On the Free Boundary Problem for the Creeping Flow

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Abstract— **In the paper the fluid flow of large viscosity and low Reynolds number is considered in bounded areas. The linearized 2D Navier-Stokes equation (Stokes system) is studied in the rectangular area partly filled with the heavy fluid. The case of the solenoidal body force is considered. The solutions of the Stokes system are obtained with the appropriate boundary conditions. It is proved that for the given pressure the solution is uniquely defined. The profiles of free surfaces are constructed for the different pressure.**

Index Terms—**Stokes-Flow, Free-Boundary**

I. Introduction

We study 2D viscous fluid flow in a bounded reservoir partly filled with the very viscous fluid (oil or polymers for example) for the low Reynolds number (Re<<1). This type of fluids are widely used in MEMS (microelectromechanical systems) devices [11], [12], [14]. In this case the Navier-Stokes equation can be linearized and reduced to the Stokes system [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. The flow is called the Stokes flow. We consider the stationary system when the body force is solenoidal and the pressure is a harmonic function and is constant at the free boundary, the normal components of the tension are also constant at the free boundary. The Stokes system is reduced to the stationary system.

Our purpose is to define pressure, velocity and free surface.

II. Statement of the Problem

In the Cartesian coordinate system $0xy$ we consider the area D bounded by the lines $x=-a,\ x=a,\ y=0, a=const>0$, and the unknown line $y=\varphi(x), \varphi(x)>0,\ \varphi(x)\in C^1[-a,a]$. It is assumed that D is filled with a fluid of large viscosity. In this case the velocity components of the flow satisfy the following system (Stokes system) [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]

$$\frac{\partial V_x}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}=F_x+\nu\Delta V_x, \tag{1}$$

$$\frac{\partial V_y}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial y}=F_y+\nu\Delta V_y, \tag{2}$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0\ , \tag{3}$$

where $\vec{V}\ (V_x, V_y)$ is the velocity, $\vec{F}\ (F_x, F_y)$ is the body force, P is the pressure, ρ is the density, ν is the viscosity.

We admit , that body force is solenoidal i.e.

$$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=0\ . \tag{4}$$

and consider the stationary case of (1), (2), (3) for the stationary pressure, velocity $\vec{V}\ ^0(V_x^0, V_y^0)$ and body force $\vec{F}\ ^0(F_x^0, F_y^0)$

$$\frac{1}{\rho}\frac{\partial P}{\partial x}=F_x^0+\nu\Delta V_x^0, \tag{5}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial y}=F_y^0+\nu\Delta V_y^0, \tag{6}$$

$$\frac{\partial V_x^0}{\partial x}+\frac{\partial V_y^0}{\partial y}=0\ . \tag{7}$$

The system (5), (6), (7) is valid in the domain D and the following boundary conditions hold [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]

$$V_x^0\Big|_{x=\pm a}=V_y^0\Big|_{x=\pm a}=V_x^0\Big|_{y=0}=V_y^0\Big|_{y=0}=0\ , \tag{8}$$

$$P\Big|_{x=-a}=P\Big|_{x=a}=f_0(y);\ P\Big|_{y=o}=C_0>0;$$
$$P\Big|_{y=\varphi(x)}=P_0=const<C_0, \tag{9}$$

$$\sigma_{nn}\Big|_{y=\varphi(x)}=-P_0;\quad \sigma_{n\tau}\Big|_{y=\varphi(x)}=0; \tag{10}$$

where $\sigma_{\rm nn}$ and $\sigma_{n\tau}$ are the tension's normal and tangential components correspondingly, $f_0(y)$ is some continues

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function, $f_0(0) = C_0$, C_0 is a given constant. Taking into account [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]

$$\sigma_{\rm nn} = \left(-P + 2\mu\frac{\partial V_x}{\partial x}\right)\cos n_1 + \left(-P + 2\mu\frac{\partial V_y}{\partial y}\right)\cos n_2, \tag{11}$$

where

$$\cos n_1 = -\frac{\varphi_x'}{\sqrt{1+(\varphi_x')^2}};\ \cos n_2 = \frac{1}{\sqrt{1+(\varphi_x')^2}}. \tag{12}$$

From (4), (5), (6), (7), one obtains

$$\Delta P = \rho\, div \vec{F} = 0. \tag{13}$$

We have to solve the following

PROBLEM 1. To find in D the functions V_x^0, V_y^0 having continues first order derivatives and satisfying the system of equations (5), (6), (7) with the boundary conditions (8), (9), (10).

