Mathematical Investigation of Area and Mass Moments of Inertia of Beams

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Abstract— **Mass moment of inertia (MMI) and Area moment of inertia (AMI) are two different concepts often confused to mean the same thing, as they are usually both referred to as moment of inertia. In this paper, difference between the two concepts was established and their relationship explored analytically using parallel axis theorem and models. In particular, moment of inertia of wood beam with circular, circular hollow, rectangular, and rectangular hollow cross sections, side by side with their AMI, were computed and plotted, with the aid of MAPLE, to see if there is a direct or an indirect relationship between the two concepts. In addition to the fact that the results are consistent with the ones in the literature, the MMI of all the beams considered are greater than their AMI. Also the AMI for the beams about another axis is greater than the solid's moment of inertia about the axis through the solid's centre of mass, given the shortest distance between the axes.**

Index Terms— **Mathematical Investigation, Area moment of inertia, Mass moment of Inertia, Beams**

I. Introduction

THERE are two concepts that can be referred to as "moment of inertia". One refers to bending resistance; the other refers to resistance to angular acceleration. AMI is a measure for the bending resistance of a shape about a certain axis. Usually, it is applied to a beam structure. The two concepts: the AMI and the MMI, are inaccurately known by the term "moment of inertia". The AMI, is also known as the following: second moment of area, and second area moment [1]. AMI is the one which is most commonly encountered in structural engineering. It is usually denoted as I, arises from studying the bending of beams. Beam's bending stress σ can be represented as:

$$\sigma = M\,y / I,$$

Manuscript received February 17, 2019; revised April 17, 2019. This work was supported fully by the Covenant University, Ota, Nigeria. Mathematical Investigation of Area and Mass Moments of Inertia of Beams
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Where, M is the applied bending moment, y is the distance from the neutral axis to where the bending stress is sought, and I is the AMI of the beam's cross section. This inertia is roughly analogous to MMI represented by J, in that it describes the distribution of section area about the centroid of the section.

In beam analysis, the maximum bending stress is encountered at the outer part of the beam, which is located at the longest distance from the neutral axis. Thus for a given structural section, I can be calculated and y is known from the geometry, so the bending stress formula can be re-written as σ = M / SM, where SM is the section modulus of the beam cross section and SM = I / y units of L3 [1,2,3].

The MMI of a body, from Newton's laws of motion, shows that it takes an applied effort to change the motion of a body [4,5]. For bodies in rectilinear motion, the equation F = ma describes how the acceleration of a body of mass m will change when a certain external force F is applied. For bodies in rotational motion, the corresponding equation is T = J α, where J is the MMI of the body (and which has units of ML^2) and α is the angular acceleration (in radians per second2) produced by the applied torque T (units of force times distance.

The MMI or J is an intrinsic property of the body, the value of which is influenced by the distribution of mass about the centre of gravity. The MMI is used to calculate the dynamics of bodies undergoing motion [6]. Beam naturally resists loads applied to their axis. They are characterized by their manner of support [7,8,9]. In this paper, the AMI and MMI were mathematically investigated and relationships drawn.

II. Area Moment of Inertia vs. Mass Moment of Inertia

The major difference between the MMI and AMI include their units and their usage. The MMI and AMI are, respectively, used as a rotational analogy of mass, and for beam equations. Both however, are often represented by I [10].

The formulas for computing the MMI and AMI of circular, hollow circular, rectangular beam, and Hollow rectangular beams, are represented in table 1 as follows [10,11, 12]::

Table 1: Formulas for mass and area moments of inertia different beams.

