Hidden Population Size Estimation under the Zero Truncated Poisson Shanker Model

Ratchaneewan Wongprachan *

Abstract—**Modeling the size of unknown populations has been discussed on the basic capture-recapture method. As a result of heterogeneity of capture probability, a mixture of Poisson distributions has been considered for developing the population size estimator. In this study, I have proposed the Horvitz Thompson estimator to model the hidden population size based on the zero truncated Poisson Shanker model. Simulated data were constructed under various parameters for overdispersed data. The efficiency of the new estimator is measured and compared with alternative estimators. The behavior of the estimator is presented using the Relative Root Mean Square Error, the Relative Bias and the Relative Variance. The simulation study shows that the new estimator performs well with the estimated value closed to the real population size.**

Keywords: Population size estimation, Zero truncated Poisson Shanker, Mixed Poisson, Horvitz-Thompson, Capture-Recapture method

1 Introduction

Population size estimation under capture-recapture method has been used to model the hidden data; for example, to model the number of animal population, the number of unseen species, the number of hidden drug users, the number of drink-driving offenders, the number of illegal immigrants etc. The commonly used distribution of count datas; for example, the Poisson distribution has been proposed to model the total population size under the same capture probability. This model is appropriate for equidispersed count data, equality of the mean and the variance. However, there is the heterogeneity of capture probability in practice. Therefore, a mixture of Poisson distribution is used as a flexible model for heterogeneous data which includes overdispersed and underdispersed data.

Suppose there are n units captured independently with replacement from population size N. Let X be a random variable which follows the Poisson model and its parameter λ denotes a random variable for the mixing distribution. Let f_x be the frequency of units captured x times, for $x = 0, 1, 2, \ldots$. For example, f_0 is unobserved frequency, f_1 is the frequency of captured once, f_2 is the frequency of captured twice and so on. Therefore, the number of units captured in the sample is defined by $n = \sum_{x>0} f_x$, the number of observation in capture-recapture data is defined by $K = \sum_{x>0} x f_x$, and $f_0 = N - n$. In this study, I have investigated the mixed Poisson distribution based on one parameter called the Poisson Shanker distribution. A new estimator using the Horvitz Thompson approach has been proposed based on the zero truncated Poisson Shanker model. The behavior of the new estimator is addressed in a simulation study which shows its efficiency compared to alternative estimators.

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2 Zero truncated Poisson Shanker model

Poisson Shanker distribution is an alternative model of the Poisson mixture model and uses the Shanker distribution for the mixing parameter [7]. Assume that X is a random variable with the Poisson distribution

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \tag{1}$$

where $x = 0, 1, 2, \ldots$ and $\lambda > 0$. When λ is a random variable following the Shanker distribution with the probability density function

$$g(\lambda) = \frac{\alpha^2}{\alpha^2+1}\,(\alpha+\lambda)\,e^{-\alpha\lambda}, \tag{2}$$

where $\lambda > 0$ and $\alpha > 0$, the random variable X is distributed as the Poisson Shanker distribution

$$p_x = \frac{\alpha^2}{\alpha^2+1}\left[\frac{x+(\alpha^2+\alpha+1)}{(\alpha+1)^{x+2}}\right], \tag{3}$$

where $x = 0, 1, 2, \ldots$ and $\alpha > 0$. In capture-recapture study, p_x is the probability of a unit captured x times. As a result of unknown the zero counts, f_0, the zero truncated distribution is considered in order to develop the estimator. The probability mass function of the zero truncated Poisson Shanker (ZTPS) is given by

$$\begin{aligned} p_x^+ &= \frac{p_x}{1-p_0} \\ &= \frac{\dfrac{\alpha^2}{\alpha^2+1}\left[\dfrac{x+(\alpha^2+\alpha+1)}{(\alpha+1)^{x+2}}\right]}{1-\dfrac{\alpha^2(\alpha^2+\alpha+1)}{(\alpha^2+1)(\alpha+1)^2}} \qquad (4) \\ &= \frac{\alpha^2(\alpha^2+\alpha+x+1)}{(\alpha+1)^x(\alpha^3+\alpha^2+2\alpha+1)}, \end{aligned}$$

where $x = 1, 2, \ldots, \alpha > 0$ and p_0 is defined as the undetected probability. The mean, variance and dispersion of a random variable X distributed as the ZTPS are given by

