text{for all} \quad k \in \mathbb{N} & & \text{and} \\ b_1^{(1,1)} := \infty & \text{and} \quad b_k^{(1,1)} := 0 & \text{for all} \quad k \geq 2 & \end{array}$$

and

$$\begin{array}{llll} a_1^{(1,3)} := 0 & \text{for all} \quad k \in \mathbb{N} & & \text{and} \\ b_1^{(1,3)} := -\infty & \text{and} \quad b_k^{(1,3)} := 0 & \text{for all} \quad k \geq 2. & \end{array}$$

For $n \geq 2$: for all $k \geq 2$,

i) if $0 \times 2^{n-2} < l \leq 1 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := \infty & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l)} \\ b_1^{(n,l)} := 0 & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l)} \end{array}$$

ii) if $1 \times 2^{n-2} < l \leq 2 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := 0 & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-2^{n-2})} \\ b_1^{(n,l)} := \infty & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-2^{n-2})} \end{array}$$

iii) if $2 \times 2^{n-2} < l \leq 3 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := 0 & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-2^{n-2})} \\ b_1^{(n,l)} := \infty & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-2^{n-2})} \end{array}$$

iv) if $3 \times 2^{n-2} < l \leq 4 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := -\infty & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-2\times 2^{n-2})} \\ b_1^{(n,l)} := 0 & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-2\times 2^{n-2})} \end{array}$$

v) if $4 \times 2^{n-2} < l \leq 5 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := -\infty & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-2\times 2^{n-2})} \\ b_1^{(n,l)} := 0 & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-2\times 2^{n-2})} \end{array}$$

vi) if $5 \times 2^{n-2} < l \leq 6 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := 0 & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-3\times 2^{n-2})} \\ b_1^{(n,l)} := -\infty & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-3\times 2^{n-2})} \end{array}$$

vii) if $6 \times 2^{n-2} < l \leq 7 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := 0 & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-3\times 2^{n-2})} \\ b_1^{(n,l)} := -\infty & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-3\times 2^{n-2})} \end{array}$$

viii) if $7 \times 2^{n-2} < l \leq 8 \times 2^{n-2}$ then

$$\begin{array}{lll} a_1^{(n,l)} := \infty & \text{and} & a_k^{(n,l)} := a_{k-1}^{(n-1,l-4\times 2^{n-2})} \\ b_1^{(n,l)} := 0 & \text{and} & b_k^{(n,l)} := b_{k-1}^{(n-1,l-4\times 2^{n-2})}. \end{array}$$

Remark 3: Notice that for all Cartesian form $\sum(a_k + b_k i)$, it follows that $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are sequences of transreal numbers which have just a finite number of non-zero elements. Because of this, $\sum_{k=1}^{\infty}(a_k + b_k i)$ is nothing more than a finite sum. So that we do not need to worry about convergence of series.

Proposition 4: Given $z \in \mathcal{A}$ and $\sum(a_k + b_k i)$ its Cartesian form, it follows that

$$z = \sum_{k=1}^{\infty}(a_k + b_k i)\,.$$

Proof: The result holds immediately from Definition 2. ■

Definition 5: Let $D \subset \mathbb{C}^T$ and $f : D \to \mathbb{C}^T$ such that $f(D) \subset \mathcal{A}$. For each $w \in D$, denote the Cartesian form of $f(w)$ as $\sum(a_k(w) + b_k i(w))$. For each $k \in \mathbb{N}$, denote as u_k the function $u_k : D \to \mathbb{R}^T$ where $u_k(w) = a_k(w)$ for all $w \in D$ and as v_k the function $v_k : D \to \mathbb{R}^T$ where $v_k(w) = b_k(w)$ for all $w \in D$. Of course, $f = \sum_{k=1}^{\infty}(u_k + v_k i)$. We call $\sum(u_k + v_k i)$ the *Cartesian form of* f.

Definition 6: For each $z \in \mathbb{C}^T$ we define $(z_n)_{n\in\mathbb{N}}$, named *the related sequence to* z, in the following way: If $z \in \mathcal{A}$ then define $z_n := z$ for all $n \in \mathbb{N}$; if $z \notin \mathcal{A}$ then take $\theta \in (\pi, 3\pi]$ such that $z = \infty e^{i\theta}$ and, for each $n \in \mathbb{N}$, take $l_n := \max\left\{t \in \mathbb{N};\ \frac{t\pi}{2^n} < \theta\right\}$ and define $z_n := \infty e^{i\frac{l_n\pi}{2^n}}$.

