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Formulas for Partial Fraction Expansion of Proper Rational Functions of Certain Types

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***Abstract*—In this paper, we present formulas to expand partial fraction of proper rational functions of types $\frac{1}{x^n(x^m+a)}$ and $\frac{1}{x^n(x^m-a)}$. The idea of formulas is to save the time and effort to solve these particular types. Applications of these formulas in engineering mathematics are also included in the examples given below.**

***Index Terms*—partial fraction expansion, integral calculus, inverse Laplace transforms, differential equations.**

I. Introduction

Partial fraction expansion is a technique by which a fraction can be decomposed as a sum of two or more simpler fractions. The problem of partial fraction is generally used in the study of integral calculus, differential equations and some areas of applied mathematics. For solving it, the method of undetermined coefficient is used as common approach. If these particular types are solved using common approach, sufficient number of unknown coefficients are involved which makes this approach really time consuming. So, by analyzing the pattern in the solution of such types of functions, we have derived a generalized solution for these particular types.

II. Theoretical Background

These are the following results on which the existence of partial fraction expansions of the given proper rational functions are based. We can refer to [1] for proof.

Theorem 2.1. Let any function F(x) can be written as,

$$F(x) = \frac{P(x)}{Q(x)}$$

Where P(x) and Q(x) are polynomials in x such that deg. P(x) < deg. Q(x).

For the function $\frac{1}{x^n(x^m+a)}$, P(x) = 1, Q(x) = $x^n(x^m+a)$

Where m & n are integers.

We may assume without loss of generality that m divides n by multiplying by suitable power of x.

On solving this function by 'The method of undetermined coefficient'.

$$F(x) = \frac{P(x)}{Q(x)} = \frac{1}{x^n(x^m+a)} = \left(\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \cdots + \frac{A_n}{x^n}\right) + \left(\frac{B_1x^{m-1} + B_2x^{m-2} + \cdots + B_{m-2}x + B_{m-1}}{x^m+a}\right)$$

Where $A_1, A_2, A_3 \ldots . A_n$ and $B_1, B_2, B_3 \ldots . B_{m-1}$ are unknown coefficients of partial fraction expansion.

This approach involves calculation for sufficient number of unknown coefficients. That's why it becomes time consuming in particular problems.

Theorem 2.2. Partial fraction expansion of this particular type in theorem 2.1 is solved directly by putting the values of m, n and a in following formulas. Provided m and n are integers and m divides n by multiplying suitable power of x.

<u>Formula(I):</u>

$$\frac{1}{x^n(x^m+a)} = \frac{1}{ax^n} - \frac{1}{a^2x^{n-m}} + \frac{1}{a^3x^{n-2m}} - \frac{1}{a^4x^{n-3m}} + \ldots\ldots\ldots + (-1)^{(\frac{n}{m}-1)}.\frac{1}{a^{n/m}x^m} + (-1)^{\frac{n}{m}}.\frac{1}{a^{n/m}(x^m+a)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ i}$$

<u>Formula(II):</u> Similarly, we can write generalized formula for the function $\frac{1}{x^n(x^m-a)}$,

$$\frac{1}{x^n(x^m-a)} = -\frac{1}{ax^n} - \frac{1}{a^2x^{n-m}} - \frac{1}{a^3x^{n-2m}} - \frac{1}{a^4x^{n-3m}} - \ldots\ldots - \frac{1}{a^{n/m}x^m} + \frac{1}{a^{n/m}(x^m-a)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ ii}$$

Note that, number of terms in expansion = $\left(\frac{n}{m}+1\right)$.

x could even be like y^π and it would give partial fraction in terms of y^π.

Since m divides n by a suitable integer. So we can take,

$\frac{n}{m} = r$ or n=mr , where r is integer.

Note that number of terms in expansion = (r+1)

Now above these formulas i and ii can also be written as,

$$\frac{1}{x^{mr}(x^m+a)} = \frac{1}{ax^{mr}} - \frac{1}{a^2x^{m(r-1)}} + \frac{1}{a^3x^{m(r-2)}} - \frac{1}{a^4x^{m(r-3)}} + \ldots + (-1)^{(r-1)}.\frac{1}{a^rx^m} + (-1)^r.\frac{1}{a^r(x^m+a)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ iii}$$

$$\frac{1}{x^{mr}(x^m-a)} = -\frac{1}{ax^{mr}} - \frac{1}{a^2x^{m(r-1)}} - \frac{1}{a^3x^{m(r-2)}} - \frac{1}{a^4x^{m(r-3)}} - \ldots\ldots\ldots - \frac{1}{a^rx^m} + \frac{1}{a^r(x^m-a)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ iv}$$

Proof. Proof of these formulas can be easily done by mathematical induction method.

