On the Stokes Flow in Pipes with the Polygonal Cross-Section

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Abstract—**In the paper an unsteady incompressible fluid flow in a prismatic pipe is studied for the low Reynolds number. The linearized Navier-Stokes equation (the Stokes equation) is considered with the suitable initial-boundary conditions. It is assumed that the pressure exponentially depends on time . The Stokes equation is reduced to the system of linear integral equations with the weakly singular kernel. The existence and uniqueness of the solutions of those equations is proved and the approximate solutions are obtained by means of the conformal mapping and the stepwise approximation methods. The example of the pipe with the hexagonal cross-section is considered.**

Index Terms—**Conformal mapping, Integral equations, Stokes flow in pipes, Step-wise approximation**

I. Introduction

The pipes with the polygonal cross-section are widely used in technological processes. However, it is important to define the velocity of the fluid flow in such pipes not only experimentally, but also analytically.

Study of the Stokes system for the pipes begins in the XIX century. The solutions of the Stokes system for the incompressible fluid flow in pipes with circular, elliptical, rectangular, triangular cross-sections were obtained by Poiseuille (1840), and Boussinesq (1868), [2], [6], [20]. Exact solutions for the flow in porous circular pipes were derived by S. Tsangaris, D. Kondaxakis and, N. Vlachakis in 2007 [30]. Exact solutions for the axi-symmetric Stokes system for the fluid flow over the ellipsoidal bodies in the infinite channel are given in [11, 13]. By means of those solutions, transportation of oxygen with the help of the single erythrocyte in a capillary was described [13]. We obtained non-smooth solutions of the Stokes system for the fluid flow over and inside the rectangular infinite prism in [17]. The numerical treatment of the Stokes system is given in [4], [7], [24], [25], [28], [29].

As we know, solutions of Stokes system for pipes with an arbitrary polygonal cross-section have not been obtained yet and this is our goal.

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We assume that the pressure is controlled and the fluid flow in a pipe is slow with a low Reynolds number. In this case the flow is called the Stokes flow and the velocity components of the flow subject to the Stokes system [1], [2],
[5], [6], [15], [16], [18], [20]—[30]

$$\frac{\partial V_x}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}=F_x+\nu\Delta V_x, \quad (1)$$

$$\frac{\partial V_y}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial y}=F_y+\nu\Delta V_y, \quad (2)$$

$$\frac{\partial V_z}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial z}=F_z+\nu\Delta V_z, \quad (3)$$

with the equation of continuity

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 , \quad (4)$$

where t is a time, $\vec{V}\ (V_x, V_y, V_z)$ is the velocity of the fluid, $\vec{F}\ (F_x, F_y, F_z)$ is the body force, P is the pressure, ρ is the density, ν is the viscosity of the fluid.

We consider the system (1), (2), (3) with the initial-boundary conditions

$$V_x\big|_S=V_y\big|_S=V_z\big|_S=0 \quad , \quad (5)$$

$$V_x(x,y,z,0)=V_x^0(x,y,z,0),$$
$$V_y(x,y,z,0)=V_y^0(x,y,z,0), \quad (6)$$
$$V_z(x,y,z,0)=V_z^0(x,y,0),$$

where S is the boundary of the pipe, $V_x^0(x,y,z,0),\ V_y^0(x,y,z,0), V_z^0(x,y,0),$ are some double-differentiable functions.

The pressure P satisfies the equation [1], [2], [5], [6], [15], [16], [18], [20]—[30]

$$\Delta P=\rho\, div\vec{F}. \quad (7)$$

In the paper, the specific pressure (depends on time exponentially) is considered. Using the Poisson formula and conformal mapping methods we reduce the system (1), (2), (3), (4) to the system of Fredholm integral equations [3], [8], [10], [19], [23]. We solve this system by the stepwise

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approximation method and obtain the velocity components. Hence, we define the velocity of the fluid. The example is given for a pipe with a hexagonal cross-section (the pipes that have hexagonal cross-sections are more convenient for transportation).