Proposition 7: For all $z \in \mathbb{C}^T$, the related sequence to z converges to z.

Proof: Let $z \in \mathbb{C}^T$ be arbitrary. If $z \in \mathcal{A}$ then the result is immediate; if $z \notin \mathcal{A}$ then take $\theta \in (\pi, 3\pi]$ such that $z = \infty e^{i\theta}$ and take $\left(\infty e^{i\frac{l_n\pi}{2^n}}\right)_{n\in\mathbb{N}}$, the related sequence

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to z. Notice that, by Definition 6, for all $n \in \mathbb{N}$, $\frac{l_n \pi}{2^n} < \theta \leq \frac{(l_n+1)\pi}{2^n}$ whence $0 < \theta - \frac{l_n \pi}{2^n} \leq \frac{\pi}{2^n}$. Taking n tending to infinity in the latter inequality, we have that $\lim_{n\to\infty} \frac{l_n \pi}{2^n} = \theta$ whence $\lim_{n\to\infty} \infty e^{i\frac{l_n \pi}{2^n}} = \infty e^{i\theta} = z$. ∎

Definition 8: Let $D \subset \mathbb{C}^T$ and $f : D \to \mathbb{C}^T$ be arbitrary. We define $(f_n)_{n\in\mathbb{N}}$, named *the related sequence of functions to* f, in the following way: for each $w \in D$, take $(z_n)_{n\in\mathbb{N}}$, the related sequence to $f(w)$. For each $n \in \mathbb{N}$, define $f_n : D \to \mathbb{C}^T$ such that $f_n(w) = z_n$.

The metric d and the homeomorphism φ used henceforth are defined in [9].

Proposition 9: Let $D \subset \mathbb{C}^T$ be arbitrary. For all $f : D \to \mathbb{C}^T$, the related sequence of functions to f converges uniformly to f.

Proof: Let $D \subset \mathbb{C}^T$, $f : D \to \mathbb{C}^T$ and $(f_n)_{n\in\mathbb{N}}$ be the related sequence of functions to f. Let positive $\varepsilon \in \mathbb{R}$ be arbitrary. For each $w \in D$ with $f(w) \notin \mathcal{A}$, denote $f(w) = \infty e^{i\theta(w)}$, where $\theta(w) \in (\pi, 3\pi]$, and denote the related sequence to $f(w)$ as $\left(\infty e^{i\frac{l_n(w)\pi}{2^n}}\right)_{n\in\mathbb{N}}$. As the function $g : \mathbb{R} \to \mathbb{C}$, where $g(x) = e^{ix}$ for all $x \in \mathbb{R}$, is uniformly continuous in $[0, 3\pi]$, it follows that there is a positive $\delta \in \mathbb{R}$ such that $|e^{ix} - e^{iy}| < \varepsilon$ whenever $x, y \in [0, 3\pi]$ and $|x - y| < \delta$. Let $m \in \mathbb{N}$ such that $\frac{\pi}{2^m} < \delta$. It follows that if $n \geq m$ then $\left|\theta(w) - \frac{l_n(w)\pi}{2^n}\right| = \theta(w) - \frac{l_n(w)\pi}{2^n} \leq \frac{\pi}{2^n} < \delta$ for all $w \in D$ with $f(w) \notin \mathcal{A}$ whence $d(f_n(w), f(w)) = \left|\varphi(f_n(w)), \varphi(f(w))\right| = \left|\frac{1}{1+\frac{1}{\infty}} e^{i\frac{l_n(w)\pi}{2^n}} - \frac{1}{1+\frac{1}{\infty}} e^{i\theta(w)}\right| = \left|e^{i\frac{l_n(w)\pi}{2^n}} - e^{i\theta(w)}\right| < \varepsilon$ for all $w \in D$ with $f(w) \notin \mathcal{A}$. Furthermore, $d(f_n(w), f(w)) = d(f(w), f(w)) = 0 < \varepsilon$ for all $n \in \mathbb{N}$ and for all $w \in D$ with $f(w) \in \mathcal{A}$. Whatever, if $n \geq m$ then $d(f_n(w), f(w)) < \varepsilon$ for all $w \in D$. ∎

