

The calculation of axisymmetric duct geometries for incompressible rotational flow with blockage effects.

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Abstract

In this paper a numerical algorithm is described for solving the boundary value problem associated with axisymmetric, inviscid, incompressible and irrotational flow with a circumferentially arranged cascade of aerofoils placed in the duct. The governing equations are formulated in terms of the stream function $\psi(x,y)$ and the function $\phi(x,y)$ as independent variables where for irrotational flow $\phi(x,y)$ can be recognized as the velocity potential function, for rotational flow $\phi(x,y)$ ceases being the velocity potential function but does remain orthogonal to the stream lines. A numerical method based on finite differences solving a Poisson type equation on a uniform mesh is employed. The technique described is capable of tackling the so-called inverse problem where the velocity wall distributions are prescribed from which the duct wall shape is calculated, as well as the direct problem where the velocity distribution on the duct walls are calculated from prescribed duct wall shapes. Results for the case of prescribing the radius i.e. the so called Dirichlet boundary conditions are given. A downstream condition is prescribed such that cylindrical flow, that is flow which is independent of the axial coordinate, exists. The numerical results are obtained by using Green's function for the Laplace equation on a rectangle. The presence of the blades has a bearing on the rate of mass flow and thus alters the usual equation of continuity

1. Introduction

Designers of ducts require numerical techniques for calculating wall shapes from a prescribed velocity distribution. The objective of the prescribed velocity is typically to avoid boundary layer separation. At inlet a velocity is prescribed to allow for a vorticity to be present calculated from $\underline{\omega} = \underline{\nabla} \wedge \underline{v}$ where the \wedge denotes the usual

cross product of vectors, $\underline{\omega}$ is the vorticity vector and \underline{v} the velocity vector respectively.

The objective of the present paper is to describe a numerical algorithm for solving the boundary value problem that arises when the independent variables are ϕ and ψ , where ϕ may be identified as the velocity potential function (for irrotational flow only), for flow with vorticity, ϕ ceases being the velocity potential function but does remain orthogonal to ψ which may be identified as the stream function. The dependent variable y , is the radial coordinate and x is the axial coordinate. The numerical technique is based on the finite difference scheme on a uniform mesh.

2. The Design Plane

As shown in Pavlika [4] when the independent variables are $\phi(x,y)$ and $\psi(x,y)$ where the $\phi(x,y)$ and $\psi(x,y)$ have been previously defined it can be shown that the governing partial differential equation that the radius satisfies is given by:

$$\frac{\partial}{\partial \phi} \left(\frac{A}{B} \frac{\partial y}{\partial \phi} \right) + \frac{\partial}{\partial \psi} \left(\frac{B}{A} \frac{\partial y}{\partial \psi} \right) = 0 \quad (1)$$

with the speed calculated from

$$\frac{1}{q^2} = \frac{1}{A^2} \left(\frac{\partial y}{\partial \psi} \right)^2 + \frac{1}{B^2} \left(\frac{\partial y}{\partial \phi} \right)^2 \quad (2)$$

and completion of the physical coordinates are provided from

$$dx = \frac{B}{A} \frac{\partial y}{\partial \psi} d\phi - \frac{A}{B} \frac{\partial y}{\partial \phi} d\psi$$

where x is the axial coordinate and A and B satisfy their own first order quasi-linear hyperbolic partial differential equations with characteristics parallel to the ϕ and ψ axes which maps the

physical flow field into an infinite strip in the (ϕ, ψ) plane. In fact the A and B satisfy:

$$\frac{\partial}{\partial \phi}(\log(A)) = \frac{\eta}{q^2} B \quad (3)$$

and

$$\frac{\partial}{\partial \psi}(\log(B)) = -\frac{\omega_\alpha}{q^2} A \quad (4)$$

Regarding temporarily η , ω_α and q as known functions of ϕ and ψ the system (3) and (4) as previously mentioned is quasi-linear hyperbolic with characteristics parallel to the ϕ and ψ axes which maps the physical flow field into an infinite strip in the (ϕ, ψ) plane. Bearing in mind the freedom available in the stream wise variation of ϕ and the cross stream variation of ψ , suitable values of A can be prescribed along one ϕ characteristic and those of B can be prescribed along one ψ characteristic.