By (13) we obtain $\Delta P = 0$, i.e. P is a harmonic function in D satisfying conditions (9) [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20].

The function $P_1 = P - C_0$ is also harmonic in D and according to (9) satisfies the boundary conditions

$$P\big|_{x=-a} = P\big|_{x=a};\ P\big|_{y=o} = 0;\ P\big|_{y=\varphi(x)} = P_0 - C_0.$$

Hence, we can continue the function $P_1(x, y)$ through the lines $x = -a$ and $x = a$ and we obtain the harmonic periodic function in the stripe bounded with the lines $y = 0$ and periodic line $\varphi_0(x)\ (\varphi_0(x) = \varphi(x); -a \le x \le a)$. The function $P\ (x, y)$ is also periodic in this stripe.

Let $P_1 = P\ (x, y) - C_0$ be an imaginary part of the holomorphic complex function $\psi(z)$;

$$z_0 = \psi(z) = Q(x, y) + iP\ (x, y) - iC_0;\ z = x + iy,$$

$$\frac{\partial P_1}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P_1}{\partial y} = -\frac{\partial Q}{\partial x},$$

$$Q\big|_{x=-a} = -\omega_1;\ Q\big|_{x=a} = \omega_1;$$

The function $z_0 = \psi(z)$ is a conformal mapping of the area D on the rectangle

$$D_0 = \{-\omega_1/2 \le Q \le \omega_1/2,\ P_0 - C_0 \le P_1 \le 0\};$$
$$\omega_1 = const;$$

of z_0 plane. The inverse function

$$z = \psi_0(z_0) = \psi^{-1}(z); \tag{14}$$

is also holomorphic and satisfies the boundary conditions

$$\operatorname{Im}\psi^{-1}(z)\big|_{P=0} = 0;\ \operatorname{Im}\psi^{-1}(z)\big|_{P=P_0-C_0} = \varphi_0; \tag{15}$$

By using the Villa formula and (15) one obtains [4], [5], [7], [10], [19]

$$\psi_0(z_0) = \frac{1}{\pi}\int_0^{2\omega_1}[\varphi_0(t)]\, K(t, z_0)\, dt + C_3, \tag{16}$$

where

$$K(t, z_0) = [\varsigma(t - z_0 - i\omega_2) - \varsigma(t - i\omega_2)], \tag{17}$$

$\varphi_0 = \varphi(\psi_0(z_0))$, is unknown function of the Muskhelishvili-Kveselava H^* class [2], [15], ς is the Weierstrass "zeta-function" for the fundamental periods $2\omega_1$ and $2i\omega_2$, $\omega_2 = C_0 - P_0$, ω_1 and C_3 are the real constants, ω_1 is a point corresponding to the point $x = 2a$ [4], [5], [15], [19].

From (16) one obtains the relationship between the pressure and the free boundary

$$P^{-1}(x, y) - C_0 = \operatorname{Im}\frac{1}{\pi}\int_0^{2\omega_1}[\varphi_0(t)]\, K(t, z_0)\, dt.$$

Hence from the viewpoint of mathematics the pressure can be any periodic harmonic function.

REMARK 1. When the body forces vector is not solenoidal, the pressure satisfies the Poisson equation

$$\Delta P = \rho\, div \vec{F}.$$

Analogously to the previous results by means of the conformal mapping (14) and Poisson's formula we can find the inverse function $P_*^{-1}(x, y)$ [2]

$$P_*^{-1}(x, y) = -\frac{\rho}{2\pi}\times$$

$$\int_{D_0} G\left(x^*, y^*, x_0^*, y_0^*\right)\left|\psi_0'\right|^2 div \vec{F}\, dx_1^* dy_1^* + P^{-1}(x, y)$$

where $G\left(x^*, y^*, x_0^*, y_0^*\right)$ is the Green function for the rectangle D_0 which is given below by the formula (32).

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III. Solution of the Problem

Let us suppose that $P(x,y)$ is known function. Consequently, the profile of a free boundary can be obtained

from the formula $P(x,\varphi(x)) = P_0$.