We will solve Problem 1 by means of two different methods: in Chapter 3 we obtain the solution directly in 0xyz space, while in Chapter 4 we use the conformal mapping method and obtain the solution in new variables (from our viewpoint this method is more convenient for the numerical treatment).

II. STATEMENT OF THE PROBLEM

We study 3D Stokes flow in the finite prismatic pipe.

In the cartesian coordinate system $Oxyz$ we consider the prismatic area $D_z = \{D_0 \times [0, l]\}; 0 \le z \le l; l > 0;$ with the boundary S and with the cross-section D_0, where D_0 is the simply connected region of xOy plane bounded by a piecewise-smooth line $\varphi(x, y)$.

Let us suppose

$$P(x, y, z, t) = a\exp(-\alpha t)(l - z)P_0(x, y), \qquad (8)$$

$$F_x = \exp(-\alpha t)(l - z)F_x^0(x, y),$$
$$F_y = \exp(-\alpha t)(l - z)F_y^0(x, y),$$
$$F_z = \exp(-\alpha t)F_z^0(x, y), \qquad (9)$$

where $F_x^0(x, y), F_y^0(x, y), F_z^0(x, y)$ are continuous functions , and P_0 is the double differentiable function in D_0, a and $\alpha > 0$ are some given constants.

Besides, we admit

$$V_x = \exp(-\alpha t)(l - z)V_x^0(x, y),$$
$$V_y = \exp(-\alpha t)(l - z)V_y^0(x, y), \qquad (10)$$
$$V_z = \exp(-\alpha t)V_z^0(x, y),$$

where functions $V_x^0(x, y),\ V_y^0(x, y),\ V_z^0(x, y)$ are to be determined.

By (5), (6), (8), (9), (10) the system (1), (2), (3), (4) becomes

$$\Delta V_x^0 + \frac{\alpha}{\nu} V_x^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial x} - \frac{1}{\nu}F_x^0, \qquad (11)$$

$$\Delta V_y^0 + \frac{\alpha}{\nu} V_y^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial y} - \frac{1}{\nu}F_y^0, \qquad (12)$$

$$\Delta V_z^0 + \frac{\alpha}{\nu} V_z^0 = -\frac{a}{\rho\nu}P_0(x, y) - \frac{1}{\nu}F_z^0, \qquad (13)$$

$$\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} = 0, \qquad (14).$$

with the initial-boundary conditions

$$V_x^0\Big|_{\varphi(x,y)} = V_y^0\Big|_{\varphi(x,y)} = V_z^0\Big|_{\varphi(x,y)} = 0, \qquad (15)$$

$$V_x(x, y, z, 0) = (l - z)\, V_x^0(x, y),$$
$$V_y(x, y, z, 0) = (l - z)\, V_y^0(x, y), \qquad (16).$$
$$V_z(x, y, z, 0) = V_z^0(x, y),$$

The equation (7) takes the form

$$\frac{\partial^2 P_0}{\partial x^2} + \frac{\partial^2 P_0}{\partial y^2} = \rho\,(\frac{\partial F_y^0}{\partial x} + \frac{\partial F_x^0}{\partial y}). \qquad (17)$$

Our goal is to solve the following problem

PROBLEM 1. In the area D_0 find the functions V_x^0, V_y^0, V_z^0 having continuous second order derivatives and satisfying the system (11), (12), (13), (14) with the boundary condition (15) and the condition (17).

III. SOLUTION OF PROBLEM 1

By means of the condition (15) and Poisson's formula the system (11), (12), (13), (14) can be reduced to the system of Fredholm integral equations [3], [15], [16], [25]

$$V_x^0 - \frac{\alpha}{2\pi\nu}\int_{D_0} G(x, y, x_1, y_1)\ V_x^0\ dx_1 dy_1 = -\frac{1}{2\pi}\int_{D_0} G(x, y, x_1, y_1)\,\Psi_1\ dx_1 dy_1, \qquad (18)$$

$$V_y^0 - \frac{\alpha}{2\pi\nu}\int_{D_0} G(x, y, x_1, y_1)\ V_y^0\ dx_1 dy_1 = -\frac{1}{2\pi}\int_{D_0} G(x, y, x_1, y_1)\,\Psi_2\ dx_1 dy_1, \qquad (19)$$