S/N	Object	MMI	AMI
1	Solid circular beam	$I = \frac{1}{2}MR^2$	$I_{xc} = \frac{\pi(d)^4}{64}$
2	Hollow circular beam	$I = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$	$I_{xch} = \frac{\pi}{64}\left[(d_2)^4 - (d_1)^4\right]$
3	Solid rectangular beam	$I = \frac{1}{12}M(a^2 + b^2)$	$I_{xr} = \frac{1}{12}b_1h^3$
4	Hollow rectangular beam	$I = \frac{M}{12}(B^3 + H^3) - \frac{M}{12}(B_1^3 + H_1^3)$	$I_{xrh} = \frac{1}{12}BH^3 - \frac{1}{12}B_1H_1{}^3$

'I' stands for both mass and area moment of inertia. They both have the same unit: $kg \cdot m^2$.
M stands for total mass of the rotating object, with the unit kg
L is the rod's total length (m)
a is the plate's length (m)
b is the plate's breath (m)
R_1 is the cylinder's inner radius (m)
R_2 is the cylinder's outer radius (m)
R is the cylinder or sphere's radius (m)
d is the diameter of the circular cross section
b_1 is the horizontal distance of the cross section
h is the height of the cross section
d_2 is the diameter of the outer circle of the cross section
d_1 is the diameter of the inner circle of the cross section
B is the horizontal distance of the outer rectangle of the cross section
H is the height of the outer rectangle of the cross section
B_1 stands for the horizontal distance of the inner rectangle of the cross section
H_1 is the height of the inner rectangle of the cross section
I_{xc} is the moment of inertia of wood beam with circular cross section.
I_{xr} is the moment of inertia of wood beam with rectangular cross section.
I_{xch} is the moment of inertia of wood beam with circular hollow cross section.
I_{xrh} is the moment of inertia of wood beam with rectangular hollow cross section.

A. *Theorem: The parallel axis theorem*

Huygens–Steiner theorem, also known as parallel axis theorem, states that the moment of inertia is minimal when the rotation axis passes through the centre-of-mass and increases as the rotation axis is moved further from the centre-of-mass [13,14,15,16]. This theorem is used for determining the AMI of rigid bodies about any axis, when its moment of inertia, about an axis parallel to it is known, through the object's centre of mass and the shortest distance (D) between the axes [9].

$$I_{x}' = I_x + AD^2 \qquad (1)$$

III. Numerical Analysis

For the purpose of numerical analysis, the following parameters values are considered:
L=5, d=3, D=2, M=10, R=1.5 , R_1=2, R_2=3, a=2, b=5, b_1=2, h=5,B=2, H=5,B_1=1, H_1=2.5, d_1=4, d_2=6

A. *For a solid circular beam*

$$I = \frac{1}{2}MR^2 \qquad (2)$$

=11.25

$$I_{xc} = \frac{\pi(d)^4}{64} \qquad (3)$$

=3.98

$$I_{x}' = I_x + AD^2 \qquad (4)$$

= 32.27

B. *For a hollow circular beam*

$$I = \frac{1}{2}M(R_1{}^2 + R_2{}^2) \qquad (5)$$

= 65

$$I_{xch} = \frac{\pi}{64}\left[(d_2)^4 - (d_1)^4\right] \qquad (6)$$

=51.06

$$I_{x}' = I_x + AD^2 \qquad (7)$$

= 113.92

C. *For a solid rectangular beam*

$$I = \frac{1}{12}M(a^2 + b^2) \qquad (8)$$

= 24.17

$$I_{xr} = \frac{1}{12}b_1h^3 \qquad (9)$$

= 20.83

$$I_{x}' = I_x + AD^2 \qquad (10)$$

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$= 60.83$

D. For a hollow rectangular beam

$$I = \frac{M}{12}(B^3 + H^3) - \frac{M}{12}(B_1^3 + H_1^3) \quad (11)$$

$= 96.98$

$$I_{xrh} = \frac{1}{12}BH^3 - \frac{1}{12}B_1H_1^3 \quad (12)$$

$= 19.53$

$$Ix' = Ix + AD^2 \quad (13)$$

$= 49.53$

IV. RESULTS AND DISCUSSION

The value of AMI is less than the value of MMI for all the different types of beams considered in this study. From the results it can be deduced that the AMI of different beams, with rotation axis away from the centre of mass, is greater than the one with the rotation axis passing through the centre of mass. It was also observed that the value of MMI of solid circular beam is less than that of hollow circular beam. Similarly, the value of the MMI of the hollow rectangular beam is greater than that of the solid rectangular beam.