By (5), (7), (8), (11) for the definition of $V_x^0(x,y)$ we have to solve the following Poisson equation

$$\Delta V_x^0 = \frac{1}{\rho\nu}\frac{\partial P}{\partial x} - \frac{1}{\nu}F_x^0 \equiv \Phi_1(x,y), \tag{18}$$

with the boundary conditions

$$V_x^0\Big|_{x=\pm a} = V_x^0\Big|_{y=0} = 0\,, \tag{19}$$

$$2\mu\frac{\partial V_x^0}{\partial x}(\cos n_1 - \cos n_2) =$$
$$-P_0 + P_0\cos n_1 + P_0\cos n_2,\ on\,\varphi(x),$$

or

$$\frac{\partial V_x^0}{\partial x}\left(1+\varphi_x^{'}\right) = -\frac{P_0}{2\mu}\left(1-\varphi_x^{'} + \sqrt{1+(\varphi_x^{'})^2}\right). \tag{20}$$

For the definition of $V_y^0(x,y)$ we have to solve the following Poisson equation

$$\Delta V_y^0 = \frac{1}{\rho\nu}\frac{\partial P}{\partial y} - \frac{1}{\nu}F_y^0 \equiv \Phi_2(x,y), \tag{21}$$

with the boundary conditions

$$V_y^0\Big|_{x=\pm a} = V_y^0\Big|_{y=0} = 0\,, \tag{22}$$

$$2\mu\frac{\partial V_y^0}{\partial y}(\cos n_2 - \cos n_1) =$$
$$-P_0 + P_0\cos n_1 + P_0\cos n_2,\ on\ \varphi(x),$$

or

$$\frac{\partial V_y^0}{\partial y}\left(1+\varphi_x^{'}\right) = \frac{P_0}{2\mu}\left(1-\varphi_x^{'} + \sqrt{1+(\varphi_x^{'})^2}\right). \tag{23}$$

By (7), (18), (19), (20) the function $\frac{\partial V_x^0}{\partial x}$ satisfies the equation

$$\Delta\frac{\partial V_x^0}{\partial x} = \frac{\partial \Phi_1(x,y)}{\partial x}\,, \tag{24}$$

with the boundary conditions

$$\frac{\partial V_x^0}{\partial x}\Big|_{x=\pm a} = \frac{\partial V_x^0}{\partial x}\Big|_{y=0_1} = 0\,, \tag{25}$$

$$\frac{\partial V_x^0}{\partial x} = -\frac{P_0}{2\mu\left(1+\varphi_x^{'}\right)}\left(1-\varphi_x^{'} + \sqrt{1+(\varphi_x^{'})^2}\right) \equiv \varphi_1(x)$$

on $\varphi(x)$. (26)

By means of the mapping $\psi_0(z_0)$ given by the formula (14) we can consider system (24), (25), (26) in z_0 plane

$$\Delta V_x^* = \left|\psi_0^{'}(z_0)\right|^2 \Phi_1^*\,, \tag{27}$$

$$V_x^*\Big|_{Q=0} = V_x^*\Big|_{Q=\omega_1} = 0\,, \tag{28}$$

$$V_x^* = -\frac{P_0}{2\mu\left(1+\varphi_x^{'}\right)}\left(1-\varphi_x^{'} + \sqrt{1+(\varphi_x^{'})^2}\right) \equiv$$
$$\varphi_1(x(Q,P))\ on\ P = P_0 - C_0, \tag{29}$$

where

$$V_x^* = \frac{\partial V_x^0(Q,P)}{\partial x};\ \ \Phi_1^* = \frac{\partial \Phi_1(x(Q,P),y(Q,P))}{\partial x}\ .$$

Hence, by means of the conformal mapping $\psi_0(z_0)$ the problem (24), (25), (26) is equivalently reduced to problem (27), (28), (29). The solution of the problem (27), (28), (29) is well-known and is given by the formula [2], [15]

$$V_x^* = -\frac{1}{2\pi}\int_{D_0} G\left(x^*, y^*, x_0^*, y_0^*\right)\ \left|\psi_0^{'}\right|^2 \Phi_1^*\ dx_1^* dy_1^*$$
$$+U_x^*, \tag{30}$$

where

$$U_x^* = \mathrm{Re}\frac{1}{\pi i}\int_0^{2\omega_1}[\varphi_1(t)]\,K(t,z_0)\,dt, \tag{31}$$

is a harmonic function, $K(t,z_0)$ is given by (17), and G is the Green function for the rectangle D_0,

$$G\left(x^*, y^*, x_0^*, y_0^*\right) =$$
$$\frac{1}{2}\log\frac{(x^*-x_0^*)^2 + (y^*+y_0^*)^2}{(x^*-x_0^*)^2 + (y^*-y_0^*)^2}, \tag{32}$$