3. The Numerical Algorithm in the Design Plane

Rewriting the partial differential equation that y satisfies i.e. equation (1) as:

$$\frac{\partial}{\partial \phi} \left(C \frac{\partial y}{\partial \phi} \right) + \frac{\partial}{\partial \psi} \left(\frac{1}{C} \frac{\partial y}{\partial \psi} \right) = c \quad (5)$$

where $C = \frac{A}{B}$, for problems posed in the design plane $c=0$, the value of C will vary depending on whether the flow field is irrotational or swirl free etc. Equation (5) will be re-written as a Poisson equation that is as:

$$\nabla^2 y = \frac{c}{C} + \left(1 - \frac{1}{C^2}\right) \frac{\partial^2 y}{\partial \psi^2} - \left(\frac{\partial}{\partial \phi} \log g_e |C| \right) \frac{\partial y}{\partial \phi} - \frac{1}{C} \frac{\partial}{\partial \psi} \left(\frac{1}{C} \right) \frac{\partial y}{\partial \psi} \quad (6)$$

where ∇^2 is the usual Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial \psi^2} + \frac{\partial^2}{\partial \phi^2}$$

so that

$$\nabla^2 y = g \left(\frac{\partial^2 y}{\partial \psi^2}, \frac{\partial y}{\partial \phi}, \frac{\partial y}{\partial \psi}, C, c \right)$$

where g is a function of the arguments shown as defined by expression (6). Writing in finite difference form using central differences gives:

$$\underline{Y}^{(i-1)} + A \underline{Y}^{(i)} + \underline{Y}^{(i+1)} = \underline{E}^{(i)}, \quad (7)$$

where

$$\underline{Y}^{(i-1)} = \begin{bmatrix} y_{i-1,1} \\ y_{i-1,2} \\ \vdots \\ y_{i-1,N} \end{bmatrix}, \quad \underline{Y}^{(i)} = \begin{bmatrix} y_{i,1} \\ y_{i,2} \\ \vdots \\ y_{i,N} \end{bmatrix}, \quad \underline{Y}^{(i+1)} = \begin{bmatrix} y_{i+1,1} \\ y_{i+1,2} \\ \vdots \\ y_{i+1,N} \end{bmatrix}$$

$$\underline{E}^{(i)} = h^2 \begin{bmatrix} g_{i,1} - y_{i,0} \\ g_{i,2} \\ \vdots \\ g_{i,N} - y_{i,N+1} \end{bmatrix}$$

and

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 & \cdot \\ 1 & -4 & 1 & & \\ 0 & 1 & -4 & \cdot & \cdot \\ & & & \cdot & 1 \\ & & & 1 & -4 \end{bmatrix}$$

on a uniform mesh with $\Delta\phi = \Delta\psi = h$.

4. Direct Solution of the Difference Equations

The matrix-vector equation (7) is

$$\underline{Y}^{(i-1)} + A \underline{Y}^{(i)} + \underline{Y}^{(i+1)} = \underline{E}^{(i)}$$

With all of order (NxN) , and column vectors $\underline{Y}^{(i)}$ and $\underline{E}^{(i)}$ of order N . To solve the vector recurrence relation a speculation is made that the $\underline{Y}^{(i-1)}$ vector can be related linearly to the $\underline{Y}^{(i)}$ vector as follows:

$$\underline{Y}^{(i-1)} = B^{(i)} \underline{Y}^{(i)} + \underline{K}^{(i)} \quad (8)$$

where the $B^{(i)}$ and the $\underline{K}^{(i)}$ are at present unknown matrices and column vectors respectively. Substituting (8) into (7) gives

$$\begin{aligned} (W^{(i)} B^{(i)} + A) \underline{Y}^{(i)} &= \underline{E}^{(i)} - W^{(i)} \underline{K}^{(i)} - E^{(i)} \underline{Y}^{(i+1)} \\ \Rightarrow \underline{Y}^{(i)} &= - (W^{(i)} B^{(i)} + A)^{-1} E^{(i)} \underline{Y}^{(i+1)} \\ &\quad + (W^{(i)} B^{(i)} + A)^{-1} (\underline{E}^{(i)} - W^{(i)} \underline{K}^{(i)}) \end{aligned}$$

but

$$\underline{Y}^{(i)} = B^{(i+1)} \underline{Y}^{(i+1)} + \underline{K}^{(i+1)}$$