<u>Formula(I)</u>

$$\frac{1}{x^{mr}(x^m+a)} = \frac{1}{ax^{mr}} - \frac{1}{a^2x^{m(r-1)}} + \frac{1}{a^3x^{m(r-2)}} - \frac{1}{a^4x^{m(r-3)}} + \ldots + (-1)^{(r-1)}.\frac{1}{a^rx^m} + (-1)^r.\frac{1}{a^r(x^m+a)}$$

Checking both sides for r=1.

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Then number of terms in expansion = (r+1) = 2

$$\frac{1}{x^m(x^m+a)}=\frac{1}{ax^m}-\frac{1}{a(x^m+a)}$$
$$=\frac{1}{a}\left[\frac{1}{x^m}-\frac{1}{(x^m+a)}\right]$$
$$=\frac{1}{a}\left[\frac{(x^m+a)-x^m}{x^m(x^m+a)}\right]$$
$$=\frac{1}{x^m(x^m+a)}$$

Hence, it is true for r = 1.

Now if formula is true for r = k then it is true for r = k+1 also.

Hence, checking both sides for r = k + 1,

$$\frac{1}{x^{m(k+1)}(x^m+a)}=\frac{1}{ax^{m(k+1)}}-\frac{1}{a^2x^{mk}}+\frac{1}{a^3x^{m(k-1)}}-\frac{1}{a^4x^{m(k-2)}}+\ldots..+(-1)^k.\frac{1}{a^{k+1}x^m}+(-1)^{k+1}.\frac{1}{a^{k+1}(x^m+a)}$$
$$=\frac{1}{ax^{m(k+1)}}-\frac{1}{a}\left[\frac{1}{ax^{mk}}-\frac{1}{a^2x^{m(k-1)}}+\frac{1}{a^3x^{m(k-2)}}-\frac{1}{a^4x^{m(k-3)}}+\ldots\ldots\ldots-(-1)^k.\frac{1}{a^kx^m}-(-1)^{k+1}.\frac{1}{a^k(x^m+a)}\right]$$
$$=\frac{1}{ax^{m(k+1)}}-\frac{1}{a}\left[\frac{1}{x^{mk}(x^m+a)}\right]$$
$$=\frac{1}{a}\left[\frac{1}{x^{m(k+1)}}-\frac{1}{x^{mk}(x^m+a)}\right]$$
$$=\frac{1}{a}\left[\frac{x^m+a-x^m}{x^{m(k+1)}(x^m+a)}\right]$$
$$=\frac{1}{x^{m(k+1)}(x^m+a)}$$

Therefore, this formula is also true for r = k + 1.

Formula(II)

$$\frac{1}{x^{mr}(x^m-a)}=-\frac{1}{ax^{mr}}-\frac{1}{a^2x^{m(r-1)}}-\frac{1}{a^3x^{m(r-2)}}-\frac{1}{a^4x^{m(r-3)}}-\ldots\ldots..-\frac{1}{a^rx^m}+\frac{1}{a^r(x^m-a)}$$

Checking both sides for r = 1.

then number of terms in expansion = r + 1 = 2

$$\frac{1}{x^m(x^m-a)}=-\frac{1}{ax^m}+\frac{1}{a(x^m-a)}$$
$$=\frac{1}{a}\left[-\frac{1}{x^m}+\frac{1}{(x^m-a)}\right]$$
$$=\frac{1}{a}\left[\frac{-x^m+a+x^m}{x^m(x^m-a)}\right]$$
$$=\frac{1}{x^m(x^m-a)}$$

Hence, it is true for r = 1.

Now, if formula is true for r = k then it is also true for r = k + 1

Hence, checking both sides for r = k + 1

$$\frac{1}{x^{m(k+1)}(x^m-a)}=-\frac{1}{ax^{m(k+1)}}-\frac{1}{a^2x^{mk}}-\frac{1}{a^3x^{m(k-1)}}-\frac{1}{a^4x^{m(k-2)}}-\ldots\ldots-\frac{1}{a^{(k+1)}x^m}+\frac{1}{a^{(k+1)}(x^m-a)}$$
$$=-\frac{1}{ax^{m(k+1)}}+\frac{1}{a}\left[-\frac{1}{ax^{mk}}-\frac{1}{a^2x^{m(k-1)}}-\frac{1}{a^3x^{m(k-2)}}-\ \ldots\ldots-\frac{1}{a^kx^m}+\frac{1}{a^k(x^m-a)}\right]$$
$$=-\frac{1}{ax^{m(k+1)}}+\frac{1}{a}\left[\frac{1}{x^{mk}(x^m-a)}\right]$$
$$=\frac{1}{a}\left[-\frac{1}{x^{m(k+1)}}+\frac{1}{x^{mk}(x^m-a)}\right]$$
$$=\frac{1}{a}\left[\frac{-x^m+a+x^m}{x^{m(k+1)}(x^m-a)}\right]$$
$$=\frac{1}{x^{m(k+1)}(x^m-a)}$$

Therefore, this formula is also true for r = k + 1.