Definition 10: Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{C}^T$ such that $f([a, b]) \subset \mathcal{A}$ and take $\sum (u_k + iv_k)$ its Cartesian form. We say that f is *integrable in* $[a, b]$ if and only if u_k and v_k are integrable in $[a, b]$ for all $k \in \mathbb{N}$. If f is integrable in $[a, b]$, the *integral of* f *in* $[a, b]$ is defined as

$$\int_a^b f(t)\ \mathrm{d}t = \sum_{k=1}^{\infty} \left(\int_a^b u_k(t)\ \mathrm{d}t + i \int_a^b v_k(t)\ \mathrm{d}t \right).$$

Definition 11: Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to \mathbb{C}^T$ and $(f_n)_{n\in\mathbb{N}}$ be the related sequence of functions to f. We say that f is *integrable in* $[a, b]$ if and only if f_n is integrable in $[a, b]$ for all $n \in \mathbb{N}$ and $\left(\int_a^b f_n(t)\ \mathrm{d}t\right)_{n\in\mathbb{N}}$ is convergent. If f is integrable in $[a, b]$, the *integral of* f *in* $[a, b]$ is defined as

$$\int_a^b f(t)\ \mathrm{d}t = \lim_{n\to\infty} \int_a^b f_n(t)\ \mathrm{d}t.$$

Remark 12: Notice that if f has Cartesian form then definitions 10 and 11 give the same result.

Definition 13: Let $D \subset \mathbb{C}^T$. A *path in* D is a continuous function $\gamma : [a, b] \to D$ where $a, b \in \mathbb{R}$ and $a < b$. The image of the function γ is denoted by $|\gamma|$.

Remark 14: For every path γ, either $|\gamma| = \{\Phi\}$ or $\Phi \notin |\gamma|$. Indeed, as γ is continuous, Φ is an isolated point and images of connected sets by continuous functions are connected ones, if $\Phi \in |\gamma|$ then $|\gamma| = \{\Phi\}$.

Now we define the derivative of a path. If $\gamma(t) \in \mathbb{C}$ then we have already the usual definition $\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h)-\gamma(t)}{h}$. If $\gamma(t) = \Phi$ then $\gamma \equiv \Phi$ and we define $\gamma'(t) = \Phi$. We have a difficulty when $\gamma(t) \in \mathbb{C}^T_\infty$. If $\gamma(t) \in \mathbb{C}^T_\infty$ we have two possibilities: either there is a neighbourhood U of t such that $\gamma(U) \subset \mathbb{C}^T_\infty$ or $\gamma(U) \cap \mathbb{C} \neq \emptyset$ for all neighbourhoods U of t. In the first case, for all $s \in U$, $\gamma(s) = \infty e^{i\theta}$ for some $\theta \in \mathbb{R}$. Hence, as γ is continuous, $\gamma(U)$ is an arc of the circle at infinity. Thus there is a path β in $\mathbb{C}$ such that $\gamma(s) = \infty\beta(s)$ for all $s \in U$ and we define $\gamma'(t) = \infty\beta'(t)$. In the second case, $t \in \overline{\gamma^{-1}(\mathbb{C})}$ so if γ is differentiable in $\gamma^{-1}(\mathbb{C})$ we define $\gamma'(t) = \lim_{s\to t} \gamma'(s)$ if this limit exist.

Definition 15: Let $\gamma : [a, b] \to D \subset \mathbb{C}^T$ be a path and $t \in [a, b]$. Henceforth $\gamma'_{\mathbb{C}}(t)$ denotes the usual complex derivative of γ in t. We say that γ *is differentiable in* t if and only if one of the following conditions holds:

i) $\gamma(t) \in \mathbb{C}$ and γ is differentiable in t in the usual sense. In this case we define the *derivative of* γ *in* t as the usual derivative of γ in t, that is, $\gamma'(t) := \gamma'_{\mathbb{C}}(t)$.
ii) $\gamma(t) = \Phi$. In this case we define the *derivative of* γ *in* t as Φ, that is, $\gamma'(t) := \Phi$.
iii) $\gamma(t) \in \mathbb{C}^T_\infty$ and there is a path β in $\mathbb{C}$ and a neighbourhood U of t such that $\gamma(s) = \infty\beta(s)$ for all $s \in U \cap [a, b]$. In this case we define the *derivative of* γ *in* t as $\infty\beta'(t)$, that is, $\gamma'(t) := \infty\beta'(t)$.
iv) $\gamma(t) \in \mathbb{C}^T_\infty$ and $t \in \overline{E}$, where E is the set of all elements s from $[a, b]$ such that $\gamma(s) \in \mathbb{C}$ and γ is differentiable in s, and there is $\lim_{s\to t} \gamma'_{\mathbb{C}}(s)$. In this case we define the *derivative of* γ *in* t as $\lim_{s\to t} \gamma'_{\mathbb{C}}(s)$, that is, $\gamma'(t) := \lim_{s\to t} \gamma'_{\mathbb{C}}(s)$.

