A Combined Laplace Transform and Boundary Element Method for Unsteady Modified Helmholtz Type Problems of Anisotropic Quadratically Graded Materials

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Abstract—**In this paper a BEM is used to solve a variable coefficient modified Helmholtz type equation numerically. Some examples are considered to show the accuracy of the numerical solutions.**

Keywords: anisotropic functionally graded materials, unsteady modified Helmholtz type problems, Laplace transform, boundary element method

1 Introduction

We will consider initial boundary value problems governed by a modified Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=\alpha(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial t} \quad (1)$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial \mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial \mu}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial \mu}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial \mu}{\partial x_2}\right)-\beta^2\mu=\alpha\frac{\partial \mu}{\partial t}$$

Equation (1) is usually used to model infiltration problems (see for examples [1–8]).

During the last decade functionally graded materials (FGMs) have become an important topic, and numerous studies on FGMs for a variety of applications have been reported. Authors commonly define an FGM as an inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. Therefore equation (1) is relevant for FGMs.

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Recently Azis and co-workers had been working on steady state problems of *anisotropic inhomogeneous* media for several types of governing equations, for examples [9–13] for the diffusion convection equation, [14–20] for the diffusion convection reaction equation, [21–25] for the Helmholtz equation and [26–29] for the Laplace type equation.

This paper is intended to extend the recently published works in [3–8] for steady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=0$$

to unsteady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form (1).

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

2 The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu(\mathbf{x},t)$ and its derivatives to (1) are sought which are valid for time interval $t \geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu(\mathbf{x},t)$ $(\mathbf{x}=(x_1,x_2))$ is specified and on $\partial\Omega_2$

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_i}n_j \quad (2)$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu(\mathbf{x},0)=0 \quad (3)$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation,

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and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x}, t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11} = \kappa_{22}$ and $\kappa_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3 The boundary integral equation

The coefficients $\kappa_{ij}, \beta^2, \alpha$ are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij} g(\mathbf{x}) \quad (4)$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \quad (5)$$

$$\alpha(\mathbf{x}) = \overline{\alpha} g(\mathbf{x}) \quad (6)$$

where the $\overline{\kappa}_{ij}, \overline{\beta}^2, \overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$, $\beta^2(\mathbf{x})$ and $\alpha(\mathbf{x})$ are quadratically graded by taking $g(\mathbf{x})$ as an quadratic function

$$g(\mathbf{x}) = [c_0 + c_i x_i]^2 \quad (7)$$

where c_0 and c_i are constants. Therefore (7) satisfies

$$\overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0 \quad (8)$$

Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g \frac{\partial \mu}{\partial x_j}\right) - \overline{\beta}^2 g\mu = \overline{\alpha} g \frac{\partial \mu}{\partial t} \quad (9)$$

Let

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x}) \psi(\mathbf{x}, t) \quad (10)$$

therefore substitution of (4) and (10) into (2) gives

$$P(\mathbf{x}, t) = -P_g(\mathbf{x}) \psi(\mathbf{x}, t) + g^{1/2}(\mathbf{x}) P_\psi(\mathbf{x}, t) \quad (11)$$

where

$$P_g(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \qquad P_\psi(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial \psi}{\partial x_j} n_i$$

Also, (9) may be written in the form

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left[g \frac{\partial \left(g^{-1/2}\psi\right)}{\partial x_j}\right] - \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g \frac{\partial \left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} + g\psi \frac{\partial g^{-1/2}}{\partial x_j}\right) - \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i} = -g^{-1} \frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} - \psi \frac{\partial g^{1/2}}{\partial x_j}\right) - \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Rearranging and neglecting the zero terms give

$$g^{1/2} \overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Equation (8) then implies

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \overline{\beta}^2 \psi = \overline{\alpha} \frac{\partial \psi}{\partial t} \quad (12)$$

Taking the Laplace transform of (10), (11), (12) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x}, s) = g^{1/2}(\mathbf{x}) \mu^*(\mathbf{x}, s) \quad (13)$$

$$P_{\psi^*}(\mathbf{x}, s) = [P^*(\mathbf{x}, s) + P_g(\mathbf{x}) \psi^*(\mathbf{x}, s)] g^{-1/2}(\mathbf{x}) \quad (14)$$