$$E(X) = \frac{(\alpha+1)^2(\alpha^2+2)}{\alpha(\alpha^3+\alpha^2+2\alpha+1)}, \qquad (5)$$

$$Var(X) = \frac{(\alpha+1)^2(\alpha^5+\alpha^4+5\alpha^3+4\alpha^2+6\alpha+2)}{\alpha^2(\alpha^3+\alpha^2+2\alpha+1)^2}, \qquad (6)$$

and

$$D(X) = \frac{\alpha^5+\alpha^4+5\alpha^3+4\alpha^2+6\alpha+2}{\alpha(\alpha^2+2)(\alpha^3+\alpha^2+2\alpha+1)}. \qquad (7)$$

Dispersion of the ZTPS model represents overdispersed count data or the variance greater than the mean when $\alpha < 1.24166$. Additionally, it can be described as underdispersed or the variance less than the mean when $\alpha > 1.24166$, and equidispersed or the variance equal to the mean when $\alpha = 1.24166$, respectively. Figure 1 shows the probability mass function of the ZTPS distribution under various parameters. It seems that the distribution of X exhibits long right tail when α is larger.

3 Estimator

3.1 Horvitz-Thompson estimator based on the ZTPS distribution

Horvitz-Thompson estimator has been used for fitting the zero-truncated model in this study. Assume that the probability of each unit caught on any occasion can be defined by $1 - p_0$ and there are n units in the sample. Then, the total target population estimated by the Horvitz-Thompson approach is defined by

$$\widehat{N} = \frac{n}{1-p_0},$$

Under the ZTPS distribution, the Horvitz-Thompson estimator can be written as

$$\begin{aligned} \widehat{N}_{ps} &= \frac{n}{1-\dfrac{\alpha^2(\alpha^2+\alpha+1)}{(\alpha^2+1)(\alpha+1)^2}} \qquad (8) \\ &= \frac{n(\alpha+1)^2(\alpha^2+1)}{\alpha^3+\alpha^2+2\alpha+1}. \end{aligned}$$

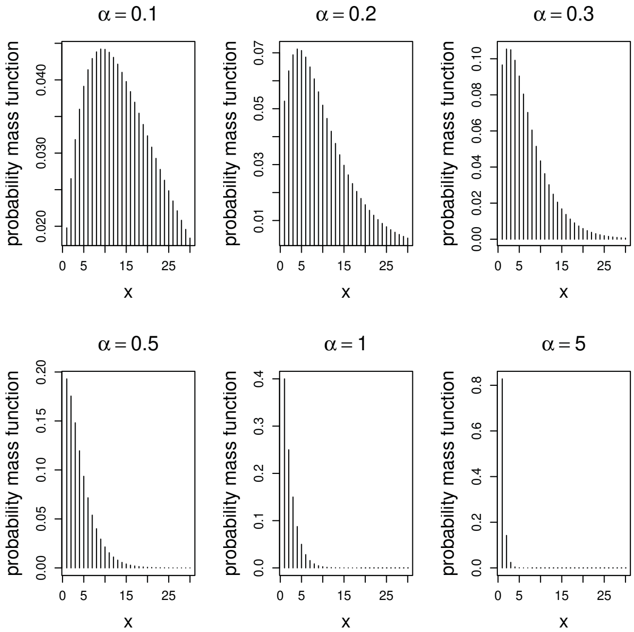

Figure 1: Probability mass function of the ZTPS distribution under $\alpha = 0.1, 0.2, 0.3, 0.5, 1$ and 5

In practice, the unknown parameter α is replacing by $\widehat{\alpha}$ using the Maximum Likelihood Estimation method (MLE) in equation (8) for $\widehat{N}_{ps}$. The likelihood function of the ZTPS distribution is defined by

$$L(\alpha) = \prod_{x=1}^{m}\left[\frac{\alpha^2(\alpha^2+\alpha+x+1)}{(\alpha+1)^x(\alpha^3+\alpha^2+2\alpha+1)}\right]^{f_x} \qquad (9)$$

and the log-likelihood function can be written as

$$\begin{aligned} \ell(\alpha) &= \sum_{x=1}^{m}\ln\left[\frac{\alpha^2(\alpha^2+\alpha+x+1)}{(\alpha+1)^x(\alpha^3+\alpha^2+2\alpha+1)}\right]^{f_x} \\ &= 2n\ln(\alpha) + \sum_{x=1}^{m} f_x \ln(\alpha^2+\alpha+x+1) - K\ln(\alpha+1) \\ &\quad -n\ln(\alpha^3+\alpha^2+2\alpha+1). \end{aligned} \qquad (10)$$

where $K = \sum_{x=1}^{m} x f_x$, $n = \sum_{x=1}^{m} f_x$, and m is a maximum number of times of capture. The estimated parameter α can be found by solving the first derivative of the log-likelihood function in equation (11) and setting it equal to zero.

