Numerical Integral Equation Methods of Average Run Length on Extended EWMA Control Chart for Autoregressive Process

Kotchaporn Karoon, Yupaporn Areepong*, and Saowanit Sukparungsee

***Abstract*—The aim of this research is to develop the Numerical Integral Equation (NIE) methods for evaluating the Average Run Length (*ARL*) on Extended Exponentially Weighted Moving Average (Extended EWMA) control chart for autoregressive process in the case of exponential white noise. Besides, this is also extended to compare efficiency of Extended EWMA with EWMA procedure. The performance of Extended EWMA control chart is better than the performance of EWMA control chart for all magnitudes of change. To demonstrate its capability, the proposed approach was applied by using the real data of an environment field.**

***Index Terms*— Extended EWMA, Average Run Length, Autoregressive process, Numerical Integral Equation**

I. Introduction

THE statistical process control (SPC) plays a vital role in monitoring, detecting changes in a process, and uses for measuring, controlling, and improving quality in areas such as industrial and manufacturing, finance and economics, health and medicine, and others (see [1]-[4]).

The Shewhart control chart was the first to be reported and is widely used for monitoring processes and detecting shifts in the process mean. It is useful for detecting large changes in the process mean, but its performance is degraded when the changes are small [5].

In several research, the Cumulative Sum (CUSUM) control chart [6] and the Exponentially Weighted Moving Average (EWMA) control chart [7] have been proposed as good alternative to the Shewhart control chart for detecting small shift (see [8]-[11]). Patel and Divecha [12] proposed the modified EWMA control chart that is effective at detecting small and abrupt changes in the process mean for observations that are independent and normally distributed or autocorrelated. Later, Khan et al. [13] redesigned the modified EWMA control statistic. Recently, Naveed et al. [14] proposed the extended EWMA control chart that performed better than other control charts for detecting small shifts in the mean of a monitored process.

The characteristic of control chart is Average Run Length (*ARL*). The ARL_0 is usually measured when the process is in-control and should be large whereas the *ARL* is correctly signaled to be out-of-control and should be as small as possible. The evaluation methods of *ARL* have been described in previous literature such as Monte Carlo simulations (MC), Markov Chain approach (MCA), Martingale approach (MA) and Numerical Integral Equation approach (NIE). Champ and Rigdon [15] studied CUSUM and EWMA charts using the Markov Chain and integral equation approaches evaluate the *ARL*. Mastrangelo and Montgomery [16] evaluated the performance of EWMA control charts for serially correlated processes by using the Monte Carlo simulation technique. Lu and Reynolds [17] used integral equation to compute *ARL* when the observations can be modeled to AR(1) and ARMA(1,1) processes plus random error. Sukparungsee and Novikov [18] used martingale approach for analytic approximation of *ARL* on EWMA control chart. Peerajit [19] studied the numerical integral equation method of *ARL* on CUSUM control chart. Supharakonsakun et al. [20] evaluated the *ARL* by NIE method on modified EWMA and compared efficiency with EWMA control chart. The performance of modified EWMA chart was found to be superior to EWMA procedure for all cases. Furthermore, Peerajit et al. [21] derived explicit analytical solutions for *ARL* of CUSUM control chart for a long-memory SARFIMA process with exponential white noise and compared focusing on the performance using NIE method both methods had similarly excellent agreement. After that, Sunthornwat and Areepong [22] derived explicit formulas of *ARL* on CUSUM control chart for seasonal and non-seasonal moving average processes with exogenous variables and evaluated against the NIE method. The results were compared with *ARL* on EWMA control chart. Recently, Phanthuna et al. [23] evaluated *ARL* of the modified EWMA control chart for the trend AR(1) process with exponential white noise and compared with the performance of *ARL* using by NIE method. As a result, the performance of modified EWMA control chart is better than EWMA control chart for small and moderated shifts. Frequently, an autoregressive process is used on control charts and it can be applied with real data such as environmental, economic, and others. As previously mentioned, indicate that the *ARL* is useful for efficiency comparing of the control charts and the NIE methods is easier to calculate the *ARL*.

However, derivation of NIE methods for the *ARL* on Extended EWMA control chart has not previously been reported. The aim of the study is to propose a numerical

Manuscript received March 23, 2021; revised April 07, 2021. The research was funding by King Mongkut's University of Technology North Bangkok contract no. KMUTNB-BasicR-64-02.

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integral equation of *ARL* on Extended EWMA control chart for autoregressive (AR(P)) process with exponential white noise. In addition, this is also extended to compare the performance with EWMA control chart.