Thus equating coefficients implies

$$B^{(i+1)} = - (W^{(i)} B^{(i)} + A)^{-1} E^{(i)} \quad (9)$$

and

$$\underline{K}^{(i+1)} = (W^{(i)} B^{(i)} + A)^{-1} (\underline{E}^{(i)} - W^{(i)} \underline{K}^{(i)})$$

For $i=0$ this gives

$$\underline{Y}^{(0)} = B^{(1)} \underline{Y}^{(1)} + \underline{K}^{(1)} \quad (10)$$

To determine the $\underline{K}^{(1)}$, if the first iterate $B^{(1)} = \underline{0}$ then

$$\underline{K}^{(1)} = \underline{Y}^{(0)}$$

The matrix and vector sequences are now defined by equations (9) and (10) for $i=1$ to M . The $\underline{Y}^{(i)}$ vectors are now calculated starting from right to left (as $\underline{Y}^{(M+1)}$ is known) using

$$\underline{Y}^{(M)} = \underline{B}^{(M+1)} \underline{Y}^{(M+1)} + \underline{K}^{(M+1)}$$

5. Axisymmetric Flow in the Absence of Body Forces

Here numerical solutions to inviscid axisymmetric flow with a constant axial velocity component given by $u_x(y) = \beta$, where β is a constant will be given. The swirl velocity $u_\alpha(y)$, will be of the form $u_\alpha(y) = \frac{l}{y}$ where l is a constant and the term l/y represents the so-called free vortex term.

6. The flow equations in the physical plane(y, α, x).

Adopting cylindrical polar coordinates with y being the radial coordinate, α the circumferential and x the axial coordinate, defining velocity components u_y , u_α and u_x with corresponding vorticity components ω_y , ω_α , ω_x in the direction of increasing y , α and x respectively, then the equation of motion with unit density becomes:

$$\frac{D\underline{u}}{Dt} = -\underline{\nabla} \cdot \underline{p} \quad (11)$$

Where $\frac{D}{Dt}$ is the material derivative. Equation (11) can be written using well known vector identities as:

$$\begin{aligned} \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} - \frac{u_\alpha^2}{y} &= -\frac{\partial p}{\partial y} \\ \frac{\partial u_\alpha}{\partial t} + u_x \frac{\partial u_\alpha}{\partial x} + u_y \frac{\partial u_\alpha}{\partial y} - \frac{u_\alpha u_y}{y} &= 0 \\ \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= -\frac{\partial p}{\partial x} \end{aligned} \quad (12)$$

Furthermore

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} = -\underline{\nabla} \cdot \underline{p}$$

can be written (once again using an appropriate vector identity as)

$$\frac{\partial \underline{u}}{\partial t} + (\underline{\omega} \wedge \underline{u}) = -\underline{\nabla} \left(p + \frac{1}{2} q^2 \right). \text{ Thus}$$

for steady flow Crocco's form of the equation of motion is obtained, i.e.

$$(\underline{u} \wedge \underline{\omega}) = \underline{\nabla} H \quad (13)$$

where H is the total head defined by $H = p + \frac{1}{2} q^2$.

Calculating the cross product on the left hand side of equation (13), gives

$$\begin{aligned} \frac{\partial H}{\partial y} &= u_\alpha \omega_x - u_x \omega_\alpha \\ 0 &= u_x \omega_y - u_y \omega_x \\ \frac{\partial H}{\partial x} &= u_y \omega_\alpha - u_\alpha \omega_x \end{aligned} \quad (14)$$

In addition for axisymmetric flow the vorticity vector $\underline{\omega}$ becomes

$$\underline{\omega} = \underline{\nabla} \wedge \underline{u} = \left\{ -\frac{\partial u_\alpha}{\partial x} \right\} \underline{y} + \left\{ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right\} \underline{\alpha} + \left\{ \frac{1}{y} \frac{\partial (y u_\alpha)}{\partial y} \right\} \underline{x} \quad (15)$$

The equation of continuity becomes

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial (y u_x)}{\partial x} + \frac{\partial (y u_y)}{\partial y} = 0$$