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{2n}{\alpha} + (2\alpha+1)\sum_{x=1}^{m}\frac{f_x}{\alpha^2+\alpha+x+1} - \frac{K}{\alpha+1} \\ &\quad -\frac{n(3\alpha^2+2\alpha+2)}{\alpha^3+\alpha^2+2\alpha+1}. \end{aligned} \qquad (11)$$

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3.2 Alternative estimators

3.2.1 Chao estimator

The Chao estimator is a well-known nonparametric method for modeling the size of unknown population in capture-recapture analysis [4], proposed as a lower bound for N refer to [3]; [2]; [5]; [8]. Terms of the frequency of unit caught once and twice are used for the Chao estimator as follow

$$\widehat{N}_c = n + \frac{f_1^2}{2f_2}. \quad (12)$$

When f_2 equals zero, this estimator is modified to

$$\widehat{N}_c = n + \frac{f_1(f_1-1)}{2(f_2+1)}. \quad (13)$$

3.2.2 Estimator under the zero truncated Poisson model

Van der Heijen, Cruyff and Houwelingen introduced the Horvitz-Thompson approach to develop the estimator under the zero truncated Poisson distribution (ZTPoi), which is defined by

$$\widehat{N}_p = \frac{n}{1-e^{-\lambda}}, \quad (14)$$

where λ is approximated using MLE, refer to [6].

4 Confidence intervals estimation

Not only point estimation, but also confidence interval estimation has been addressed in this study. The (1-α)100% confidence interval for N_{ps} based on the ZTPS model is constructed by

$$\widehat{N}_{ps} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\widehat{N}_{ps})}. \quad (15)$$

Variance estimation by Böhning [1] has been used for estimating the variance of the new estimator in equation (8). A simple variance approximation by the conditional technique is used to construct the variance of N as follows

$$Var(\widehat{N}_{ps}) = E\left[Var\left(\widehat{N}_{ps}|n\right)\right] + Var\left(E\left[\widehat{N}_{ps}|n\right]\right). \quad (16)$$

4.1 Estimation of $E\left[Var\left(\widehat{N}_{ps}|n\right)\right]$

This part arises from estimating the parameter α of the ZTPS distribution from the sample unit of size n. Assume that the term of $E\left[Var\left(\widehat{N}_{ps}|n\right)\right]$ can be approximated by $Var\left(\widehat{N}_{ps}|n\right)$ using the δ method and $h(\alpha) = \frac{1}{1-p_0}$. Then,

$$\widehat{V}ar\left(\widehat{N}_{ps}|n\right) = n^2\left(\frac{\partial}{\partial\alpha}\frac{1}{h(\alpha)}\right)^T Var(\alpha)\left(\frac{\partial}{\partial\alpha}\frac{1}{h(\alpha)}\right), \quad (17)$$

where

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{1}{h(\alpha)} &= \frac{\partial}{\partial\alpha}\frac{1}{1-p_0} \\ &= \frac{\partial}{\partial\alpha}\frac{(\alpha+1)^2(\alpha^2+1)}{(\alpha^3+\alpha^2+2\alpha+1)} \\ &= \frac{\alpha(\alpha+1)(\alpha^4+\alpha^3+5\alpha^2+3\alpha+2)}{(\alpha^3+\alpha^2+2\alpha+1)^2}.\end{aligned} \quad (18)$$