II. Extended EWMA Control Chart for AR(P) Process

The EWMA control chart was originally proposed by Robert [7]. The EWMA control chart can be expressed by the recursive equation below.

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t, t = 1,2,... \quad (1)$$

where λ is exponential smoothing parameters with ($0<\lambda\le 1$) The upper control limit (*UCL*) and Lower control limit (*LCL*) of EWMA control chart are given by

$$UCL = \mu_0 \pm L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \quad (2)$$

where μ_0 is the target mean, σ is the process standard deviation and *L* is suitable in control limit width.

The Extended EWMA control chart was proposed by Naveed et al. [14]. It is developed form the EWMA control chart. The Extended EWMA control chart can be expressed by the recursive equation below.

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}, t = 1,2,... \quad (3)$$

where λ_1 and λ_2 are exponential smoothing parameters with ($0<\lambda_1\le 1$) and ($0\le\lambda_2<\lambda_1$) and $E_0 = u$ is the initial value. The upper control limit (*UCL*) and Lower control limit (*LCL*) of the Extended EWMA control chart are given by

$$UCL = \mu_0 \pm Q\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \quad (4)$$

where μ_0 is the target mean, σ is the process standard deviation and *Q* is suitable in control limit width.

The autoregressive (AR(P)) process can be described by

$$X_t = \eta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_p X_{t-p} + \varepsilon_t, \varepsilon_t \sim Exp(\alpha)$$

where η is a constant, ϕ_i is an autoregressive coefficient ($-1\le\phi_i\le 1$), ε_t is the error term of time t ,and the stopping time of the Extended EWMA control chart is given by

$$\tau_b = \inf\{t \ge 0 : E_t > b\}, b > \mathrm{u} \quad (5)$$

where τ_b is the stopping time, b is *UCL*. The *ARL* for the AR(p) process is given by

$$ARL = L(u) = E_\infty(\tau_b) \ge T$$

where θ is the change-point time and $E_\theta(.)$ is the expectation. Meanwhile, the stopping time of the EWMA control chart is given by

$$\tau_h = \inf\{t \ge 0 : Z_t > h\}, h > \mathrm{u} \quad (6)$$

where τ_h is the stopping time, *h* is *UCL*. The *ARL* for the AR(P) process is given by

$$ARL = L(u) = E_\infty(\tau_h) \ge T$$

III. Numerical Integral Equation (NIE) Methods of *ARL* on Extended EWMA Control Chart

Let $L(u)$ denote *ARL* for AR(P) process, the Extended EWMA statistics E_t can be written as:

$$E_t = (1-\lambda_1+\lambda_2)Z_{t-1} + (\lambda_1\phi_1-\lambda_2)X_{t-1}$$
$$+\lambda_1\phi_2 X_{t-2} + \lambda_1\phi_3 X_{t-3} + ... + \lambda_1\phi_p X_{t-p} + \lambda_1\eta + \lambda_1\varepsilon_t$$

If $\varepsilon_t \ge 0$ *LCL*=0 and *UCL*=*b*, respectively, and then the function $L(u)$ can be derived by Fredholm integral equation of the second kind. $L(u)$ is defined as follows:

$$L(u) = 1 + \int L(E_1) f(\varepsilon_1) d\varepsilon_1 \quad (7)$$

Consequently, the function $L(u)$ is obtained as follows:

$$L(u) = 1 + \frac{1}{\lambda_1}\int_0^b L(y) f\left(\frac{y-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)X_{t-1}}{\lambda_1} -\phi_2 X_{t-2} - ... - \phi_p X_{t-p} - \eta\right) dy \quad (8)$$

Equation (8) can be approximated by using of numerical quadrature rules (see [24]) which can be calculated using many methods. In this research, there are four methods, namely Gaussian Rule, Midpoint Rule, Simpson's Rule, and Trapezoidal Rule.

A. Gaussian Rule

The numerical method to solve the NIE uses the quadrature rule approach which approximates the integral by finite sum of areas of rectangles with base b/m with heights chosen as the values of *f* at midpoints of the one-sided interval. The approximation for an integral is evaluated by the quadrature rule as follows:

$$\int_0^b W(y) f(y) dy \approx \sum_{j=1}^m w_j f(A_j).$$

Let $L(a_i)$ be a numerical approximation to the integral equation which can be found as the solution of linear equations as follows:

$$\tilde{L}(a_i) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^m w_j L(a_j) f\left(\frac{a_j-(1-\lambda_1+\lambda_2)a_i-(\lambda_1\phi_1-\lambda_2)X_{t-1}}{\lambda_1} -\phi_2 X_{t-2} - ... - \phi_p X_{t-p} - \eta\right)$$