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi^*}{\partial x_i \partial x_j} - \left(\overline{\beta}^2 + s\overline{\alpha}\right) \psi^* = 0 \quad (15)$$

where s is the variable of the Laplace-transformed domain. A boundary integral equation for the solution of (15) is given in the form

$$\eta(\mathbf{x}_0) \psi^*(\mathbf{x}_0, s) = \int_{\partial\Omega} [\Gamma(\mathbf{x}, \mathbf{x}_0) \psi^*(\mathbf{x}, s) - \Phi(\mathbf{x}, \mathbf{x}_0) P_{\psi^*}(\mathbf{x}, s)] \, dS(\mathbf{x}) \quad (16)$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(a, b) \in \Omega$, $\eta = \frac{1}{2}$ if $(a, b) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a, b). The so called fundamental solution Φ in (16) is any solution of the equation

$$\overline{\kappa}_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} - \left(\overline{\beta}^2 + s\overline{\alpha}\right) \Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \overline{\kappa}_{ij} \frac{\partial \Phi(\mathbf{x}, \mathbf{x}_0)}{\partial x_j} n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \ln R & \text{if } \overline{\beta}^2 + s\overline{\alpha} = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2 + s\overline{\alpha} < 0 \\ \frac{-K}{2\pi} K_0(\omega R) & \text{if } \overline{\beta}^2 + s\overline{\alpha} > 0 \end{cases} \quad (17)$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \frac{1}{R} \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{-\imath K\omega}{4} H_1^{(2)}(\omega R) \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{K\omega}{2\pi} K_1(\omega R) \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \end{cases}$$

$$\begin{cases} \text{if } \overline{\beta}^2 + s\overline{\alpha} = 0 \\ \text{if } \overline{\beta}^2 + s\overline{\alpha} < 0 \\ \text{if } \overline{\beta}^2 + s\overline{\alpha} > 0 \end{cases}$$

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where $K = \ddot{\tau}/D$, $\omega = \sqrt{|\overline{\beta}^2 + s\overline{\alpha}|/D}$, $D = \left[\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\dot{\tau} + \overline{\kappa}_{22}\left(\dot{\tau}^2 + \ddot{\tau}^2\right)\right]/2$, $R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2}$, $\dot{x}_1 = x_1 + \dot{\tau}x_2$, $\dot{a} = a + \dot{\tau}b$, $\dot{x}_2 = \ddot{\tau}x_2$, and $\dot{b} = \ddot{\tau}b$. where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\tau + \overline{\kappa}_{22}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (17) are given by

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \frac{1}{R}\left(\dot{x}_1 - \dot{a}\right) \\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}\left(\dot{x}_1 - \dot{a}\right)\right] + \ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right] \end{aligned}$$

Use of (13) and (14) in (16) yields

$$\eta g^{1/2}\mu^* = \int_{\partial\Omega}\left[\left(g^{1/2}\Gamma - P_g\Phi\right)\mu^* - \left(g^{-1/2}\Phi\right)P^*\right]dS \qquad (18)$$

This equation provides a boundary integral equation for determining μ^* and its derivatives at all points of Ω.

Knowing the solutions $\mu^*(\mathbf{x}, s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (18), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x}, t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned} \mu(\mathbf{x}, t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \mu^*(\mathbf{x}, s_m) \\ \frac{\partial\mu(\mathbf{x}, t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial\mu^*(\mathbf{x}, s_m)}{\partial x_1} \qquad (19) \\ \frac{\partial\mu(\mathbf{x}, t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial\mu^*(\mathbf{x}, s_m)}{\partial x_2} \end{aligned}$$

where

$$\begin{aligned} s_m &= \frac{\ln 2}{t}m \\ V_m &= (-1)^{\frac{N}{2}+m} \times \\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m, \frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2} - k\right)!k!(k-1)!(m-k)!(2k-m)!} \end{aligned}$$

A simple script is developed and embedded into the main FORTRAN code to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any number N.

4 Numerical examples

In order to justify the analysis derived in the previous sections, we will consider two problems of an analytical solution and without a simple analytical solution. For both problems we take

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= 0.75 - 0.15x_1 - 0.35x_2 \\ \overline{\kappa}_{ij} &= \begin{bmatrix} 1 & 0.2 \\ 0.2 & 0.5 \end{bmatrix} \\ \overline{\beta}^2 &= 1 \end{aligned}$$

For a simplicity, a unit square (depicted in Figure 1) will be taken as the geometrical domain.

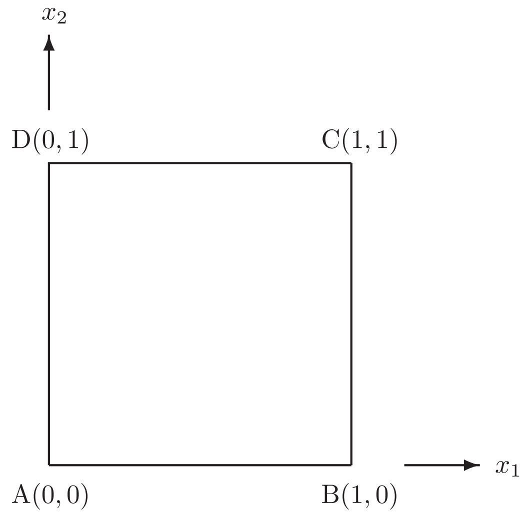

Figure 1: The domain Ω

4.1 Problem 1

Another aspect that will be justified is the accuracy of the numerical solutions. The analytical solution is assumed to be

$$\mu(\mathbf{x}, t) = \frac{[1 - \exp(-1.75t)](0.25 - 0.15x_1 - 0.1x_2)}{0.75 - 0.15x_1 - 0.35x_2}$$