Definition 16: Let $\gamma : [a, b] \to D$ be a path. We say that γ is smooth when γ is differentiable and γ' is continuous in $[a, b]$.

Definition 17: Let $\gamma : [a, b] \to \mathbb{C}^T$ be a smooth path and $f : |\gamma| \to \mathbb{C}^T$ be a function. We say that f is *integrable on* γ if and only if $(f \circ \gamma)\gamma'$ is integrable in $[a, b]$. If f is integrable on γ, the *integral of* f *on* γ is defined as

$$\int_\gamma f(z)\ \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t.$$

Proposition 18: Every complex function of complex variable is integrable in the usual sense if and only if it is integrable in the transcomplex sense. In other words: let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $f : |\gamma| \to \mathbb{C}$ be a function, it follows that f is integrable on γ in the usual sense if and only if f is integrable on γ in the transcomplex sense. Furthermore both integrals have the same value.

Proof: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $f : |\gamma| \to \mathbb{C}$ be a function. As $((f \circ \gamma)\gamma')([a, b]) \subset \mathbb{C}$, there are

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functions $u:[a,b]\to\mathbb{R}$ and $v:[a,b]\to\mathbb{R}$ such that $u+vi$ is the Cartesian form of $(f\circ\gamma)\gamma'$.

It follows that f is integrable on γ in the usual sense if and only if $(f\circ\gamma)\gamma'$ is integrable in $[a,b]$ in the usual sense if and only if u and v are integrable in $[a,b]$ in the usual sense if and only if u and v are integrable in $[a,b]$ in the transreal sense ([7], Proposition 49.a) if and only if, by Definition 10, $(f\circ\gamma)\gamma'$ is integrable in $[a,b]$ in the transcomplex sense if and only if, by Definition 17, f is integrable on γ in the transcomplex sense. And

$$\underset{\mathbb{C}}{\int_\gamma} f(z)\ \mathrm{d}z=\underset{\mathbb{C}}{\int_a^b}((f\circ\gamma)\gamma')(t)\ \mathrm{d}t=\underset{\mathbb{R}}{\int_a^b}u(t)\ \mathrm{d}t+i\underset{\mathbb{R}}{\int_a^b}v(t)\ \mathrm{d}t$$

$$=\int_a^b u(t)\ \mathrm{d}t+i\int_a^b v(t)\ \mathrm{d}t=\int_a^b((f\circ\gamma)\gamma')(t)\ \mathrm{d}t=\int_\gamma f(z)\ \mathrm{d}z$$

where $\underset{\mathbb{C}}{\int_\gamma} f(z)\ \mathrm{d}z$ denotes the integral of f on γ in the usual sense and $\underset{\mathbb{C}}{\int_a^b}((f\circ\gamma)\gamma')(t)\ \mathrm{d}t$ denotes the integral of $(f\circ\gamma)\gamma'$ in $[a,b]$ in the usual sense and $\underset{\mathbb{R}}{\int_a^b}u(t)\ \mathrm{d}t$ and $\underset{\mathbb{R}}{\int_a^b}v(t)\ \mathrm{d}t$ denote, respectively, the integral of u and v in $[a,b]$ in the usual sense. ■

Example 19: Let us calculate the integral of $|z|$ along the semi-straight line from 0 to ∞i.