$; i = 1,2,...,m$

Let $R_{m\times m}$ be a matrix, the definition of the m to m^{th} element of the matrix *R* is given by

$$[R_{ij}] \approx \frac{1}{\lambda_1} w_j f\left(\frac{a_j-(1-\lambda_1+\lambda_2)a_i-(\lambda_1\phi_1-\lambda_2)X_{t-1}}{\lambda_1} -\phi_2 X_{t-2} - ... - \phi_p X_{t-p} - \eta\right)$$

Finally, substituting, a_i by u in $\tilde{L}(a_i)$, then the numerical approximation equation (8) for the function $\tilde{L}(u)$ is as follows:

$$\tilde{L}(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^m w_j L(a_j) f\left(\frac{a_j-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)X_{t-1}}{\lambda_1} -\phi_2 X_{t-2} - ... - \phi_p X_{t-p} - \eta\right) \quad (9)$$

where $a_j = \frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j = \frac{b}{m}; j = 1,2,...,m.$

B. Midpoint Rule

Given $f(A_j) = f\left(\frac{a_j-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)X_{t-1}}{\lambda_1} -\phi_2 X_{t-2} - ... - \phi_p X_{t-p} - \eta\right)$

The approximation for the integral is given by

$$\tilde{L}_M(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f(A_j) \quad (10)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}$; $j = 1, 2, ..., m$.

C. Simpson's Rule

By using the Simpson's Rule, *ARL* can be solved as follows:

$$\tilde{L}_S(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{2m+1} w_j L(a_j) f(A_j) \quad (11)$$

where $a_j = jw_j$ and $w_j = \frac{4}{3}\left(\frac{b}{2m}\right)$; $j = 1, 3, ..., 2m-1,$

$w_j = \frac{2}{3}\left(\frac{b}{2m}\right)$; $j = 2, 4, ..., 2m-2,$

in other cases, $w_j = \frac{1}{3}\left(\frac{b}{2m}\right)$.

D. Trapezoidal Rule

By using the Trapezoidal Rule, *ARL* can be solved as follows:

$$\tilde{L}_T(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m+1} w_j L(a_j) f(A_j) \quad (12)$$

where $a_j = jw_j$ and $w_j = \frac{b}{m}$; $j = 1, 2, ..., m-1,$

in other cases, $w_j = \frac{b}{2m}$.

IV. Numerical Results

In this section, the *ARL* was approximated by NIE method using the Gaussian Rule, Midpoint Rule, Simpson's Rule, and Trapezoidal Rule on Extended EWMA control chart with m = 500 nodes. The numerical results are computed by MATHEMATICA. In this research, the initial parameter values are studied at ARL_0 = 370 in cases of AR(2) process and ARL_0 = 500 in cases of AR(3) process, and given $\lambda_1 = 0.05, 0.10, \lambda_2 = 0.01$. The 'in-control' process had a parameter value as $\alpha = \alpha_0 = 1$ with shift size ($\delta = 0$). In contrast, the 'out-of-control' process was presented with parameter values as $\alpha_1 = \alpha_0(1+\delta)$ with shift sizes δ = 0.001, 0.003, 0.005, 0.010, 0.050, 0.100, 0.500, and 1.000, respectively were determined. The coefficient parameters of the process $\phi_1 = \phi_2 = 0.1$ were used for the AR(2) process, and $\phi_1 = \phi_2 = \phi_3 = 0.1$ were used for the AR(3) process.

The results showed that the *ARL* values of Extended EWMA control chart using the NIE methods that used the Gaussian rule (9) give results close to the midpoint rule (10), Simpson's rule (11), and trapezoidal rule (12) both AR(2) process in TABLE I and AR(3) process in TABLE II. The analytical results with the computational (CPU) times for AR(2) process of NIE using four methods, namely the Gaussian rule, the midpoint rule, the Simpson's rule, and the trapezoidal rule take approximately 2.4–3.3 seconds, 2.2-2.6 seconds, 8–10 seconds, and 2.1–2.2 seconds, respectively. While, the analytical results with the CPU times for AR(3) process of NIE method using four methods, namely the Gaussian rule, the midpoint rule, the Simpson's rule, and the trapezoidal rule take approximately 2.6–3.7 seconds, 2.7-3.2 seconds, 9–11 seconds, and 2.4 –2.9 seconds, respectively.

As mentioned above, the results indicating that the *ARL* on Extended EWMA control chart of NIE using four methods gave the same analysis results. Besides, the NIE method using the trapezoidal rule takes less CPU times than others. The entries inside the parentheses are the CPU times in seconds and the bolds are the least CPU times.