Considering the Fisher information to approximate the variance of its parameter, $\widehat{V}ar(\alpha)$ can be derived as

$$\begin{aligned}\widehat{V}ar(\alpha) &= -\left(\frac{\partial^2\ell}{\partial\alpha^2}\right)^{-1} \\ &= \left(\frac{2n}{\alpha^2} - A + (2\alpha+1)^2B - \frac{K}{(\alpha+1)^2} - C\right)^{-1}\end{aligned} \quad (19)$$

where

$$A = \sum_{x=1}^{m}\frac{2f_x}{(\alpha^2+\alpha+x+1)},$$

$$B = \sum_{x=1}^{m}\frac{f_x}{(\alpha^2+\alpha+x+1)^2},$$

and

$$C = \frac{n(3\alpha^4+4\alpha^3+2\alpha^2-2\alpha+2)}{(\alpha^3+\alpha^2+2\alpha+1)^2}.$$

Replacing equation (18) and (19) in equation (17), the term of $\widehat{V}ar(\widehat{N}_{ps}|n)$ can be written as

$$\widehat{V}ar(\widehat{N}_{ps}|n) = \left[\frac{n\alpha(\alpha+1)(\alpha^4+\alpha^3+5\alpha^2+3\alpha+2)}{(\alpha^3+\alpha^2+2\alpha+1)^2}\right]^2\widehat{V}ar(\alpha). \quad (20)$$

4.2 Estimation of $Var\left(E\left[\widehat{N}_{ps}|n\right]\right)$

This term arises from the binomial random variable for n by sampling from the unknown population size N with probability of success $1-p_0$. It provides $Var(n) = N(1-p_0)p_0$, where $N(1-p_0)$ can be replaced by n. Assume that $E\left[\widehat{N}_{ps}|n\right]$ can be approximated by $\widehat{N}_{ps}$, then

$$\begin{aligned}Var\left(E\left[\widehat{N}_{ps}|n\right]\right) &\approx Var\left(\frac{n}{1-p_0}\right) \\ &= \frac{np_0}{(1-p_0)^2} \\ &= \frac{n\left[\frac{\alpha^2(\alpha^2+\alpha+1)}{(\alpha^2+1)(\alpha+1)^2}\right]}{\left[1-\frac{\alpha^2(\alpha^2+\alpha+1)}{(\alpha^2+1)(\alpha+1)^2}\right]^2} \\ &= \frac{n\alpha^2(\alpha^2+\alpha+1)(\alpha+1)^2(\alpha^2+1)}{(\alpha^3+\alpha^2+2\alpha+1)^2}\end{aligned} \quad (21)$$

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Replacing equation (20) and (21) in equation (16), the variance estimation of $\widehat{N}_{ps}$ can be written as

$$Var(\widehat{N}_{ps}) = \left[\frac{n\alpha(\alpha+1)(\alpha^4+\alpha^3+5\alpha^2+3\alpha+2)}{(\alpha^3+\alpha^2+2\alpha+1)^2}\right]^2 \widehat{V}ar(\alpha) + \frac{n\alpha^2(\alpha^2+\alpha+1)(\alpha+1)^2(\alpha^2+1)}{(\alpha^3+\alpha^2+2\alpha+1)^2}. \quad (22)$$

Similarly, this approach has been used for $\widehat{N}_c$ and $\widehat{N}_p$ shown in equations (23); refer to [1] and [6]

$$Var(\widehat{N}_c) = \frac{1}{4}\frac{f_1^4}{f_2^3} + \frac{f_1^3}{f_2^2} + \frac{1}{2}\frac{f_1^2}{f_2} - \frac{1}{4}\frac{f_1^4}{nf_2^2} - \frac{1}{2}\frac{f_1^4}{f_2(2nf_2+f_1^2)}, \quad (23)$$

and

$$Var(\widehat{N}_p) = \left[\frac{ne^{-\lambda}}{(1-e^{-\lambda})^2}\right]^2 \widehat{V}ar(\lambda) + \frac{ne^{-\lambda}}{(1-e^{-\lambda})^2}, \quad (24)$$

where $\widehat{V}ar(\lambda) = \left[\frac{n}{\lambda(1-e^{-\lambda})} - \frac{ne^{-\lambda}}{(1-e^{-\lambda})^2}\right]^{-1}$.