$$V_z^0 - \frac{\alpha}{2\pi\nu}\int_{D_0} G(x, y, x_1, y_1)\ V_z^0\ dx_1 dy_1 = -\frac{1}{2\pi}\int_{D_0} G(x, y, x_1, y_1)\,\Psi_3\ dx_1 dy_1, \qquad (20)$$

where G is Green's function for the Laplace equation in the area D_0 ,

$$\Psi_1 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial x} - \frac{1}{\nu}F_x^0,$$

$$\Psi_2 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial y} - \frac{1}{\nu}F_y^0,$$

$$\Psi_3 = -\frac{a}{\rho\nu}P_0(x, y) - \frac{1}{\nu}F_z^0.$$

If we take the first order derivatives of equations (11), (12) and use the formula (17) we obtain

$$\Delta\left(\frac{\partial V_x^0}{\partial x}+\frac{\partial V_y^0}{\partial y}\right)+\frac{\alpha}{\nu}\left(\frac{\partial V_x^0}{\partial x}+\frac{\partial V_y^0}{\partial y}\right)=0. \qquad (21)$$

According to (14) equation (21) has only trivial solution. It means that $\frac{\alpha}{2\pi\nu}$ is not the eigenvalue of the integral equations (18), (19), (20) and hence they have unique solutions. Applying the Banach theorem we conclude [3]:

If $\frac{\alpha}{2\pi\nu}<\frac{1}{M}$, where

$$\int_{D_0}\left|G(x,y,x_1,y_1)\right|\ dx_1dy_1\leq M;\ (x,y)\in D_0,$$

then there exists the unique solutions of equations (18), (19), (20) which are given by the formulas

$$V_x^0=\lim_{n\to\infty}V_{xn};\ V_y^0=\lim_{n\to\infty}V_{yn};\ V_z^0=\lim_{n\to\infty}V_{zn}, \quad (22)$$

where

$$V_{x0}=-\frac{1}{2\pi}\int_{D_0}G(x,y,x_1,y_1)\,\Psi_1\ dx_1dy_1, \qquad (23)$$

$$V_{xn}=V_{x0}+\frac{\alpha}{2\pi\nu}\int_{D_0}G(x,y,x_1,y_1)\,V_{x(n-1)}\ dx_1dy_1,$$

$$V_{y0}=-\frac{1}{2\pi}\int_{D_0}G(x,y,x_1,y_1)\,\Psi_2\ dx_1dy_1, \qquad (24)$$

$$V_{yn}=V_{y0}+\frac{\alpha}{2\pi\nu}\int_{D_0}G(x,y,x_1,y_1)\,V_{y(n-1)}\ dx_1dy_1.$$

$$V_{z0}=-\frac{1}{2\pi}\int_{D_0}G(x,y,x_1,y_1)\,\Psi_3\ dx_1dy_1, \qquad (25)$$

$$V_{zn}=V_{z0}+\frac{\alpha}{2\pi\nu}\int_{D_0}G(x,y,x_1,y_1)\,V_{z(n-1)}\ dx_1dy_1.$$

If $\frac{\alpha}{4\pi^2\nu}$ is rather small, then the solutions of the system (18), (19), (20) will be given by (30), (31), (32) and hence the solutions of the system (11), (12), (13), (14) will be given by (10).

We define the profile of the velocity by the formula

$$\left|\vec{V}\right| \qquad (26)$$

$$=\exp(-\alpha t)\sqrt{(l-z)^2\left(V_x^0\right)^2+(l-z)^2\left(V_y^0\right)^2+\left(V_z^0\right)^2}.$$

The Green function G is given by the formula [23]

$$G(x,y,x_1,y_1)=-\ln\left|f(x,y,x_1,y_1)\right|;$$
$$f(x_1,y_1,x_1,y_1)=0, \qquad (27)$$

where $f(x,y,x_1,y_1)$ is the conformal mapping of the area D_0 at the unit circle.