III. Examples

We are now demonstrating some solved examples on the basis of these formulas, including its application in certain topics in engineering mathematics such as integral calculus, inverse Laplace transformation and differential equations.

Example 3.1 Find partial fraction decomposition of Algebraic Function $\frac{1}{x^{21}(x^3+1)}$.

Solution. On comparing this function with the generalized form $\frac{1}{x^n(x^m+a)}$,

$n=21,\ m=3\ and\ a=1.$

Here, n is a multiple of m

Now, no. of terms $=(\frac{n}{m}+1)=\left(\frac{21}{3}+1\right)=8$

On applying formula we get,

$$\frac{1}{x^{21}(x^3+1)}=\frac{1}{x^{21}}-\frac{1}{x^{18}}+\frac{1}{x^{15}}-\frac{1}{x^{12}}+\frac{1}{x^9}-\frac{1}{x^6}+\frac{1}{x^3}-\frac{1}{x^3+1}$$

Example 3.2 Find partial fraction decomposition of Algebraic Function $\frac{1}{x^{18}(x^2-1)}$.

Solution. On comparing this function with the generalized form $\frac{1}{x^n(x^m-a)}$.

$n=18,\quad m=2\ and\ a=1$

Here, n is a multiple of m

Now, no. of terms $=(\frac{n}{m}+1)=\left(\frac{18}{2}+1\right)=10$

On applying formula we get,

$$\frac{1}{x^{18}(x^2-1)}=-\frac{1}{x^{18}}-\frac{1}{x^{16}}-\frac{1}{x^{14}}-\frac{1}{x^{12}}-\frac{1}{x^{10}}-\frac{1}{x^8}-\frac{1}{x^6}-\frac{1}{x^4}-\frac{1}{x^2}+\frac{1}{x^2-1}$$

Example 3.3 Find partial fraction decomposition of Algebraic Function $\frac{1}{x^{4\pi}(x^\pi+1)}$.

Solution. On comparing this function with the generalized form $\frac{1}{x^n(x^m+a)}$,

$n=4\pi,\ m=\pi\ and\ a=1.$

Here, n is a multiple of m

Now, no. of terms $=(\frac{n}{m}+1)=5$

On applying formula we get,

$$\frac{1}{x^{4\pi}(x^\pi+1)}=\frac{1}{x^{4\pi}}-\frac{1}{x^{3\pi}}+\frac{1}{x^{2\pi}}-\frac{1}{x^\pi}+\frac{1}{x^\pi+1}$$

Example 3.4 Find integration of function $\frac{1}{x^{12}(x^2+4)}$

Solution. On comparing this function with the generalized form $\frac{1}{x^n(x^m+a)}$,

$n=12,\ m=2\ and\ a=4.$

Here, n is a multiple of m

Now, no. of terms $=(\frac{n}{m}+1)=7$

On applying formula we get,

$$\frac{1}{x^{12}(x^2+4)}=\frac{1}{4x^{12}}-\frac{1}{4^2x^{10}}+\frac{1}{4^3x^8}-\frac{1}{4^4x^6}+\frac{1}{4^5x^4}-\frac{1}{4^6x^2}+\frac{1}{4^6(x^2+4)}$$

$$\int\frac{dx}{x^{12}(x^2+4)}=\frac{1}{4}\left(\frac{-1}{11x^{11}}\right)-\frac{1}{4^2}\left(\frac{-1}{9x^9}\right)+\frac{1}{4^3}\left(\frac{-1}{7x^7}\right)-\frac{1}{4^4}\left(\frac{-1}{5x^5}\right)+\frac{1}{4^5}\left(\frac{-1}{3x^3}\right)-\frac{1}{4^6}\left(\frac{-1}{x}\right)+\frac{1}{4^6}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]+c$$

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Example 3.5 Find the value of $\int \frac{dx}{x^{10}(x^2-4)}$

Solution. On comparing this function with the generalized form $\frac{1}{x^n(x^m-a)}$,