5 Simulation Results

A simulation study has been carried out to evaluate the performance of Horvitz-Thompson estimator under the ZTPS model and the results have been compared to the ZTPoi model and the Chao estimator. Here, heterogeneous data have been simulated using the negative binomial distribution (NB) under the condition of population size $N = 250, 500, 1000, 2000$ with the parameter $\mu = 1, 3, 5$ and $s = 1, 1.25, 1.5, 2$

$$p_x = \frac{\Gamma(x+s)}{\Gamma(x+1)\Gamma(s)}\left(\frac{s}{s+\mu}\right)^s\left(\frac{\mu}{s+\mu}\right)^x, x = 0, 1, 2, \ldots, s.$$

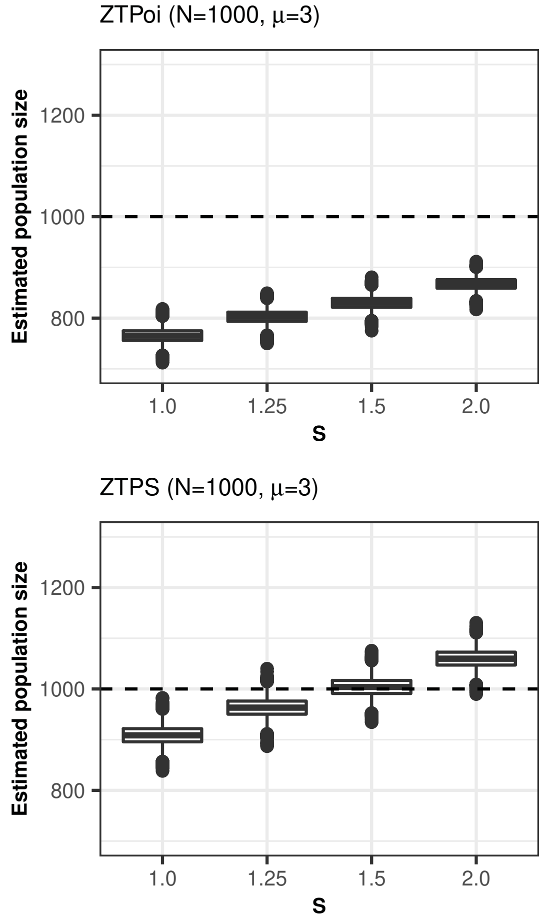

Figure 2: Boxplot of the estimated Population size $N = 1000$ based on the ZTPoi and ZTPS distribution for data from NB(μ, s)

All estimators have been fitted 10000 times to approximate point and interval estimation of N. Figure 2 shows the estimated population size using the ZTPoi and ZTPS estimators for the count data from the negative binomial with $\mu = 3$ and $s = 1, 1.25, 1.5, 2$. It is found that the ZTPS esitmator gives the approximation closed to $N = 1000$ with smaller bias. In this simulation study, the performance of estimators has been evaluated by Relative Root Mean Square Error (RRMSE), Relative Bias (RBias) and Relative Variance (RVar) as follows:

$$RRMSE(\widehat{N}) = \frac{1}{N}\sqrt{E(\widehat{N}-N)^2} \quad (25)$$

$$RBias(\widehat{N}) = \frac{1}{N}[E(\widehat{N}) - N] \quad (26)$$

$$RVar(\widehat{N}) = \frac{1}{N^2}[E(\widehat{N} - E(\widehat{N}))^2] \quad (27)$$

Table 1 represents RBias, RVar and RRMSE to measure the performance of the estimators of N. Referring to the performance in terms of RRMSE, it is found that the estimator under the ZTPoi model provides the highest RRMSE for all situations, while the estimator under ZTPS model outperforms with the smallest RRMSE under almost all conditions. As shown in Figure 3, the ZTPS estimator provides good result especially when $\mu > 1$. In addition to that, the lower bound estimator known as the Chao estimator is comparable to the ZTPS estimator when $s \geq 2$, whereas the ZTPoi estimator results in less accuracy compared to others.

According to the behavior of estimators based on the RBias criterion, it is found that the negative RBias for ZTPoi and Chao estimators, especially ZTPoi gives the worst RBias. The performance of ZTPS provides better approximation with RBias close to zero.

Considering the RVar criterion, it is shown that ZTPoi provides the smallest RVar, whereas ZTPS produces bigger RVar than ZTPoi for all situations. However, the performance of ZTPS can be improved for large μ. Although, the ZTPoi performs the best in terms of RVar, the estimated population size is an underestimate under

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all conditions.