For the polygonal areas with the n-angles the function $f(x,y,x_1,y_1)$ is the inverse function of the integral

$$z=C_1\int_0^f(t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2-1}\Lambda\ (t-a_n)^{\alpha_n-1}dt+C_2,$$

where $\alpha_i,\ \ i=0,\Lambda\ ,n;$ are the angles of the polygon, $a_i,\ \ i=0,\Lambda\ ,n;$ are the points of the circle corresponding to the vertices of the polygon and C_1,C_2 are the definite constants [8], [9], [19], [23].

In the next chapter we study Problem 1 by means of the conformal mapping method.

IV. SOLUTION OF PROBLEM 1 BY MEANS OF THE CONFORMAL MAPPING

Let us consider a conformal mapping $f_1(w)$ of the rectangle

$D\{-a_0/2\leq\xi\leq a_0/2;0\leq\eta\leq b_0\}$

of $w=\xi+i\eta$ plane on the area D_0 (Fig.1).

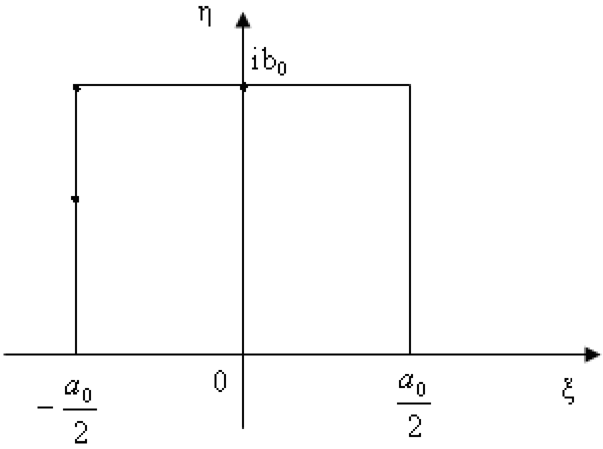

Fig.1. The rectangle $D\{-a_0/2\leq\xi\leq a_0/2;0\leq\eta\leq b_0\}$.

By the mapping $f_1(w)$ the system (1), (2), (3) becomes

$$\Delta V_x^0+\frac{\alpha}{\nu}\left|f_1^{'}(w)\right|^2V_x^0$$
$$=\left|f^{'}{}_1(w)\right|^2\frac{1}{\rho\nu}\frac{\partial P_0}{\partial x}\frac{\partial x}{\partial\xi}-\left|f^{'}{}_1(w)\right|^2\frac{1}{\nu}F_x^0, \qquad (28)$$

$$\Delta V_y^0+\frac{\alpha}{\nu}\left|f_1^{'}(w)\right|^2V_y^0$$
$$=\left|f^{'}{}_1(w)\right|^2\frac{1}{\rho\nu}\frac{\partial P_0}{\partial y}\frac{\partial y}{\partial\eta}-\left|f_1^{'}(w)\right|^2\frac{1}{\nu}F_y^0, \qquad (29)$$

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$$\Delta V_z^0 + \frac{\alpha}{\nu}\left|f'_1(w)\right|^2 V_z^0$$
$$= -\left|f'_1(w)\right|^2 \frac{a}{\rho\nu} P_0(\xi,\eta) - \left|f_1'(w)\right|^2 \frac{1}{\nu} F_z^0, \qquad (30)$$

$$\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} = 0, \qquad (31)$$

with the boundary conditions

$$V_x^0\Big|_{S_0} = V_y^0\Big|_{S_0} = V_z^0\Big|_{S_0} = 0, \qquad (32)$$

S_0 is the boundary of the area D .