7. The Blockage effect: deriving the additional flow equation due to the circumferentially arranged aerofoils.

In deriving the additional flow equation the effect of the circumferentially arranged blades placed in the duct must be considered. The blades effect the rate of mass flow η , considering figure 7.1, with $k=k(x,y)$ representing the blockage effect, the mass flow into and out of the fluid element is:

Face A:

$$\left(2u_y + \frac{\partial u_y}{\partial x} \delta x \right) \delta x \left(2k + \frac{\partial k}{\partial x} \delta x \right) + O((\delta x)^2)$$

Face B:

$$-\left(2u_x + 2 \frac{\partial u_x}{\partial x} \delta x + \frac{\partial u_x}{\partial y} \delta y \right) *$$

$$\delta y \left(2k + 2 \frac{\partial k}{\partial x} \delta x + \frac{\partial k}{\partial y} \delta y \right)$$

$$+O((\delta x)^2) + O((\delta y)^2)$$

Face C:

$$-\left(2u_y + \frac{\partial u_y}{\partial x} \delta x + 2 \frac{\partial u_y}{\partial y} \delta y\right)^*$$

$$\delta x \left(2k + 2 \frac{\partial k}{\partial y} \delta y + \frac{\partial k}{\partial x} \delta x\right)$$

$$+O((\delta x)^2) + O((\delta y)^2)$$

Face D:

$$\left(2u_x + \frac{\partial u_x}{\partial y} \delta y\right) \delta y \left(2k + \frac{\partial k}{\partial y} \delta y\right)$$

$$+O((\delta y)^2)$$

summing these terms, using the principle of conservation of mass and taking the limit as $\delta x \rightarrow 0, \delta y \rightarrow 0$ gives:

$$\lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}} \left(\frac{\text{mass flow}}{4\delta x \delta y} \right) = \frac{\partial(ku_x)}{\partial x} + \frac{\partial(ku_y)}{\partial y} = 0$$

which may be identified as the continuity equation for two dimensional compressible flow with the density term being replaced by the blockage factor k .

8. The Blockage Function $k(x,y)$

The blockage function $k(x,y)$ is defined to be of the form $k(x,y) = 1 - \frac{\lambda(x)}{T(y)}$ where the function

$\lambda(x)$ represents the contour shape of the aerofoil and the term $T(y)$ is a scaling factor given be

$T(y) = \frac{2\pi}{N} y$, where N is the number of blades

(arbitrary). If the axial span of the aerofoil is x_1 then the function $\lambda(x)$ is defined to have a maximum at $x_1/5$ and $\lambda(x_1/5) = x_1/10$. Furthermore $\lambda(x)$ is chosen to vanish at $x=0$ and $x=x_1$. Choosing $\lambda(x)$ to be of the form $\lambda(x) = cx^\alpha(x^\beta - x_1^\beta)$, where c is a constant. with

$\beta = 1(\text{arbitrary}) \Rightarrow \alpha = \frac{1}{4}$, applying these

conditions gives $c = -\frac{1}{8} \left(\frac{5}{x_1}\right)^{1/4}$

and $\lambda(x) = -\frac{1}{8} \left(\frac{5x}{x_1}\right)^{1/4} (x - x_1)$

9. The Design Plane counterparts

In order to compute numerical solutions in the design plane, expressions are required for the terms A, B and ω_α , thus

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -\frac{1}{y} \left(u_x \frac{\partial y}{\partial x} + u_y \right)$$

$$= -q \frac{\partial}{\partial s} (\log(y))$$

or

$$\eta = -\frac{q^2}{B} \frac{\partial}{\partial \phi} (\log(y)),$$

but

$$\eta = \frac{q^2}{B} \frac{\partial}{\partial \phi} (\log(A))$$

thus $Ay = f(\psi)$, that is $\frac{\partial \psi}{\partial n} = \frac{yq}{f(\psi)}$. The

arbitrary function $f(\psi)$ represents the freedom in the cross stream distribution of ψ and choosing $f(\psi)$ to be unity everywhere ψ can be identified as the usual Stokes stream function given by

$$\frac{\partial \psi}{\partial x} = -yu_y; \frac{\partial \psi}{\partial y} = yu_x$$