However, the MMI of the rectangular beam, both hollow and solid, are greater than that of circular beams respectively. Figure 1 depicts that the AMI, for all the beams considered, on the average, has the least maximum amplitude, followed by their MMI.

The AMI, with rotation axis away from the centre of mass, has the highest maximum amplitude. Figure 2 through figure 13 show the 3D plotting of the AMI and MMI, with rotation axis away from the centre of mass, for all the types of beam considered in this. PAT represents AMI about any axis

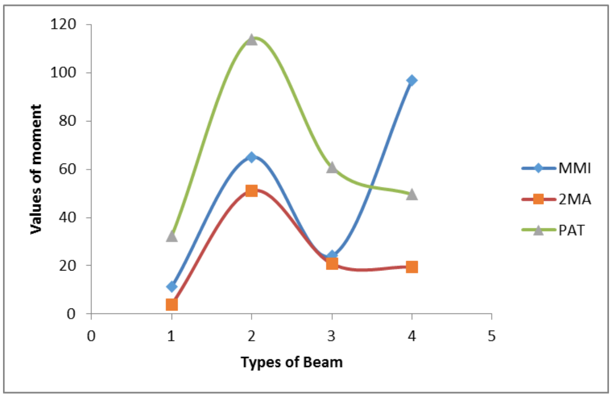

Figure 1: PAT compared to MMI and AMI/2MA of a rigid body.

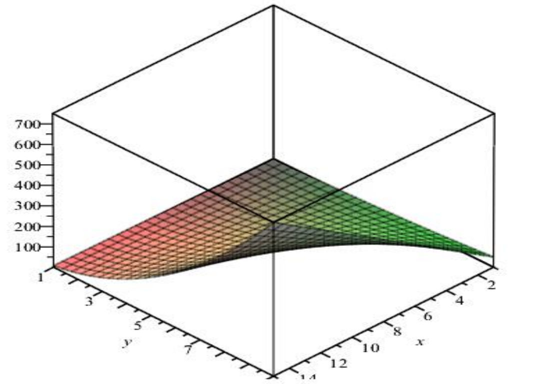

Figure 2: The MMI of circular solid beam at different values of mass and radius

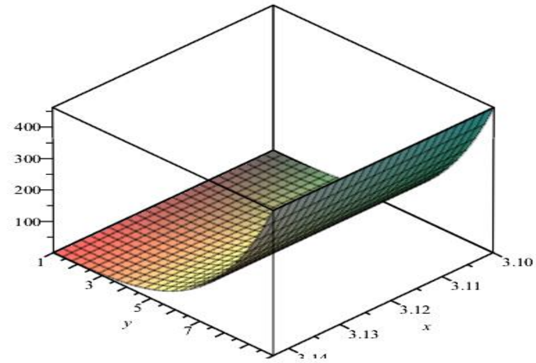

Figure 3: The AMI of circular solid beam at different values of diameter

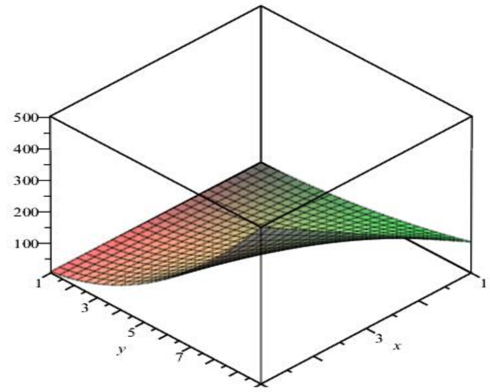

Figure 4: The AMI of solid circular beam with rotational axis away from centre of mass at different values of area and diameter