V. Performance Comparison of the *ARL* on EWMA and Extended EWMA Control Charts

In this section, the NIE method using the trapezoidal rule on Extended EWMA control chart compared to the *ARL* on EWMA control chart for AR(2) process in TABLE III when $\lambda_1 = 0.05, 0.10$, $\phi_1 = \phi_2 = 0.1$, $\eta = 0$, ARL_0 = 370, and AR(3) process in TABLE IV when $\lambda_1 = 0.05, 0.10$, $\phi_1 = \phi_2 = \phi_3 = 0.1$, $\eta = 0$, ARL_0 = 500.

The results in TABLE III presented that the Extended EWMA control chart reduced the ARL_1 more than the EWMA control when the small shift sizes ($\delta < 0.50$). The results indicated that the performance of Extended EWMA control chart is better than the EWMA control chart when the small shift sizes ($\delta < 0.50$). While, the performance of Extended EWMA control chart is close to the EWMA control chart the large shift sizes ($\delta \geq 0.50$) both $\lambda_1 = 0.05$ and $\lambda_1 = 0.10$. Likewise, the results in TABLE IV showed that the details are like to the result in TABLE III.

VI. Application

In the section, real data was applied to determine the *ARL* by the NIE method using trapezoidal rule on the Extended EWMA and the EWMA control charts for Biochemical Oxygen Demand (BOD) in terms of milligram per liter (mg/L) and Salinity (SAL) in terms of part per thousand (ppt) pollutants in the water which are important indicators of water quality. Biochemical Oxygen Demand (BOD) and Salinity (SAL) were collected quarterly form January 2011 to April 2020 as the dataset of real observations. This data is a stationary time series. The data of BOD was analyzed with condition of $ARL_0 = 370$ and fitted with AR(2) process with the significant of mean and standard deviation equals 4.818421 and 2.912309, respectively and the process coefficients, $\phi_1 = 0.550326$, $\phi_2 = -0.325831$, $\eta = 4.827637$, $\lambda_2 = 0.01$, and then the error as exponential white noise with $\alpha_0 = 2.507313$. Meanwhile, the data of SAL was analyzed with condition of $ARL_0 = 500$ and fitted with AR(3) process with the significant of mean and standard deviation 3.332432 and 5.859183, respectively, and the coefficients process $\phi_1 = 0.571799$, $\phi_2 = -0.517654$, $\phi_3 = 0.451555$, $\eta = 3.428618$, $\lambda_2 = 0.01$, and then the error as exponential white noise with $\alpha_0 = 4.397613$.

The results for the *ARL* of the EWMA and Extended EWMA control charts on AR(2) for dataset of real observations in TABLE V are agreement to the simulation results in TABLE III, and then these control charts on AR(3) dataset of real observations in TABLE VI are agreement to the simulation results in TABLE IV. The results of the performance comparison showed that the Extended EWMA control chart provided smaller ARL_1 than the EWMA control chart for all magnitudes of change except when the large shift sizes ($\delta \geq 0.50$), the performance of Extended EWMA

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

control chart is close to the EWMA control chart both AR(2) and AR(3) processes. These results indicated that the performance of the Extended EWMA control chart was more efficient than the EWMA control charts for all situations as illustrated in Fig. 1 and Fig.2.

Biochemical Oxygen Demand (BOD) in terms of milligram per liter (mg/L) and Salinity (SAL) in terms of part per thousand (ppt) are important indicators of water quality, Biochemical Oxygen Demand (BOD) and Salinity (SAL) values are important indicator of pollution in water, which are major environmental problem. If these values are higher than the international standard can greatly affect public health and ecosystems. Biochemical Oxygen Demand (BOD) and Salinity (SAL) values are analyzed. The 40 observations of quarterly showed form January 2011 until December 2020. The upper and lower control limits were established by (2) for the EWMA control chart, and then (4) for the Extended EWMA control chart. The detection of the process with real data for exponential smoothing parameter is shown in Fig. 3 and Fig. 4.

The results in Fig. 3 showed that *ARL* of AR(2) process on Extended EWMA control chart detected the out-of-control process at the 3rd observation whereas EWMA control chart detected it at 20th observation. For AR(3) process, the results in Fig. 4 showed that Extended EWMA control chart detected the out-of-control process at the 4th observation whereas EWMA control chart detected it at the 19th observation. Moreover, exponential smoothing parameter of 0.05 is recommended.