Equation (12), (circumferential component) gives

$$0 = u_x \frac{\partial(yu_\alpha)}{\partial x} + u_y \frac{\partial(yu_\alpha)}{\partial y}$$

Referring to the meridional plane figure 9.1, it may be deduced that

$$u_x = q \frac{\partial x}{\partial s}; u_y = q \frac{\partial y}{\partial s}$$

$$\Rightarrow \frac{\partial}{\partial s} (yu_\alpha) = 0$$

$$\therefore yu_\alpha = C(\psi)$$

where $q = \frac{ds}{dt}$. In terms of $C(\psi)$ the vorticity

vector (equation (15)) becomes

$$\underline{\omega} = \nabla \wedge \underline{u} = \left\{ -\frac{1}{y} \frac{\partial C}{\partial x} \right\} \underline{y} + \left\{ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right\} \underline{\alpha} + \left\{ \frac{1}{y} \frac{\partial C}{\partial y} \right\} \underline{x}$$

$= \omega_y \underline{y} + \omega_\alpha \underline{\alpha} + \omega_x \underline{x}$, by definition.

An expression for ω_α is required as this appears in the expression for B , so using the radial component of equation (14) gives

$$\omega_\alpha = \frac{u_x}{u_\alpha} \left(\frac{1}{y} \frac{\partial C}{\partial y} \right) - \frac{1}{u_x} \frac{\partial H}{\partial y}$$

using the Stokes' stream function this becomes

$$\omega_\alpha = \frac{C(\psi)}{y} \left(\frac{dC}{d\psi} \right) - y \frac{dH}{d\psi}$$

which is the required expression to be used in calculation of B according to definition (4). If far upstream the flow is assumed to be cylindrical so that all quantities are independent of x , then with unit density the equation of motion and the Stokes' Stream function give

$$u_y = 0; \frac{\partial p}{\partial x} = 0; \frac{\partial p}{\partial y} = \frac{u_\alpha^2}{y}; \frac{\partial \psi}{\partial x} = 0; \frac{\partial \psi}{\partial y} = yu_x$$

giving

$$\omega_\alpha = \frac{C(\psi)}{y} \left(\frac{dC}{d\psi} \right) - \frac{y}{2} \frac{d}{d\psi} (u_x^2 + u_\alpha^2) - \frac{u_\alpha^2}{u_x y}$$

With $u_x(y) = \beta$ and $u_\alpha(y) = ky + \frac{l}{y}$ as

previously defined. Once $\frac{dH}{d\psi}$ has been calculated

upstream it takes this value throughout the (ϕ, ψ) since as is self evident the expression is independent of ϕ . This last expression for ω_α is required in the calculation of B and numerical coupling with equation (1) gives the numerical solution in the design plane.

10. Downstream Conditions

Downstream a cylindrical flow condition as discussed below will be prescribed. Defining the pressure function $H(\psi)$ and the function $C(\psi)$ as

$$H(\psi) = \frac{1}{2}(u_x^2 + u_\alpha^2) + \frac{P}{\rho} \text{ and } C(\psi) = yu_\alpha$$

for cylindrical flow radial equilibrium (from equation (12) radial component gives

$$\frac{1}{\rho} \frac{dp}{dy} = \frac{u_\alpha^2}{y}$$

Integrating gives

$$\frac{1}{\rho} (p - p_{y\text{-inner}}) = \int_{y\text{-inner}} \frac{u_\alpha^2}{y} dy = \int_{y\text{-inner}} \frac{C^2(\psi)}{y^3} dy$$

Which gives $H(\psi)$ as

$$H(\psi) = \frac{1}{2}(u_x^2 + u_\alpha^2) + \frac{P_{y\text{-inner}}}{\rho}$$

$$+ \int_{y\text{-inner}} \frac{C^2(\psi)}{y^3} dy$$

$$\begin{aligned} \text{Now } \int_{y\text{-inner}} \frac{C^2(\psi)}{y^3} dy &= -\frac{1}{2} \int_{y\text{-inner}} C^2 d(1/y^2) \\ &= -\frac{1}{2} \left[\frac{C^2}{y^2} - \left(\frac{C^2}{y^2} \right)_{y\text{-inner}} \right] + \frac{1}{2} \int_{y\text{-inner}} \frac{1}{y^2} \frac{dC^2}{dy} dy \end{aligned}$$