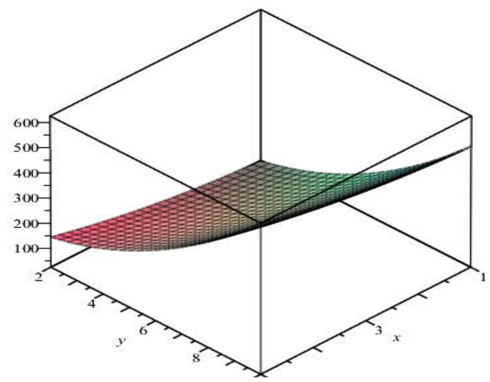

Figure 5: The MMI of hollow circular beam at different values of inner and outer diameters

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

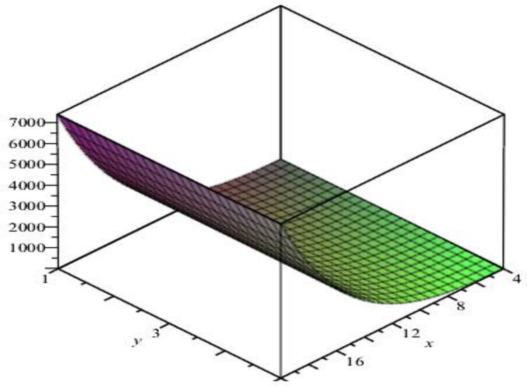

Figure 6: The AMI of hollow circular beam at different values of inner and outer radii

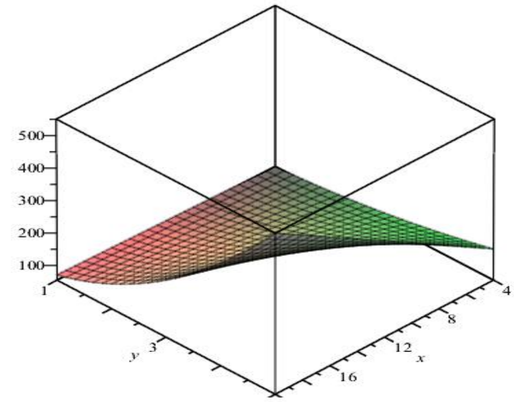

Figure 7: The AMI of hollow circular beam with rotation axis away from centre of mass at different values of area and diameter

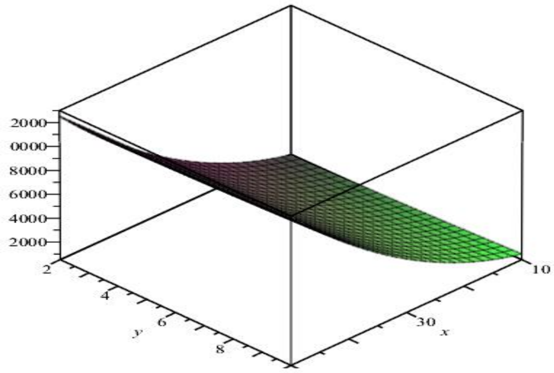

Figure 8: The MMI of solid rectangular beam at different values of lengths of both sides

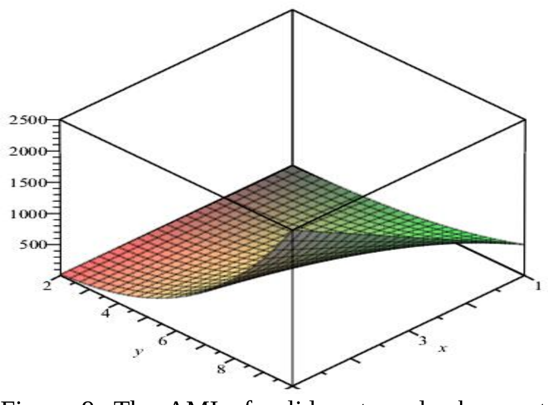

Figure 9: The AMI of solid rectangular beam at different values of horizontal distance and height