where

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$$z^* = sn\left(\frac{z_0}{C^*}\right) = x^* + iy^*,$$

$$sn\left(\frac{x_1{}^* + iy_1{}^*}{C^*}\right) = x_0{}^* + iy_0{}^*,$$

sn is the Jakobi "sinus" with the periods $4K_1$ and $2K_2$

[4], [5], [15], $C^* = \dfrac{C_0 - P_0}{2K_1}$,

$$\omega_1 = 2C^* K_1; \omega_2 = C_0 - P_0 = C^* K_2 .$$

Analogously to the previous results for the definition of the

function $V_y^* = \dfrac{\partial V_y^0}{\partial y}$ in z_0 plane by (7), (21), (22), (23)

we obtain the following system

$$\Delta V_y^* = \left|\psi_0^{'}(z_0)\right|^2 \Phi_2^* , \tag{33}$$

$$V_y^*\Big|_{Q=0} = V_y^*\Big|_{Q=\omega_1} = 0 , \tag{34}$$

$$V_y^* = -\varphi_1(x(Q,P)), \text{ on } P = P_0 - C_0, \tag{35}$$

where $\Phi_2^* = \dfrac{\partial \Phi_2(x(Q,P), y(Q,P))}{\partial x}$.

The solution of problem (33), (34), (35) is

$$V_y^* = -\frac{1}{2\pi}\int_{D_0} G\left(x^*, y^*, x_0{}^*, y_0{}^*\right)\left|\psi_0^{'}\right|^2 \Phi_2^* \, dx_1{}^* dy_1{}^*$$
$$-U_x^*, \tag{36}$$

where U_x^* is given by (31), $K(t, z_0)$ is given by (17),

$G\left(x^*, y^*, x_0{}^*, y_0{}^*\right)$ is the Green function for the rectangle
D_0 given by (32).

Hence, having find V_x^* and V_y^* we can define V_x and V_y by the formulas

$$V_x^0 = \int_0^x [V_x^*(t,y)]\, dt - \int_0^a [V_x^*(t,y)]\, dt, \tag{37}$$

$$V_y^0 = \int_0^y [V_y^*(x,t)]\, dt. \tag{38}$$

THEOREM: For the given harmonic pressure P the components of the velocity V_x, V_y of Stokes flow are uniquely defined and are given by formulas (37) and (38), where V_x^* and V_y^* are given by (30) and (36).

REMARK 2. Having find V_x, V_y the vortex will be defined by the formula [1], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]

$$\Omega = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}.$$

REMARK 3. The formula (32) can be simplified . As the

parameter ω_1 of the conformal mapping $\psi_0(z_0)$ can be

chosen arbitrary, we can choose ω_1 in such a way that the

quantity $q = \exp\left(-\pi\chi\right) \approx 0;\ \chi = \dfrac{C_0 - P_0}{\omega_1};$ is infinitely

small and the following formula is valid [4], [5], [15]

$$sn\left(\frac{z_0}{C^*}\right) = \sin\left(\frac{\pi z_0}{C_0 - P_0}\right). \tag{39}$$

For example, $q \approx 0;$ for $\omega_1 = 5(C_0 - P_0);$

$K_1 \approx 1{,}6;\ K_2 \approx 7{,}9;$ or for $\omega_1 = 3.3\times(C_0 - P_0);$

$K_1 \approx 1{,}6;\ K_2 \approx 5{,}2$ [4], [5], [14].

By means of the formula (39) and

$$\sin z = \sin x cochy + i\cos x \cosh y$$

one obtains [4], [5], [15]

$$G\left(x^*, y^*, x_0{}^*, y_0{}^*\right) = \frac{1}{2}\log\frac{(x^* - x_0{}^*)^2 + (y^* + y_0{}^*)^2}{(x^* - x_0{}^*)^2 + (y^* - y_0{}^*)^2}$$

where

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$$(x^* - x_0{}^*)^2 + (y^* + y_0{}^*)^2 = \sin^2 x + \cosh^2 y$$
$$+ \sin^2 x_1^* + \cosh^2 y_1^* + 2\cos x \cos x_1^* cochy \cosh y_1^*$$
$$+ 2\sin x \sin x_1^* \cosh y cochy_1^*,$$
$$(x^* - x_0{}^*)^2 + (y^* - y_0{}^*)^2 = \sin^2 x + \cosh^2 y$$
$$+ \sin^2 x_1^* + \cosh^2 y_1^* - 2\cos x \cos x_1^* cochy \cosh y_1^*$$
$$- 2\sin x \sin x_1^* \cosh y cochy_1^*.$$