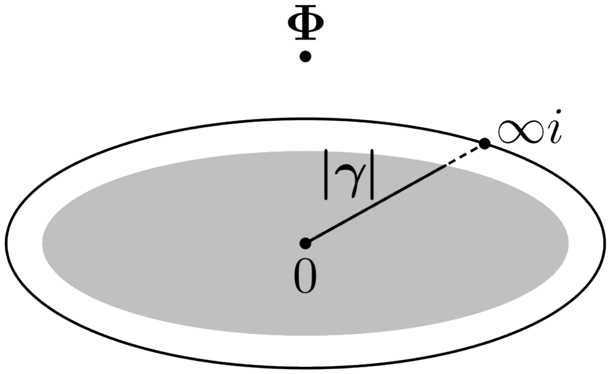

Fig. 1. A Semi-straight Line

Let $f:\mathbb{C}^T\to\mathbb{C}^T$ where $f(z)=|z|$ and $\gamma:[0,1]\to\mathbb{C}^T$ where $\gamma(t)=\frac{t}{1-t}i$. Note that γ is continuous and differentiable with $\gamma'(t)=\frac{1}{(1-t)^2}i$. Thus $\int_\gamma f(z)\ \mathrm{d}z=\int_0^1 f(\gamma(t))\gamma'(t)\ \mathrm{d}t=\int_0^1\left|\frac{t}{1-t}i\right|\frac{1}{(1-t)^2}i\ \mathrm{d}t=\int_0^1\frac{t}{1-t}\frac{1}{(1-t)^2}i\ \mathrm{d}t=i\int_0^1\frac{t}{(1-t)^3}\ \mathrm{d}t=\infty i$.

Example 20: Let us calculate the integral of $\overline{z}$ along the circle at infinity. Let $f:\mathbb{C}^T\to\mathbb{C}^T$ where $f(z)=\overline{z}$ and $\gamma:[-\pi,\pi]\to\mathbb{C}^T$ where $\gamma(t)=\infty e^{it}$. Note that γ is continuous and differentiable with $\gamma'(t)=\infty ie^{-it}$. Thus $\int_\gamma f(z)\ \mathrm{d}z=\int_{-\pi}^{\pi}f(\gamma(t))\gamma'(t)\ \mathrm{d}t=\int_{-\pi}^{\pi}\overline{\infty e^{it}}\infty ie^{it}\ \mathrm{d}t=\int_{-\pi}^{\pi}\infty e^{-it}\infty ie^{it}\ \mathrm{d}t=\int_{-\pi}^{\pi}\infty ie^{-it}e^{it}\ \mathrm{d}t=\int_{-\pi}^{\pi}\infty i\ \mathrm{d}t=i\int_{-\pi}^{\pi}\infty\ \mathrm{d}t=\infty i\times 2\pi=\infty i$.

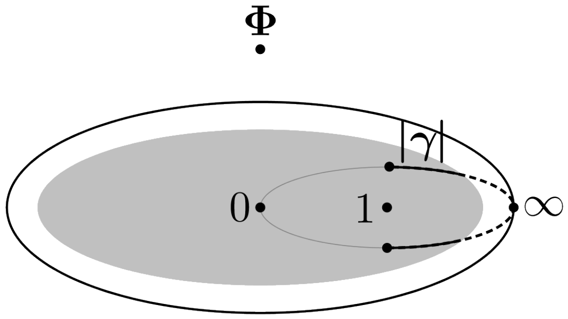

Fig. 2. A Semi-circle

Example 21: Let us calculate the integral of $\frac{1}{|z|^2}$ along C, a semi-circle of centre 1 and radius $\frac{1}{2}$.
Notice that, for all $z\in C$, $d(z,1)=\frac{1}{2}$ whence $|\varphi(z)-\varphi(1)|=\frac{1}{2}$, hence $\left|\varphi(z)-\frac{1}{2}\right|=\frac{1}{2}$. As $d(z,1)=\frac{1}{2}$ for all $z\in C$, we have that $\Phi\notin C$. Because of this, for all $z\in C$ there is $w\in\overline{B_{\mathbb{C}}(0,1)}$ such that $z=\varphi^{-1}(w)$. Thus C is made from points $\varphi^{-1}(w)$, with $w\in\mathbb{C}$, such that $\left|\varphi(\varphi^{-1}(w))-\frac{1}{2}\right|=\frac{1}{2}$, that is, $\left|w-\frac{1}{2}\right|=\frac{1}{2}$. But $\left|w-\frac{1}{2}\right|=\frac{1}{2}$ if and only if $w=\frac{1}{2}+\frac{1}{2}e^{it}$ for some $t\in\mathbb{R}$. Therefore each point of C is given by $\varphi^{-1}\left(\frac{1}{2}+\frac{1}{2}e^{it}\right)=\frac{\left|\frac{1}{2}+\frac{1}{2}e^{it}\right|}{1-\left|\frac{1}{2}+\frac{1}{2}e^{it}\right|}e^{i\mathrm{Arg}\left(\frac{1}{2}+\frac{1}{2}e^{it}\right)}$
$=\frac{1}{2-\sqrt{2+2\cos(t)}}(1+\cos(t)+i\sin(t))$ for some $t\in\mathbb{R}$.