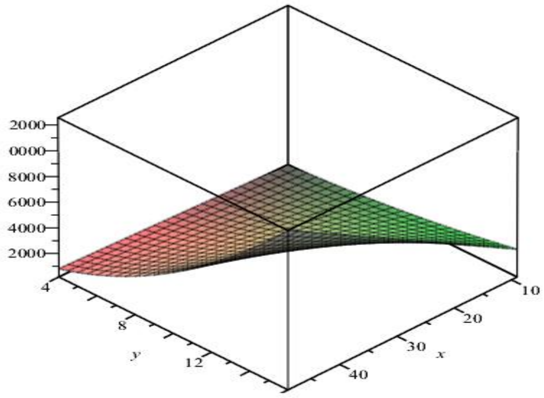

Figure 10: The AMI of solid rectangular beam with rotation axis away from centre of mass at different values of area and diameter.

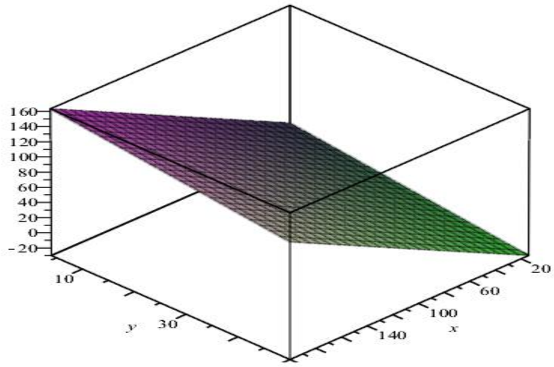

Figure 11: The MMI of hollow rectangular beam at different horizontal distance and height of inner and outer rectangles.

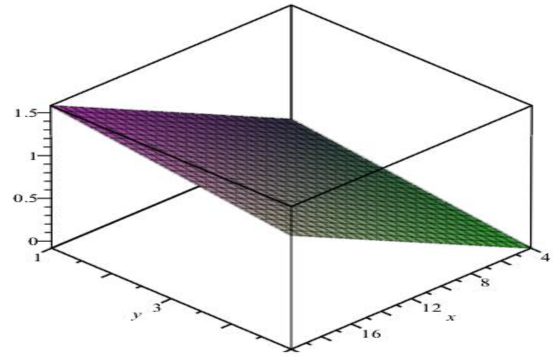

Figure 12: The AMI of hollow rectangular beam at different horizontal distance and height of inner and outer rectangles.

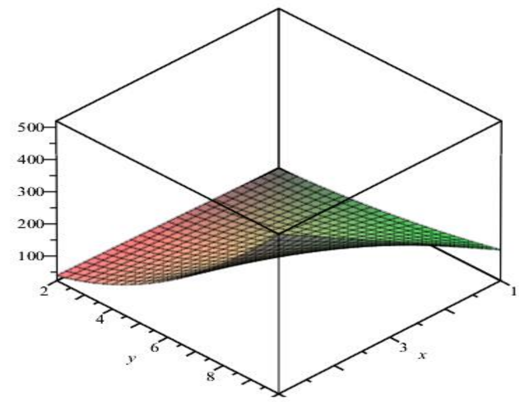

Figure 13: The AMI of hollow rectangular beam with rotation axis away from centre of mass at different area and diameter.

ISBN: 978-988-14049-2-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

V. CONCLUSION

The difference between the MMI and AMI was clearly established in this study. Both were analytically computed. The parallel axis theorem was established by considering a numerical example. The following different beams were considered: solid circular, hollow circular, solid rectangular, and hollow rectangular beams.

The AMI for the solid beams about another axis is greater than the beam's AMI the axis through the solid's centre of mass, if the perpendicular distance between the axes is known. Also the MMI of all types of beam considered are greater than the AMI of the same beams. This implies that the resistance to angular acceleration for the beams considered is greater than their resistance to bending.

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