REMARK 4. We can consider the non-stationary case, when the velocity components, body forces and the pressure are representable in the form

$$V_x = \exp(-\alpha t)\, V_x^0(x,y),\ V_y = \exp(-\alpha t)\, V_y^0,$$
$$F_x = \exp(-\alpha t)\, F_x^0(x,y), F_y = \exp(-\alpha t)\, F_y^0,$$
$$P(t,x,y) = \exp(-\alpha t)\, P(x,y),$$

where t is the time, $\alpha > 0$ is the definite constant, the system (1), (2), (3) will be reduced to the system

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = F_x^0 + \nu \Delta V_x^0 + \alpha V_x^0, \tag{40}$$
$$\frac{1}{\rho}\frac{\partial P}{\partial y} = F_y^0 + \nu \Delta V_y^0 + \alpha V_y^0, \tag{41}$$
$$\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} = 0, \tag{42}$$

with the boundary conditions (8), (9), (10). From (40), (41), (42) for the definition of $V_x^0(x,y)$ and $V_y^0(x,y)$ we obtain the Helmholtz equations

$$\Delta V_x^0 + \frac{\alpha}{\nu} V_x^0 = \frac{1}{\rho\nu}\frac{\partial P}{\partial x} - \frac{1}{\nu} F_x^0, \tag{43}$$
$$\Delta V_y^0 + \frac{\alpha}{\nu} V_y^0 = \frac{1}{\rho\nu}\frac{\partial P}{\partial y} - \frac{1}{\nu} F_y^0, \tag{44}$$

with the boundary conditions (19), (20), (22), (23).

By means of the conformal mapping (14) we can reduce the system (40), (41), (42) to the singular integral equations with the weakly singular kernel [2]

$$V_x^* + \frac{\alpha}{2\pi\nu} \int_{D_0} G\left(x^*, y^*, x_0{}^*, y_0{}^*\right) \left|\psi_0'\right|^2 V_x^*\, dx_1{}^* dy_1{}^* \tag{45}$$
$$= -\frac{1}{2\pi} \int_{D_0} G\left(x^*, y^*, x_0{}^*, y_0{}^*\right) \left|\psi_0'\right|^2 \Phi_1^*\, dx_1{}^* dy_1{}^* + U_x^*,$$

$$V_y^* + \frac{\alpha}{2\pi\nu} \int_{D_0} G\left(x^*, y^*, x_0{}^*, y_0{}^*\right) \left|\psi_0'\right|^2 V_y^*\, dx_1{}^* dy_1{}^* \tag{46}$$
$$= -\frac{1}{2\pi} \int_{D_0} G\left(x^*, y^*, x_0{}^*, y_0{}^*\right) \left|\psi_0'\right|^2 \Phi_2^*\, dx_1{}^* dy_1{}^* - U_x^*,$$

where U_x^* and $G\left(x^*, y^*, x_0{}^*, y_0{}^*\right)$ are given by (31)(32).

By means of the Banach theorem we obtain [2]:

If $\frac{\alpha}{2\pi\nu} < \frac{1}{M}$, where for $(x^*, y^*) \in D_0$,

$$\int_{D_0} \left|G\left(x^*, y^*, x_0{}^*, y_0{}^*\right)\right| \left|\psi_0'\right|^2 dx_1{}^* dy_1{}^* \le M;$$

then there exists the unique solution of equations (45) ,(46).

Hence, for any harmonic pressure the solution of the Stokes system also exists in the non-stationary case.

REMARK 5. The free boundary problem for the ideal fluid was consider by the author in [6],[8],[9],[10]. In this works has been investigated the waves with peaks -Stokes waves.

IV. Conclusion

For any harmonic pressure satisfying the condition (9) there exist the unique solution of the Stokes system. Any level line of harmonic pressure represents some wave in the creeping flow (this fact is very similar to the case of perfect fluid).

Below 2 type of waves for the different harmonic pressures are constructed by means of “Maple”.