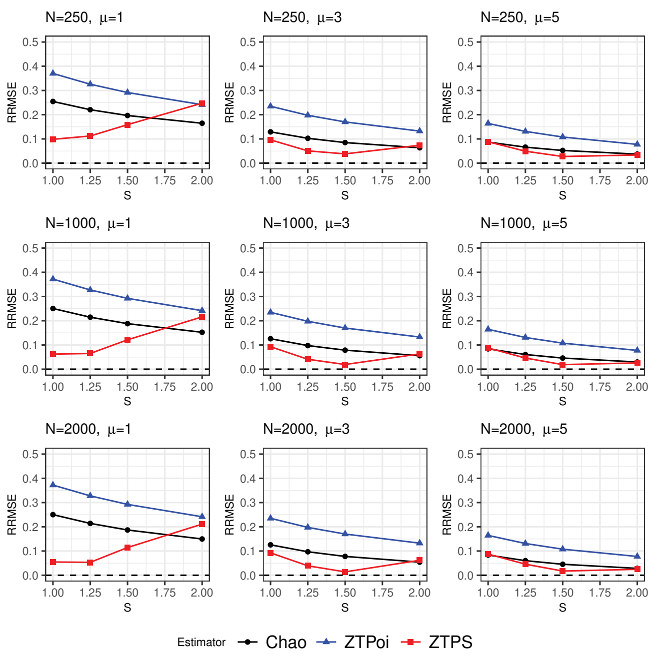

Figure 3: RRMSE of population size estimators for count data from NB(μ, s)

6 Conclusions and Future Work

The problem of estimating population size has been investigated widely in many areas including ecology, social science, medicine etc. The unknown zero frequency is an important issue for modeling. Both parametric and nonparametric statistics are used to develop the estimator for the number of hidden population. Poisson Shanker is an alternative to the mixed Poisson distribution with one parameter showing overdispersion, underdispersion and equidispersion. In this article, Horvitz Thompson estimator has been improved based on the zero truncated Poison Shanker model for modeling the population size. The ZTPS estimator outperforms alternative estimators including Chao and ZTP especially for large μ. ZTP performs poorly for heterogeneous Poisson population with the underestimated population size. In this study, I have used a one parameter model. On the other hand, distributions with more parameters would develop the efficiency of estimator. Fitting more complex models will be studied in the future.

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Table 1: RBias, RVar and RRMSE of population size estimators for count data from NB(μ, s)