Let us take $\gamma:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{C}^T$, where $\gamma(t)=\frac{1}{2-\sqrt{2+2\cos(t)}}(1+\cos(t)+i\sin(t))$, and calculate the integral of $f:\mathbb{C}^T\to\mathbb{C}^T$, where $f(z)=\frac{1}{|z|^2}$, along $|\gamma|$. Firstly, note that γ is continuous. Indeed, if $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$ then, clearly γ is continuous in t and, furthermore, $\gamma(0)=\infty$ and $\lim_{t\to 0}\gamma(t)=\infty$ whence γ is also continuous in 0. Secondly, note that γ is differentiable. In fact, clearly γ is differentiable in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$ with $\gamma'(t)=\frac{2\sin(t)\left(\sqrt{2+2\cos(t)}-4\right)}{4\left(2-\sqrt{2+2\cos(t)}\right)^2}+i\frac{8\cos(t)-\left(\sqrt{2+2\cos(t)}\right)^3}{4\left(2-\sqrt{2+2\cos(t)}\right)^2}$ and $\lim_{t\to 0}\gamma'(t)=\infty$ whence γ is differentiable in 0 with $\gamma'(0)=\infty$. Furthermore γ' is continuous. Thus γ is a smooth path.

Now, notice that $f(\gamma(t))=\frac{1}{|\gamma(t)|^2}=\frac{\left(2-\sqrt{2+2\cos(t)}\right)^2}{2+2\cos(t)}$ for all $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Thus $f(\gamma(t))\gamma'(t)=\frac{\sin(t)\left(\sqrt{2+2\cos(t)}-4\right)}{2(2+2\cos(t))}+i\frac{8\cos(t)-\left(\sqrt{2+2\cos(t)}\right)^3}{4(2+2\cos(t))}$ for all $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$ and $f(\gamma(0))\gamma'(0)=\Phi$. Therefore

$$\begin{aligned}\int_\gamma f(z)\ \mathrm{d}z &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\gamma(t))\gamma'(t)\ \mathrm{d}t\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin(t)\left(\sqrt{2+2\cos(t)}-4\right)}{2(2+2\cos(t))}\ \mathrm{d}t\\ &\quad +i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8\cos(t)-\left(\sqrt{2+2\cos(t)}\right)^3}{4(2+2\cos(t))}\mathrm{d}t\\ &= (\pi-2-\sqrt{2})i.\end{aligned}$$

Example 22: Let us calculate the integral of z along a semi-circle at infinity. Let $f:\mathbb{C}^T\to\mathbb{C}^T$ where $f(z)=z$ and $\gamma:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{C}^T$ where $\gamma(t)=\infty e^{it}$. Note that γ is continuous and continuously differentiable with $\gamma'(t)=\infty ie^{-it}$. Thus $\int_\gamma f(z)\ \mathrm{d}z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(\gamma(t))\gamma'(t)\ \mathrm{d}t=$

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$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \infty e^{it} \infty i e^{it} \ \mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \infty i e^{i2t} \ \mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \infty e^{i(2t+\pi)} \ \mathrm{d}t = \lim_{n\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_n(t) \ \mathrm{d}t$ where, for each $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $(f_n(t))_{n\in\mathbb{N}}$ is the related sequence to $\infty e^{i(2t+\pi)}$.

Now, notice that, given $n \in \mathbb{N}$, from Definition 6 it follows that, $f_n(t) = \infty e^{i\frac{(l-1)\pi}{2^n}}$ for all $t \in \left[\frac{-2^n+l-1}{2^{n+1}}\pi, \frac{-2^n+l}{2^{n+1}}\pi\right)$, for each $l \in \{1, \ldots, 2^{n+1}\}$. Hence, given $n \in \mathbb{N}$, denoting the Cartesian form of f_n as $\sum \left(u_k^{(n)} + iv_k^{(n)}\right)$ and denoting, for each $l \in \{1, \ldots, 2^{n+1}\}$ and for each $t \in \left[\frac{-2^n+l-1}{2^{n+1}}\pi, \frac{-2^n+l}{2^{n+1}}\pi\right)$, the Cartesian form of $f_n(t) = \infty e^{i\frac{(l-1)\pi}{2^n}}$ as $\sum \left(a_k^{(n,\, l-1)} + ib_k^{(n,\, l-1)}\right)$, it follows that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_n(t) \ \mathrm{d}t =$$