VII. CONCLUSION

In the study, the results showed the derivation of NIE methods for the *ARL* on Extended EWMA control chart for autoregressive process with exponential white noise. The *ARL* on Extended EWMA control chart of NIE method using the Gaussian rule was efficient as same as three methods namely NIE method using midpoint rule, Simpson's rule, and trapezoidal rule. The analytical results with the CPU times for NIE method using the trapezoidal rule takes less CPU times than others both in case AR(2) and AR(3) processes. In addition, the performance of Extended EWMA control chart is better than the EWMA control for all situations except when the large shift sizes ($\delta \geq 0.50$), the performance of Extended EWMA control chart is close to the EWMA control chart. Besides, the ARL performances on EWMA and Extended EWMA control charts were compared using real data about BOD and SAL values which are important indicators of water quality. The results present that Extended EWMA control chart performed better than the EWMA control chart for all magnitudes of change.

ACKNOWLEDGEMENTS

We are grateful to the referees for their constructive comments and suggestions which helped to improve this research. The research was funding by King Mongkut's University of Technology North Bangkok contract no. KMUTNB-BasicR-64-02.

TABLE I
COMPARISON OF *ARL* VALUES FOR AR(2) PROCESS WHEN GIVEN $\phi_1 = \phi_2 = 0.1, \lambda_2 = 0.01, \eta = 0$ FOR $ARL_0 = 370$

λ_1	δ	$\tilde{L}(u)$	$\tilde{L}_M(u)$	$\tilde{L}_S(u)$	$\tilde{L}_T(u)$
0.05	0.000	370.2816867 (2.781)	370.2816867 (2.344)	370.2817345 (8.500)	370.2818301 **(2.266)**
	0.001	217.4953218 (2.454)	217.4953218 (2.296)	217.4953447 (8.609)	217.4953904 **(2.140)**
	0.003	119.5016792 (3.204)	119.5016792 (2.266)	119.5016898 (8.656)	119.5017111 **(2.250)**
	0.005	82.59899242 (2.844)	82.59899242 (2.281)	82.59899927 (8.500)	82.59901297 **(2.171)**
	0.010	46.91029627 (3.204)	46.91029627 (2.312)	46.91029983 (8.515)	46.91030696 **(2.109)**
	0.050	11.27836031 (3.172)	11.27836031 (2.313)	11.27836099 (8.437)	11.27836237 **(2.141)**
	0.100	6.26083855 (2.999)	6.260838549 (2.328)	6.260838866 (8.453)	6.260839500 **(2.156)**
	0.500	2.12322277 (3.047)	2.123222768 (2.406)	2.123222804 (8.562)	2.123222876 **(2.218)**
	1.000	1.58415169 (3.109)	1.584151688 (2.359)	1.584151699 (8.469)	1.584151720 **(2.172)**
0.10	0.000	370.0373364 (3.281)	370.0373364 (2.766)	370.0376867 (8.859)	370.0383875 **(2.109)**
	0.001	238.6024870 (2.969)	238.6024870 (2.391)	238.6026464 (9.219)	238.6029651 **(2.125)**
	0.003	139.8263530 (2.969)	139.8263530 (2.313)	139.8264170 (8.906)	139.8265449 **(2.093)**
	0.005	99.10372993 (3.063)	99.10372993 (2.281)	99.10376662 (8.969)	99.10384000 **(2.156)**
	0.010	57.66047917 (3.157)	57.66047917 (2.376)	57.66049529 (9.297)	57.66052753 **(2.187)**
	0.050	14.09867481 (2.765)	14.09867481 (2.281)	14.09867716 (9.187)	14.09868188 **(2.250)**
	0.100	7.777103996 (2.937)	7.777103996 (2.422)	7.777105032 (8.875)	7.777107105 **(2.297)**
	0.500	2.513186717 (2.703)	2.513186717 (2.344)	2.513186834 (8.750)	2.513187068 **(2.313)**
	1.000	1.811855293 (3.140)	1.811855293 (2.624)	1.811855328 (8.687)	1.811855398 **(2.203)**

TABLE II
COMPARISON OF *ARL* VALUES FOR AR(3) PROCESS WHEN GIVEN $\phi_1 = \phi_2 = \phi_3 = 0.1, \lambda_2 = 0.01, \eta = 0$ FOR $ARL_0 = 500$