Taking into the account (32) and Poisson's formula we reduce the system (28), (29), (30), (31) to the system of integral equations [3], [15], [16],

$$V_x^0 - \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f'_1(w)\right|^2 V_x^0\, dx_1dy_1$$
$$= -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_1^*\, dx_1dy_1, \qquad (33)$$

$$V_y^0 - \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 V_y^0\, dx_1dy_1$$
$$= -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_2^*\, dx_1dy_1, \qquad (34)$$

$$V_z^0 - \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 V_z^0\, dx_1dy_1$$
$$= -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_3^*\, dx_1dy_1, \qquad (35)$$

where

$$\Psi_1^* = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial \xi} - \frac{1}{\nu}F_x^0,$$
$$\Psi_2^* = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial \eta} - \frac{1}{\nu}F_y^0,$$
$$\Psi_3^* = -\frac{a}{\rho\nu}P_0(\xi,\eta) - \frac{1}{\nu}F_z^0,$$

G_0 is Green's function for the Laplace equation in the rectangle D [23]

$$G_0(w;w_0) = -\ln\left|\frac{sn\,w - sn\,w_0}{\bar{sn}\,w - sn\,w_0}\right|, \qquad (36)$$

$sn\,w$ is the Jakobi sinus with the periods $2a_0$; $2ib_0$ [8], [9], [10], [19], [23].

As we proved previously $\frac{\alpha}{2\pi\nu}$ is not the eigenvalue of the system of integral equations (33), (34), (35) and if this constant is rather small, then the solutions of this system are given by the formula (22), where

$$V_{x0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_1^* dx_1dy_1, \qquad (37)$$

$$V_{xn} = V_{x0} + \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f'_1(w)\right|^2 V_{x(n-1)} dx_1dy_1,$$

$$V_{y0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_2^* dx_1dy_1, \qquad (38)$$

$$V_{yn} = V_{y0} + \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 V_{y(n-1)} dx_1dy_1,$$

.

$$V_{z0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 \Psi_3^* dx_1dy_1, \qquad (39)$$

$$V_{zn} = V_{z0} + \frac{\alpha}{2\pi\nu}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2 V_{z(n-1)} dx_1dy_1.$$

Below as an example we consider the case when D_0 is the hexagon (Fig.2)

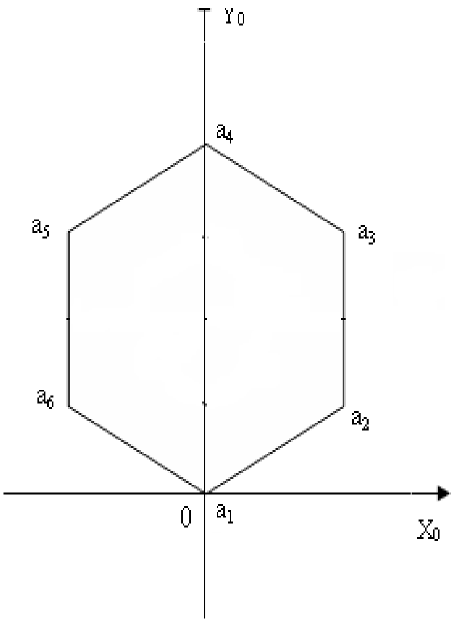

Fig.2. The hexagon with the vertices $a_1, a_2, a_3, a_4, a_5, a_6; a_1 = 0; \mathrm{Re}\, a_4 = 0$

Example. Suppose, that D_0 is the hexagon of z-plane, $z = x + iy$, with the vertices $a_1, a_2, a_3, a_4, a_5, a_6; a_1 = 0; \mathrm{Re}\, a_4 = 0$ and with the axis of symmetry a_1a_4. In this case the conformal mapping of the rectangle D on the area D_0 is given by the formula [12], [14]

$$z = f_1(w) = C\int_0^{z_0} t^{-1/3}\left(t^2-a^2\right)^{-1/3}\left(t^2-b^2\right)^{-1/3}dt,$$

$$z_0 = sn\left(\frac{w}{C_0}\right); w = C_0\int_0^{z_0}\left(1-t^2\right)^{-1/2}\left(1-k^2t^2\right)^{-1/2}dt,$$

where

$$|C| = \frac{|a_3-a_2|}{k_0}; C_0 \approx \frac{a_0}{\pi},$$

$$k_0 = \int_a^b t^{-1/3}\left(t^2-a^2\right)^{-1/3}\left(t^2-b^2\right)^{-1/3}dt,$$

sn is the Jakobi sinus with the modulus k [8], [9], [19], [23], $a=1; b=1/k$.