V. EXAMPLES

Here we consider two cases:

1)

$$P = \operatorname{Im}\left(\frac{1}{d} ar\cos\left[\cos(zd)/coch(dh)\right]\right), \tag{47}$$

where $d = \pi / a$, h (h>0) is some parameter. After simple transformations we obtain

$$4\cosh^2(Pd) \times coch^2(dh) = -2b +$$
$$\sqrt{b_0 - 2\cos(2xd) \times coch(2dh) \times coch(2yd)},$$
$$b = coch^2(dh) - \cosh^2(yd) + \sin^2(xd), \tag{48}$$
$$b_0 = coch^2(2dh) + \cosh^2(2yd) + \cos^2(2xd).$$

By the formula (48) it is easy to construct the profile of a free boundary for the different parameters by means of “Maple”. In Fig. 1. and Fig. 2.the profilis of the free boundaries are given for the different parameters

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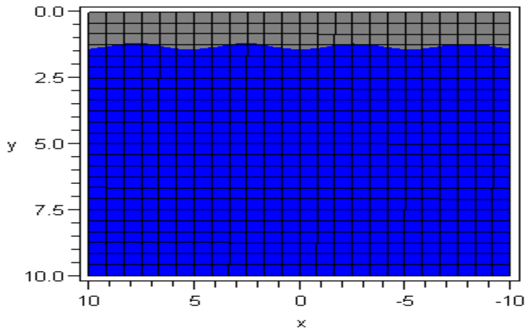

Fig. 1. The profile of the free boundary for the pressure (47) in case of $P_0 = 1, C_0 = 2, a = 5, h = 1$.

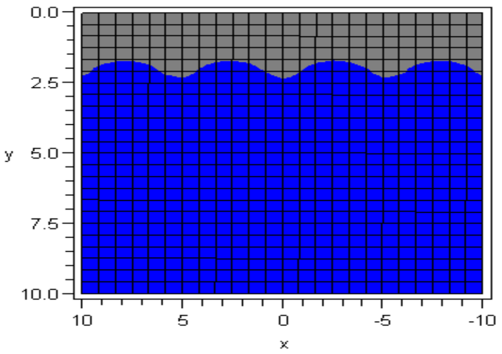

Fig. 2. The profile of the free boundary for the pressure (47) in case of $P_0 = 1, C_0 = 5, a = 5, h = 2$.

2) $$P = \operatorname{Im}\sqrt{sn^2\frac{2K_1 z}{a} + sn^2(ih)}, \qquad (49)$$

where h (h>0) is some parameter and sn is the Jakobi function with the periods $4K_1$ and $2K_2$.

After simple transformations we obtain

$$4P^2 = b + \sqrt{c^2 + \cosh^2(2yd) + \cos^2(2xd) - 2c\times\cos(2xd)\times coch(2yd)},$$

$$b = coch(2yd)\times\cos(2xd) - c,\ c = 1 + 2\sin^2 h,\ d = \frac{\pi}{a}. \qquad (50)$$

By formula (50) it is easy to construct the profile of a free boundary for the different parameters by means of “Maple”. In Fig. 3.and Fig. 4, the free boundary is given for different parameters.

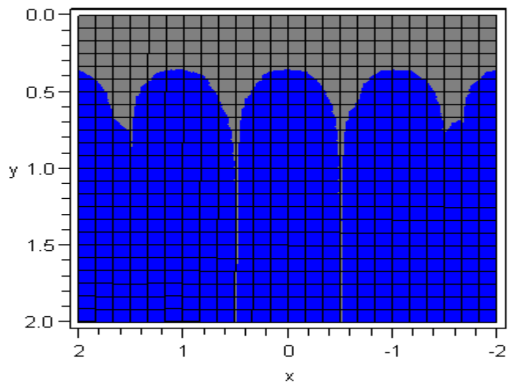

Fig. 3. The profile of the free boundary for the pressure (49) in case of $P_0 = 1; C_0 = 8; a = 1; c = 2; K_1 = 1.6; K_2 = 7.9$.

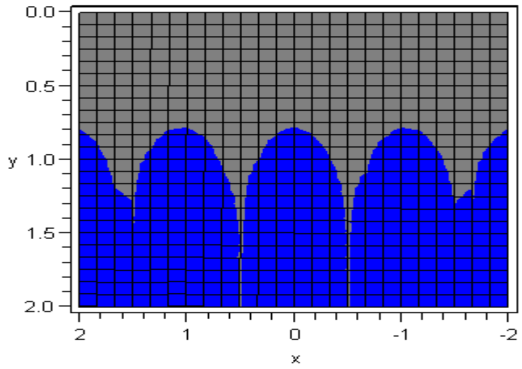

Fig. 4. The profile of the free boundary for the pressure (49) in case of $P_0 = 5; C_0 = 8; a = 1; c = 2; K_1 = 1.6; K_2 = 7.9$.

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