λ_1	δ	$\tilde{L}(u)$	$\tilde{L}_M(u)$	$\tilde{L}_S(u)$	$\tilde{L}_T(u)$
0.05	0.000	500.0084741 (2.844)	500.0084741 (2.922)	500.0085488 (10.172)	500.0086984 **(2.625)**
	0.001	256.4589010 (3.078)	256.4589010 (2.891)	256.4589295 (10.344)	256.4589866 **(2.672)**
	0.003	130.3137673 (3.157)	130.3137673 (2.798)	130.3137792 (9.812)	130.3138029 **(2.625)**
	0.005	87.59190462 (2.970)	87.59190462 (2.797)	87.59191196 (10.109)	87.59192664 **(2.624)**
	0.010	48.45562396 (2.921)	48.45562396 (2.719)	48.45562766 (9.578)	48.45563506 **(2.594)**
	0.050	11.35617866 (3.173)	11.35617866 (2.938)	11.35617935 (10.485)	11.35618074 **(2.734)**
	0.100	6.281941903 (2.985)	6.281941903 (3.125)	6.281942222 (10.109)	6.281942859 **(2.407)**
	0.500	2.124507217 (2.656)	2.124507217 (2.828)	2.124507253 (10.203)	2.124507325 **(2.516)**
	1.000	1.584621911 (2.734)	1.584621911 (2.859)	1.584621922 (10.188)	1.584621943 **(2.610)**
0.10	0.000	500.2007835 (3.125)	500.2007835 (3.093)	500.2012639 (10.609)	500.2022248 **(2.875)**
	0.001	278.8226545 (3.094)	278.8226545 (3.079)	278.8228201 (10.609)	278.8231512 **(2.578)**
	0.003	148.3064008 (3.344)	148.3064008 (3.016)	148.3064567 (10.813)	148.3065685 **(2.531)**
	0.005	101.2669728 (3.094)	101.2669728 (3.156)	101.2670030 (10.531)	101.2670634 **(2.547)**
	0.010	56.81437838 (3.000)	56.81437838 (3.046)	56.81439101 (10.531)	56.81441627 **(2.718)**
	0.050	13.41635460 (3.031)	13.41635460 (3.016)	13.41635639 (10.577)	13.41635997 **(2.828)**
	0.100	7.380674151 (3.297)	7.380674151 (2.937)	7.380674938 (10.422)	7.380676512 **(2.688)**
	0.500	2.405866112 (3.172)	2.405866112 (3.063)	2.405866200 (10.969)	2.405866376 **(2.719)**
	1.000	1.748842205 (3.672)	1.748842205 (2.999)	1.748842231 (10.141)	1.748842283 **(2.719)**

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE III
COMPARISON OF *ARL* FOR AR(2) PROCESS ON EWMA AND EXTENDED EWMA CONTROL CHARTS WHEN GIVEN $\phi_1 = \phi_2 = 0.1, \lambda_2 = 0.01, \eta = 0$ FOR $ARL_0 = 370$

δ	$\lambda_1 = 0.05$ *		$\lambda_1 = 0.10$ **	
	EWMA	Extended EWMA	EWMA	Extended EWMA
0.000	370.077	370.282	370.094	370.038
0.001	263.301	217.495	269.630	238.603
0.003	167.231	119.502	175.000	139.827
0.005	122.719	82.5990	129.717	99.1038
0.010	73.9901	46.9103	79.0515	57.6605
0.050	18.5918	11.2784	20.0176	14.0987
0.100	10.1670	6.26084	10.9047	7.77711
0.500	3.07978	2.12322	3.22161	2.51319
1.000	2.13048	1.58415	2.19833	1.81186

*h = 0.0750269 for EWMA, b = 0.0328891 for Extended EWMA
**h = 0.1561386 for EWMA, b = 0.1013575 for Extended EWMA

TABLE IV
COMPARISON OF *ARL* FOR AR(3) PROCESS ON EWMA AND EXTENDED EWMA CONTROL CHARTS WHEN GIVEN $\phi_1 = \phi_2 = \phi_3 = 0.1, \lambda_2 = 0.01, \eta = 0$ FOR $ARL_0 = 500$

δ	$\lambda_1 = 0.05$ *		$\lambda_1 = 0.10$ **	
	EWMA	Extended EWMA	EWMA	Extended EWMA
0.000	500.400	500.657	500.279	500.202
0.001	310.475	251.383	318.378	278.823
0.003	176.875	126.362	184.688	148.307
0.005	123.906	84.6312	130.304	101.267
0.010	71.2039	46.6690	75.3908	56.8144
0.050	17.0721	10.9156	18.1352	13.4164
0.100	9.32177	6.04369	9.86674	7.38068
0.500	2.88611	2.05933	2.99339	2.40587
1.000	2.02595	1.54543	2.07834	1.74884

*h = 0.0676623 for EWMA, b = 0.0297363 for Extended EWMA
**h = 0.1402343 for EWMA, b = 0.0913434 for Extended EWMA