$n = 10,\ m = 2\ and\ a = 4.$

Here, n is a multiple of m

Now, no. of terms = $(\frac{n}{m} + 1) = 6$

On applying formula we get,

$$\frac{1}{x^{10}(x^2-4)} = -\frac{1}{4x^{10}} - \frac{1}{4^2x^8} - \frac{1}{4^3x^6} - \frac{1}{4^4x^4} - \frac{1}{4^5x^2} + \frac{1}{4^5(x^2-4)}$$

$$\int \frac{dx}{x^{10}(x^2-4)} = -\frac{1}{4}\left(\frac{-1}{9x^9}\right) - \frac{1}{4^2}\left(\frac{-1}{7x^7}\right) - \frac{1}{4^3}\left(\frac{-1}{5x^5}\right) - \frac{1}{4^4}\left(\frac{-1}{3x^3}\right) - \frac{1}{4^5}\left(\frac{-1}{x}\right) + \frac{1}{4^5}\left[\frac{1}{4}\ln\left(\frac{x-2}{x+2}\right)\right] + c$$

Example 3.6 Find the inverse Laplace transformation of the rational function $\frac{1}{s^6(s-1)}$

Solution. On comparing this function with the generalized form $\frac{1}{s^n(s^m-a)}$,

$n = 6,\ m = 1\ and\ a = 1.$

Here, n is a multiple of m

Now, no. of terms = $(\frac{n}{m} + 1) = 7$

On applying formula we get,

$$\frac{1}{s^6(s-1)} = -\frac{1}{s^6} - \frac{1}{s^5} - \frac{1}{s^4} - \frac{1}{s^3} - \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}$$

$$L^{-1}\left[\frac{1}{s^6(s-1)}\right] = -\frac{t^5}{5!} - \frac{t^4}{4!} - \frac{t^3}{3!} - \frac{t^2}{2!} - \frac{t}{1!} - 1 + e^t$$

Example 3.7 Find the inverse Laplace transformation of the rational function $\frac{1}{s^8(s^2+9)}$

Solution. On comparing this function with the generalized form $\frac{1}{s^n(s^m+a)}$,

$n = 8,\ m = 2\ and\ a = 9.$

Here, n is a multiple of m

Now, no. of terms = $(\frac{n}{m} + 1) = 5$

On applying formula we get,

$$\frac{1}{s^8(s^2+9)} = \frac{1}{9s^8} - \frac{1}{9^2s^6} + \frac{1}{9^3s^4} - \frac{1}{9^4s^2} + \frac{1}{9^4(s^2+9)}$$

$$L^{-1}\left[\frac{1}{s^8(s^2+9)}\right] = \frac{1}{9}\left(\frac{t^7}{7!}\right) - \frac{1}{9^2}\left(\frac{t^5}{5!}\right) + \frac{1}{9^3}\left(\frac{t^3}{3!}\right) - \frac{1}{9^4}\left(\frac{t}{1!}\right) + \frac{1}{9^4}\left[\frac{1}{3}\sin 3t\right]$$

Example 3.8 Solve the differential equation $\mathrm{y}'' + 16\mathrm{y} = \mathrm{t}^5$

Solution. Let $F(s) = L[y(t)]$, applying Laplace transformation, we have $(s^2 + 16)F(s) = \frac{6!}{s^6}$

Or, $F(s) = 6!\left[\frac{1}{s^6(s^2+16)}\right]$

On comparing $\left[\frac{1}{s^6(s^2+16)}\right]$ with the generalized form $\frac{1}{s^n(s^m+a)}$,

$n = 6,\ m = 2\ and\ a = 16.$

Here, n is a multiple of m

Now, no. of terms = $(\frac{n}{m} + 1) = 4$

On applying formula we get,

$$\frac{1}{s^6(s^2+16)} = \frac{1}{16s^6} - \frac{1}{16^2s^4} + \frac{1}{16^3s^2} - \frac{1}{16^3(s^2+16)}$$

$$L^{-1}\left[\frac{6!}{s^6(s^2+16)}\right] = \frac{6!}{16}\left(\frac{t^5}{5!}\right) - \frac{6!}{16^2}\left(\frac{t^3}{3!}\right) + \frac{6!}{16^3}\left(\frac{t}{1!}\right) - \frac{6!}{16^3}\left[\frac{1}{4}\sin 4t\right]$$

IV. Concluding Remarks

In this paper, we have presented formulas to solve partial fraction decompositions of proper rational functions of certain types. These formulas can be of great use in saving time and energy and also have useful application in engineering mathematics. In these particular types of functions, these formulas can be used as an alternative to the method of undetermined coefficient or other classical techniques in higher school or undergraduate level.

References

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