We choose

$$\overline{\alpha} = -1/s$$

and a set of boundary conditions (see Figure 1)

P is given on side AB
P is given on side BC
μ is given on side CD
P is given on side AD

Table 1 shows the accuracy of the numerical solutions μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ solutions in the domain for Problem 1. The errors mainly occur in the fourth decimal place for the $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ solutions. The elapsed CPU time for the computation of the numerical solutions at 19×19 spatial positions and 11 time steps is 3688.109375 seconds.

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Table 1: Comparison of the numerical (Num) and the analytical (Anal) solutions at $(x_1, x_2) = (0.5, 0.5)$ for Problem 1

t	μ		$\frac{\partial \mu}{\partial x_1}$		$\frac{\partial \mu}{\partial x_2}$	
	Num	Anal	Num	Anal	Num	Anal
0.0005	0.0002	0.0002	-0.0002	-0.0002	-0.0000	-0.0000
0.5	0.1456	0.1458	-0.1312	-0.1312	-0.0144	-0.0146
1.0	0.2062	0.2066	-0.1859	-0.1859	-0.0204	-0.0207
1.5	0.2316	0.2319	-0.2086	-0.2087	-0.0230	-0.0232
2.0	0.2420	0.2425	-0.2182	-0.2182	-0.0239	-0.0242
2.5	0.2462	0.2469	-0.2222	-0.2222	-0.0241	-0.0247
3.0	0.2487	0.2487	-0.2240	-0.2238	-0.0248	-0.0249
3.5	0.2487	0.2495	-0.2246	-0.2245	-0.0243	-0.0249
4.0	0.2495	0.2498	-0.2250	-0.2248	-0.0247	-0.0250
4.5	0.2502	0.2499	-0.2251	-0.2249	-0.0252	-0.0250
5.0	0.2492	0.2500	-0.2250	-0.2250	-0.0243	-0.0250

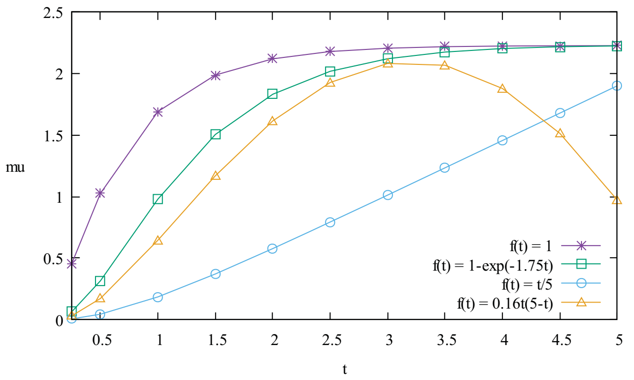

Figure 2: Solutions μ at $(x_1, x_2) = (0.5, 0.5)$ for Problem 2

4.2 Problem 2

We choose

$$\overline{\alpha} = 1$$

and boundary conditions (see Figure 1)

$$\begin{aligned} P &= f(t) \text{ on side AB} \\ P &= 0 \text{ on side BC} \\ \mu &= 0 \text{ on side CD} \\ P &= 0 \text{ on side AD} \end{aligned}$$

where $f(t)$ takes four cases

$$\begin{aligned} &\text{Case 1: } f(t) = 1 \\ &\text{Case 2: } f(t) = 1 - \exp(-1.75t) \\ &\text{Case 3: } f(t) = t/5 \\ &\text{Case 4: } f(t) = 0.16t(5-t) \end{aligned}$$

The results in Figure 2 are expected. The trends of the solutions μ mimics the trends of the exponential function $f(t) = 1 - \exp(-1.75t)$, the linear function $f(t) = t/5$ and the quadratic function $f(t) = 0.16t(5-t)$ of the boundary condition on side AB. Specifically, for the exponential function $f(t) = 1 - \exp(-1.75t)$, as time t goes to infinity, values of this function go to 1. So for big value of t, the case of $f(t) = 1 - \exp(-1.75t)$ is similar to the case of $f(t) = 1$. And the two plots of solutions μ for both cases in Figure 2 verifies this, they approach a same steady state solution as t gets bigger.

5 Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic functionally graded materials which are governed by the modified Helmholtz type equation (1). It is easy and accurate. It involves a time variable free fundamental solution and therefore that is why it would be more accurate. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution usually has singular time points.

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