$$\sum_{k=1}^{\infty} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_k^{(n)}(t) \ \mathrm{d}t + i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_k^{(n)}(t) \ \mathrm{d}t \right) =$$

$$\sum_{k=1}^{\infty} \left(\sum_{l=1}^{2^{n+1}} \int_{\frac{-2^n+l-1}{2^{n+1}}\pi}^{\frac{-2^n+l}{2^{n+1}}\pi} a_k^{(n,\, l-1)} \mathrm{d}t + i \sum_{l=1}^{2^{n+1}} \int_{\frac{-2^n+l-1}{2^{n+1}}\pi}^{\frac{-2^n+l}{2^{n+1}}\pi} b_k^{(n,\, l-1)} \mathrm{d}t \right) =$$

$$\sum_{k=1}^{\infty} \left(\sum_{l=1}^{2^{n+1}} a_k^{(n,\, l-1)} + i \sum_{l=1}^{2^{n+1}} b_k^{(n,\, l-1)} \right) =$$

$$\sum_{k=1}^{\infty} (\Phi + i\Phi) = \Phi.$$

Example 23: If $\gamma : [a, b] \to \mathbb{C}^T$ is the constant path $\gamma \equiv \Phi$ then $\int_\gamma f(z) \ \mathrm{d}z = \Phi$ for all $f : \{\Phi\} \to \mathbb{C}^T$. Indeed, $\int_\gamma f(z) \ \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \ \mathrm{d}t = \int_a^b f(\Phi) \times \Phi \ \mathrm{d}t = \int_a^b \Phi \ \mathrm{d}t = \Phi$.

IV. Conclusion

Earlier, real elementary functions, real limits and both real differential and integral calculus were extended to transreal forms. This extended real analysis to transreal analysis.

We now introduce the transcomplex integral and, incidentally, the derivative for transcomplex functions whose domain is a real interval. In future work, totalising the transcomplex derivative would complete the task of extending the main elements of complex analysis to transcomplex analysis.

Taking these results all together, a very large part of practical computation is extended to transnumbers.

References

[1] J. A. D. W. Anderson and T. S. dos Reis, "Transreal limits expose category errors in ieee 754 floating-point arithmetic and in mathematics," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA, pp 86–91.

[2] J. A. D. W. Anderson, N. Völker, and A. A. Adams, Perspex machine viii: Axioms of transreal arithmetic. In Longin Jan Lateki, David M. Mount, and Angela Y. Wu, editors, *Vision Geometry XV*, volume 6499 of *Proceedings of SPIE*, pages 2.1–2.12, 2007.

[3] T. S. dos Reis, "Proper and improper Riemann integral in a single definition," Proceeding Series of the Brazilian Society of Computational and Applied Mathematics. 5:010017-1–010017-7, 2016.

[4] T. S. dos Reis and J. A. D. W. Anderson, "Construction of the transcomplex numbers from the complex numbers," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA, pp 97–102.

[5] T. S. dos Reis and J. A. D. W. Anderson, "Transdifferential and transintegral calculus," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA, pp 92–96.

[6] T. S. dos Reis and J. A. D. W. Anderson, (2015) Transreal limits and elementary functions. In: Kim H., Amouzegar M., Ao S. (eds) Transactions on Engineering Technologies. Springer, Dordrecht, pp 209–225.

[7] T. S. dos Reis and J. A. D. W. Anderson, "Transreal calculus," IAENG International Journal of Applied Mathematics, vol. 45, no. 1, pp 51–63, 2015.

[8] T. S. dos Reis and J. A. D. W. Anderson, "Transcomplex topology and elementary functions," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2016, WCE 2016, 29 June - 1 July, 2016, London, U.K., pp 164-169.

[9] T. S. dos Reis and J. A. D. W. Anderson, "Transcomplex numbers: properties, topology and functions," Engineering Letters, vol. 25, no. 1. pp 90–103, 2017.

[10] T. S. dos Reis, W. Gomide, and J. A. D. W. Anderson, "Construction of the transreal numbers and algebraic transfields," IAENG International Journal of Applied Mathematics, vol. 46, no. 1, pp 11–23, 2016.

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