TABLE V
COMPARISON OF *ARL* FOR AR(2) ON EWMA AND EXTENDED EWMA CONTROL CHARTS FOR $ARL_0 = 370$ WHEN $\alpha_0 = 2.507313, \phi_1 = 0.550326, \phi_2 = -0.325831, \eta = 4.827637, \lambda_2 = 0.01$

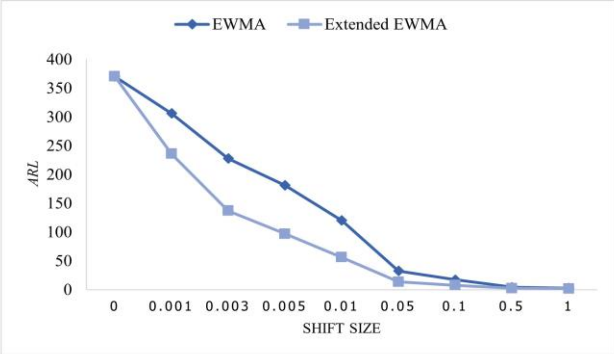
δ	$\lambda_1 = 0.05$ *		$\lambda_1 = 0.10$ **	
	EWMA	Extended EWMA	EWMA	Extended EWMA
0.000	370.044	370.051	370.039	370.060
0.001	306.245	235.819	319.342	265.478
0.003	227.725	136.991	250.548	169.861
0.005	181.267	96.7545	206.051	125.070
0.010	120.079	56.1081	142.523	75.6793
0.050	32.6381	13.7005	40.5381	19.0592
0.100	17.2941	7.57321	21.2021	10.4056
0.500	4.28503	2.47411	4.79756	3.12054
1.000	2.68913	1.79344	2.88882	2.14724

*h = 0.2656812 for EWMA, b = 0.1253684 for Extended EWMA
**h = 0.5628761 for EWMA, b = 0.3763229 for Extended EWMA

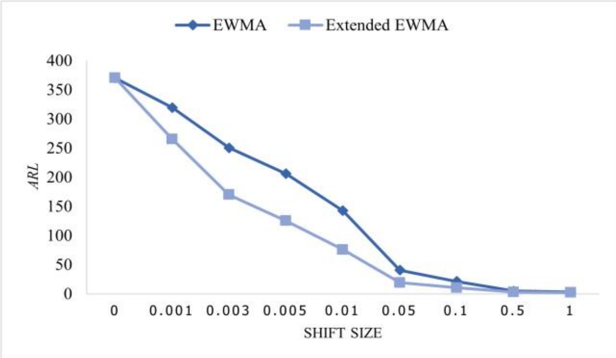
TABLE VI
COMPARISON OF *ARL* FOR AR(3) ON EWMA AND EXTENDED EWMA CONTROL CHARTS FOR $ARL_0 = 500$ WHEN $\alpha_0 = 4.397613, \phi_1 = 0.571799, \phi_2 = -517654, \phi_3 = 0.451555, \eta = 3.428618,$ $\lambda_2 = 0.01$

δ	$\lambda_1 = 0.05$ *		$\lambda_1 = 0.10$ **	
	EWMA	Extended EWMA	EWMA	Extended EWMA
0.000	500.035	500.039	500.191	500.502
0.001	377.510	306.243	396.499	344.152
0.003	253.472	172.913	280.285	212.109
0.005	190.871	120.713	216.746	153.491
0.010	118.153	69.1421	138.333	91.0734
0.050	29.6781	16.5495	35.5844	22.2538
0.100	15.6925	9.05034	18.5915	11.9876
0.500	4.04212	2.82717	4.45537	3.42439
1.000	2.58695	1.99465	2.75556	2.30028

*h = 0.4451557 for EWMA, b = 0.2885339 for Extended EWMA
**h = 0.9404860 for EWMA, b = 0.7437720 for Extended EWMA