By the mapping $f_1(w)$ we have the following correspondence of points

$a_1 \leftrightarrow 0,\ a_2 \leftrightarrow a_0/2,\ a_3 \leftrightarrow a_0/2+ib_0,\ a_4 \leftrightarrow ib_0,$
$a_5 \leftrightarrow -a_0/2+ib_0,\ a_6 \leftrightarrow -a_0/2;\ a_0, b_0 > 0.$

If $\frac{\alpha}{4\pi^2\nu}$ is rather small and taking into the account (37), (38), (39), we obtain the approximate solutions of the system (28), (29), (30)

$$V_{x0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2\Psi_1^* dx_1dy_1, (40)$$

$$V_{y0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2\Psi_2^* dx_1dy_1, (41)$$

$$V_{z0} = -\frac{1}{2\pi}\int_D G_0(\xi,\eta,x_1,y_1)\left|f_1'(w)\right|^2\Psi_3^* dx_1dy_1, (42)$$

where G_0 is given by the formula (36) and [12], [14]

$$\left|f_1'(w)\right|^2 = C_1^2\left(\frac{sn\frac{w}{C_0}}{cn\frac{w}{C_0}dn\frac{w}{C_0}}\right)^{2/3}, \qquad (43)$$

$C_1 = k^{2/3}C/C_0; C_0 = a_0/\pi$; sn, cn, dn are the Jakobi functions [8], [9], [19], [23].

If $q = e^{-\pi\chi}$, ($\chi = \frac{2b_0}{a_0}$), is sufficiently small, the following formulas are valid [8], [9], [12], [14], [19], [23]

$$\begin{aligned} sn(w/C_0) &\approx \sin\gamma\left(1+4q\cos^2\gamma\right) \approx \sin\gamma, \\ cn(w/C_0) &\approx \cos\gamma\left(1-4q\sin^2\gamma\right) \approx \cos\gamma, \\ dn(w/C_0) &\approx \left(1-8q\sin^2\gamma\right) \approx 1, \end{aligned} \qquad (44)$$

where

$$\gamma = \frac{\pi w}{a_0C_0}; b_0 = \frac{5}{3}a_0; k_0 \approx 0.34;$$

$$\sin\gamma = \sin\frac{\pi\xi}{a_0C_0}\cosh\frac{\pi\eta}{a_0C_0} + i\cos\frac{\pi\xi}{a_0C_0}sh\frac{\pi\eta}{a_0C_0}.$$

Let us assume that $\left(\frac{\pi w}{a_0C_0}\right)^3$, $\left(\frac{\pi w}{a_0C_0}\right)^3$ are negligible and

$$a_0 = 10^3; k \approx 0.02; k_0 \approx 0.34; |a_3-a_2| = 1; \nu = 1;$$

then

$$\sin\frac{\pi\xi}{a_0C_0} \approx \frac{\pi\xi}{a_0C_0}; sh\frac{\pi\eta}{a_0C_0} \approx \frac{\pi\eta}{a_0C_0}; |C| \approx 3\cdot 10^{-3};$$

$$\cos\frac{\pi\xi}{a_0C_0} \approx 1-\frac{1}{2}\left(\frac{\pi\xi}{a_0C_0}\right)^2; \cosh\frac{\pi\eta}{a_0C_0} \approx 1+\frac{1}{2}\left(\frac{\pi\eta}{a_0C_0}\right)^2$$

and by (44) the formulas (36), (43) becomes

$$G_0(w;w_0) = -\frac{1}{2}\ln\left|\frac{(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2}{(\xi_1-\xi_0)^2+(\eta_1+\eta_0)^2}\right|,$$

$$\left|f_1'(w)\right|^2 = 12\cdot 10^{-8}\left(\frac{1+\frac{1}{2}\eta_1^2-\frac{1}{2}\xi_1^2}{\eta_1^2+\xi_1^2}\right)^{2/3},$$