			RBias			RVar			RRMSE		
N	μ	s	Chao	ZTPoi	ZTPS	Chao	ZTPoi	ZTPS	Chao	ZTPoi	ZTPS
250	1	1	-0.2398	-0.3688	-0.0382	1.8348	0.2895	2.0614	0.2546	0.3703	0.0985
		1.25	-0.2021	-0.3235	0.0481	1.9539	0.3725	2.5608	0.2206	0.3258	0.1121
		1.5	-0.1750	-0.2885	0.1152	2.0129	0.4457	2.9913	0.1966	0.2916	0.1589
		2	-0.1364	-0.2366	0.2155	2.1271	0.5722	3.7161	0.1647	0.2414	0.2476
	3	1	-0.1191	-0.2347	-0.0903	0.6248	0.0177	0.2760	0.1291	0.2348	0.0962
		1.25	-0.0910	-0.1973	-0.0357	0.5629	0.0244	0.3237	0.1026	0.1975	0.0507
		1.5	-0.0719	-0.1699	0.0047	0.5127	0.0264	0.3613	0.0850	0.1702	0.0383
		2	-0.0478	-0.1318	0.0616	0.4412	0.0327	0.4190	0.0636	0.1323	0.0739
	5	1	-0.0771	-0.1640	-0.0858	0.4042	0.0023	0.1059	0.0870	0.1640	0.0882
		1.25	-0.0551	-0.1309	-0.0443	0.3299	0.0030	0.1199	0.0660	0.1310	0.0494
		1.5	-0.0403	-0.1078	-0.0151	0.2812	0.0036	0.1303	0.0524	0.1079	0.0274
		2	-0.0235	-0.0773	0.0237	0.2094	0.0044	0.1448	0.0373	0.0775	0.0338
500	1	1	-0.2441	-0.3704	-0.0415	1.6869	0.2798	2.0106	0.2509	0.3712	0.0758
		1.25	-0.2079	-0.3259	0.0433	1.7792	0.3588	2.4893	0.2163	0.3270	0.0828
		1.5	-0.1812	-0.2908	0.1104	1.8414	0.4296	2.9081	0.1911	0.2922	0.1342
		2	-0.1419	-0.2386	0.2114	1.9604	0.5542	3.6263	0.1551	0.2409	0.2279
	3	1	-0.1219	-0.2342	-0.0904	0.5709	0.0172	0.2727	0.1265	0.2343	0.0934
		1.25	-0.0927	-0.1970	-0.0357	0.5251	0.0220	0.3211	0.0982	0.1971	0.0438
		1.5	-0.0743	-0.1698	0.0043	0.4775	0.0259	0.3583	0.0805	0.1700	0.0271
		2	-0.0512	-0.1321	0.0606	0.4064	0.0322	0.4154	0.0586	0.1324	0.0671
	5	1	-0.0813	-0.1645	-0.0868	0.3562	0.0022	0.1047	0.0856	0.1645	0.0880
		1.25	-0.0570	-0.1311	-0.0447	0.3010	0.0029	0.1191	0.0621	0.1311	0.0473
		1.5	-0.0421	-0.1074	-0.0149	0.2541	0.0035	0.1294	0.0478	0.1074	0.0220
		2	-0.0254	-0.0775	0.0231	0.1903	0.0043	0.1437	0.0320	0.0776	0.0287
1000	1	1	-0.2473	-0.3714	-0.0435	1.6086	0.2748	1.9837	0.2505	0.3718	0.0623
		1.25	-0.2107	-0.3265	0.0419	1.7023	0.3534	2.4615	0.2147	0.3271	0.0650
		1.5	-0.1832	-0.2917	0.1088	1.7808	0.4241	2.8805	0.1880	0.2924	0.1213
		2	-0.1462	-0.2404	0.2076	1.8716	0.5424	3.5652	0.1524	0.2416	0.2160
	3	1	-0.1236	-0.2347	-0.0912	0.5513	0.0170	0.2710	0.1258	0.2347	0.0927
		1.25	-0.0949	-0.1976	-0.0369	0.5032	0.0217	0.3185	0.0975	0.1976	0.0410
		1.5	-0.0756	-0.1699	0.0042	0.4606	0.0258	0.3574	0.0786	0.1699	0.0194
		2	-0.0522	-0.1327	0.0597	0.3971	0.0319	0.4133	0.0558	0.1328	0.0631
	5	1	-0.0819	-0.1645	-0.0870	0.3442	0.0022	0.1042	0.0840	0.1645	0.0876
		1.25	-0.0582	-0.1309	-0.0448	0.2861	0.0029	0.1184	0.0606	0.1309	0.0461
		1.5	-0.0431	-0.1075	-0.0152	0.2426	0.0034	0.1290	0.0459	0.1075	0.0189
		2	-0.0264	-0.0775	0.0230	0.1811	0.0042	0.1433	0.0297	0.0776	0.0259
2000	1	1	-0.2487	-0.3720	-0.0448	1.5744	0.2719	1.9675	0.2502	0.3722	0.0547
		1.25	-0.2120	-0.3274	0.0403	1.6726	0.3500	2.4432	0.2140	0.3277	0.0533
		1.5	-0.1844	-0.2921	0.1079	1.7451	0.4203	2.8613	0.1868	0.2924	0.1144
		2	-0.1466	-0.2409	0.2068	1.8486	0.5389	3.5478	0.1497	0.2414	0.2110
	3	1	-0.1245	-0.2349	-0.0917	0.5404	0.0169	0.2703	0.1256	0.2350	0.0924
		1.25	-0.0956	-0.1975	-0.0369	0.4933	0.0216	0.3178	0.0969	0.1975	0.0390
		1.5	-0.0764	-0.1700	0.0038	0.4522	0.0256	0.3562	0.0778	0.1701	0.0139
		2	-0.0524	-0.1325	0.0600	0.3914	0.0319	0.4135	0.0542	0.1325	0.0617
	5	1	-0.0826	-0.1645	-0.0871	0.3358	0.0022	0.1041	0.0836	0.1645	0.0874
		1.25	-0.0589	-0.1310	-0.0449	0.2790	0.0028	0.1182	0.0601	0.1310	0.0456
		1.5	-0.0440	-0.1076	-0.0153	0.2357	0.0034	0.1288	0.0453	0.1076	0.0173
		2	-0.0267	-0.0775	0.0230	0.1776	0.0042	0.1432	0.0283	0.0775	0.0245