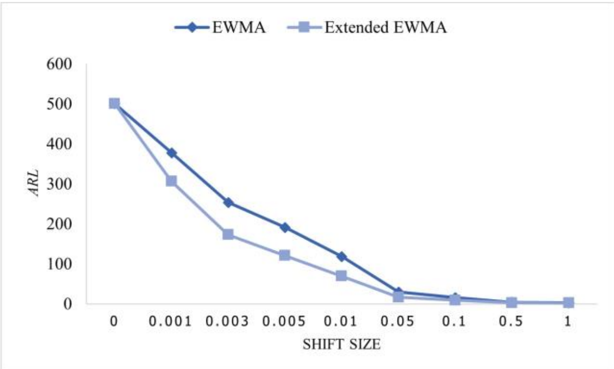


$\lambda_1 = 0.05$

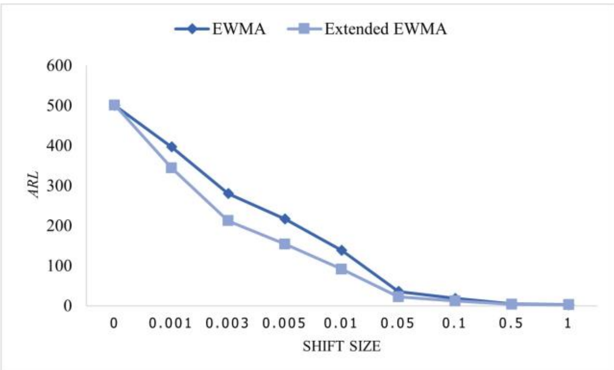


$\lambda_1 = 0.10$

Fig. 1. *ARL* for the AR(2) process on EWMA and Extended EWMA control charts with real data when given $ARL_0 = 370$.



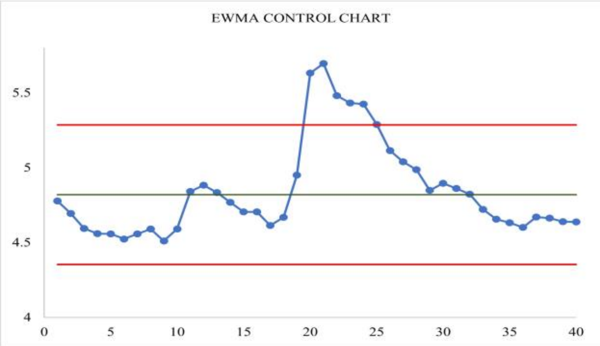
$\lambda_1 = 0.05$



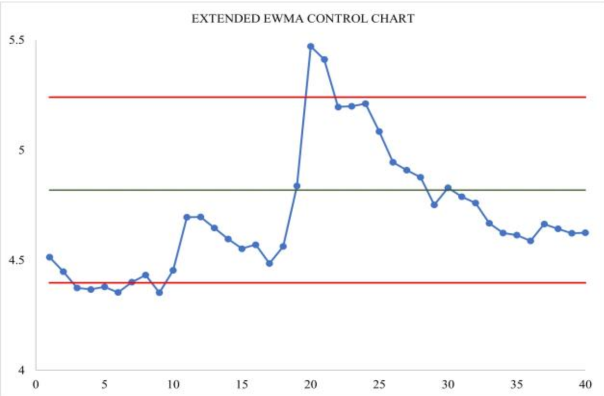
$\lambda_1 = 0.10$

Fig. 2. *ARL* for the AR(3) process on EWMA and Extended EWMA control charts with real data when given $ARL_0 = 500$.

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

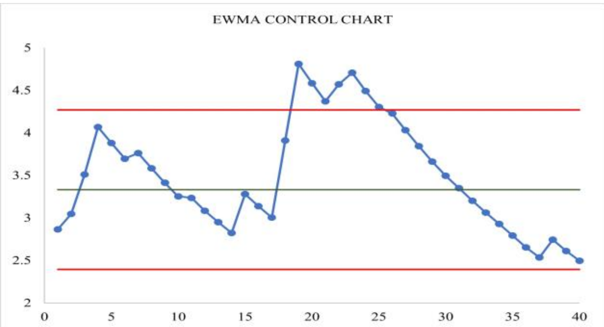


EWMA control chart

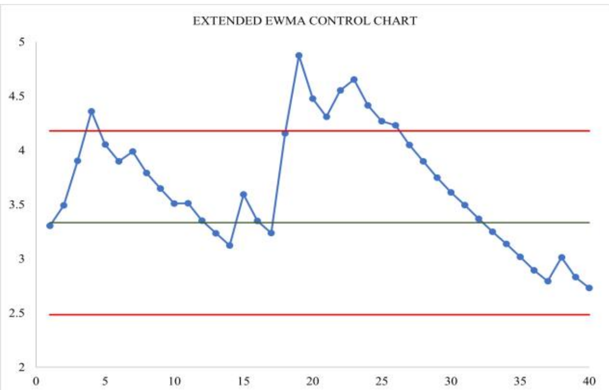


Extended EWMA control chart

Fig. 3. The detection of the AR(2) process with Biochemical Oxygen Demand (BOD) in terms of milligram per liter (mg/L) for $\lambda_1 = 0.05$.



EWMA control chart



Extended EWMA control chart

Fig. 4. The detection of the AR(3) process with Salinity (SAL) in terms of part per thousand (ppt) for $\lambda_1 = 0.05$.

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ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)