where

$$\xi_1 = \frac{\pi\xi}{a_0C_0};\ \eta_1 = \frac{\pi\eta}{a_0C_0};$$

We now suppose

$F_x^0 = d_1; F_y^0 = d_2, F_z^0 = d_3; P_0(x,y) = P_0;$

$d_1; d_2, d_3; P_0$ are the constants, then from (40), (41), (42) we define the velocity components and the velocity modulus $\left|\vec{V}\right|$ for the pipe with the hexagonal cross-section

$$V_{x0} \approx 10^{-5}d_1 \int_D \ln\left|\frac{(\xi_1-\xi_0)^2+(\eta_1+\eta_0)^2}{(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2}\right|\left(\frac{1+\frac{1}{2}\eta_1^2-\frac{1}{2}\xi_1^2}{\eta_1^2+\xi_1^2}\right)^{2/3}d\xi d\eta$$

$$,V_{y0} = 10^{-5}d_2 \int_D \ln\left|\frac{(\xi_1-\xi_0)^2+(\eta_1+\eta_0)^2}{(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2}\right|\left(\frac{1+\frac{1}{2}\eta_1^2-\frac{1}{2}\xi_1^2}{\eta_1^2+\xi_1^2}\right)^{2/3}d\xi d\eta$$

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$$V_{z0}=10^{-5}(\frac{a}{\rho}P_0+d_3)$$

$$\int_D \ln\left|\frac{(\xi_1-\xi_0)^2+(\eta_1+\eta_0)^2}{(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2}\right|\left(\frac{1+\frac{1}{2}\eta_1^2-\frac{1}{2}\xi_1^2}{\eta_1^2+\xi_1^2}\right)^{2/3} d\xi d\eta$$

$$\left|\vec{V}\right|=10^{-5}\exp(-\alpha t)$$

$$\sqrt{(l-z)^2(d_1)^2+(l-z)^2(d_2)^2+\left(\frac{a}{\rho}P_0+d_3\right)^2} \quad (45)$$

$$\int_D \ln\left|\frac{(\xi_1-\xi_0)^2+(\eta_1+\eta_0)^2}{(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2}\right|\left(\frac{1+\frac{1}{2}\eta_1^2-\frac{1}{2}\xi_1^2}{\eta_1^2+\xi_1^2}\right)^{2/3} d\xi d\eta.$$

V. Conclusion

Hence, we conclude: If the pressure is represented by the formula (8) and satisfies the equation (17) and $\frac{\alpha}{2\pi\nu}$ is not the eigenvalue of the system of integral equations (33), (34), (35) ,then there exists the unique solution of the Stokes system (1),(2), (3), (4) for the velocity components of the incompressible fluid flow in a pipe with the polygonal cross-section and this solution is given in the explicit form by the formulas (22), (23), (24), (25).

In this paper we obtained new type of solutions for the Stokes system in a prismatic pipe with no axial symmetry.

In future, we plan to obtain a numerical approximation for the formula (45).

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References

[1] G.K. Batchelor, *An Introduction to Fluid Dynamics.*Cambridge, Univ. Press, 1967.
[2] J. Boussinesq,,"Mémoire sur l'influence des Frottements dans les des Fluids". *J. Math. Pures Appl.* ***13*** *(2):* 1868, 377–424.
[3] A. Bitsadze, *Some Classes of Partial Differential Equations.* New York: Gordon and Breach Science Publishers, 1988.
[4] T.Chacon-Rebollo, R. Lewandowski , *Mathematical and numerical foundations of turbulence models and applications, Modeling and Simulation in Science, Engineering and Technology,* Springer, New-York, 2014.
[5] A. Chwang, T. Wu,"Hydromechanics of low-Reynolds-number flow", *J. Fluid Mech. 62,* 1974.
[6] P.G.Drazin, N.Riley,*The Navier-Stokes equations: a classification of flows and exact solutions,* 334, Cambridge University Press, 2006.
[7] Y.Giga , "Bounded H-calculus for the hydrostatic Stokes operator on L_p spaces and applications", *Proc.Am.Math. Soc.*N.145, 2017, 3865-3876.
[8] A. Hurvitz und R. Courant,*Vorlesungen Uber Allgemeine Functionentheorie*, Springer, Berlin, 1929.
[9] Janke-Ende-Losch, *Fafeenhonererfunctionen*. Stuttgart, 1960.
[10] N.Khatiashvili,"On the Conformal Mapping Method for the Helmholtz Equation*," Integral Methods in Science and Engineering*, (C. Constanda, Ed.), New-York, Birkhauser,Vol. 1, 2010, pp. 173-177.
[11] N.Khatiashvili, K.Pirumova, D. Janjgava, „On the Stokes flow over ellipsoidal type bodies", in *Lecture Notes in Engineering and Computer Science: World Congress on Engineering 2013*, 7-9, London, UK 2013, pp.148-151.
[12] N.Khatiashvili,„On the Hexagonal Quantum Billiard", *Sem. I. Vekua Inst. Appl. Math., REPORTS*, 2014, Vol.40, pp.14-27.
[13] N.Khatiashvili, K. Pirumova, I.Khatiashvili, V. Akhobadze ," On the Influence of the Cancer Proteins on the Blood Flow", *Rep. Enlarged Sess. Sem. I. Vekua Inst. Appl. Math.*, Tbilisi, 2015, Vol.29, pp. 60-63.
[14] N.Khatiashvili ," On the energy levels of electrons in nanostructures", in *Lecture Notes in Engineering and Computer Science*: *World Congress on Engineering 2018*, 7-9 July, London, UK, 2018, pp.14-19.
[15] N.Khatiashvili, "On the Free Boundary Problem for the crapping Flows", in *Lecture Notes in Engineering and Computer Science*: *World Congress on Engineering 2021*, 7-9 July, London, UK, 2021, pp.29-35.
[16] N.Khatiashvili , „On 2D Free Boundary Problem for the Stokes Flow", *Proc. I.Vekua Inst. Appl. Math*., Vol. 70, 2020 , pp. 61-71.
[17] N.Khatiashvili , "On the Non-Smooth Solutions of 3D Navier-Stokes Equations for the Incompressible Fluid Flows", *International Journal of Physics,* 9 (3), 2021, 178-185.
[18] S. Kim, S. Karrila ,*Microhydrodynamics: Principles and Selected Applications*. Dover, 2005.
[19] W.Koppenfels, F.Stallmann, *Praxis der Konformen Abbildung*, Springer, Berlin,1959.
[20] H. Lamb, *Hydrodynamics, (Sixth ed.),* NY, Dover,1945.
[21] L. D. Landay, E. M. Lifshitz, *Fluid Mechanics,Course of Theoretical Physics*. 6, Pergamon Press, 1987.
[22] B. Lautrup, *Physics of Continuous Matter, Second Edition,* CRC Press, 2011.
[23] M. A. Lavrentiev, B. V. Shabat, *Methods of the theory of functions in a complex variable,* Moskow: Nauka, 1987.
[24] P.L.Lions,*Mathematical Topics in Fluid Mechanicsnics,*Vol.1, Oxford University Press, NY, 1996.
[25] A.J. Majda, A.L. Bertozzi, *Vorticity and Incompressible Flow,*Vol.27, Cambridge University Press, 2002.
[26] H. Ockendon , J. Ockendon., *Viscous Flow*, Cambridge Univ. Press, 1995.
[27] L.Prandtl, *Essentials of Fluid Mechanics, third ed.,* Vol. 158, Springer, NY 2010.
[28] P.W. Schroeder, C. Lehrenfeld , A. Linke , G. Lube, Towards computable flows and robust estimates for inf-sup stable FEM applied to the time-dependent incompressible NSE, *SEMA J.,75.*,2018., pp. 629-653.
[29] R.Temam, *Navier-Stokes Equations, Theory and numerical Analysis,* AMS Chelsea, 2001.
[30] S.Tsangaris, D. Kondaxakis, N. Vlachakis, "Exact solution for flow in a porous pipe with unsteady wall suction and/or injection", *Commun. Nonlinear Sci. Numer. Simul.,* **12**. 2007, 1181–1189.

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