Therefore

$$\begin{aligned} H(\psi) &= \frac{1}{2} u_x^2 + \frac{P_{y\text{-inner}}}{\rho} + \frac{1}{2} (u_\alpha^2)_{y\text{-inner}} \\ &+ \int_{\psi=0} \frac{1}{y^2} \frac{dC^2}{d\psi} d\psi \end{aligned}$$

Suppose $u_{x,1} = u_{x,1}(\psi)$ and $u_{\alpha,1} = u_{\alpha,1}(\psi)$, where the subscript 1 denotes upstream conditions, then $u_{x,2} = u_{x,2}(\psi)$ and $u_{\alpha,2} = u_{\alpha,2}(\psi)$ are required as functions of ψ , where the subscript 2 similarly denoting downstream conditions, so that

$$\begin{aligned} \frac{1}{2} u_{x,2}^2 &= H(\psi) - \frac{P_{2,inner}}{\rho} - \frac{1}{2} (u_{\alpha,2}^2)_{inner} \\ &- \frac{1}{2} \int_{\psi=0} \frac{1}{y_1^2} \frac{dC^2}{d\psi} d\psi \end{aligned} \quad (16)$$

$$\text{and } \int_{\psi=0} \frac{d\psi}{u_{x,2}} = \frac{1}{2} (y_2^2 - y_{2,inner}^2)$$

Furthermore $C(\psi) = y_1 u_{\alpha,1} = y_2 u_{\alpha,2}$, and equation (16) now gives

$$\begin{aligned} \frac{1}{2} u_{x,2}^2 &= \frac{1}{2} u_{x,1}^2 + \frac{P_{1,inner}}{\rho} - \frac{P_{2,inner}}{\rho} + \\ &\frac{1}{2} ((u_{\alpha,1}^2)_{inner} - (u_{\alpha,2}^2)_{inner}) \\ &+ \frac{1}{2} \int_{\psi=0} \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) d(C^2) \end{aligned}$$

or

$$u_{x,2}^2 = u_{x,1}^2 + K + \int_{\psi=0} \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) d(C^2) \quad (17)$$

where

$$K = 2 \left(\frac{P_{1,inner}}{\rho} - \frac{P_{2,inner}}{\rho} \right) + (u_{\alpha,1}^2)_{inner} - (u_{\alpha,2}^2)_{inner}$$

$$\text{and } y_2^2 = y_{2,inner}^2 + 2 \int_{\psi=0} \frac{d\psi}{u_{x,2}} \quad (18)$$

with $u_{x,2}$ in this case given by (17).

11. Calculation procedure

The calculation of the downstream radii $y_2(\psi)$ follow from equation (18) with $u_{x,2}$ given by equation (17), which can be written as

$$u_{x,2}^2 = g(\psi) + K, \text{ where}$$

$$g(\psi) = u_{x,1}^2 + \int_{\psi=0}^{\psi} \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \frac{d(C^2)}{d\psi} d\psi \quad (19)$$

In order to calculate the $(n+1)^{th}$ iterate it is known that:

$$\begin{aligned} \frac{\partial}{\partial K} (y_{2,outer}^2) &= 2 \int_{\psi=0}^{\Psi} \frac{\partial}{\partial K} \left(\frac{d\psi}{\sqrt{g(\psi) + K}} \right) \\ &= - \int_{\psi=0}^{\Psi} \frac{d\psi}{(u_{x,2}^3)^{(n)}} \end{aligned}$$

but

$$\left(\frac{\partial}{\partial K} (y_{2,outer}^2) \right)^{(n)} = \frac{(y_{2,outer}^2)^{(n+1)} - (y_{2,outer}^2)^{(n)}}{K^{(n+1)} - K^{(n)}} \quad (20)$$

from which as can be seen from equation (20) the $K^{(n)}$ must be calculated iteratively with $K^{(0)}=0$, Once the $K^{(n+1)}$ has been calculated it is introduced into equation (19), giving rise to a new $(u_{x,2}^2)^{(n+1)}$ which in turn gives a new $(y_{x,2}^2)^{(n+1)}$ from equation (18) and the process repeated until some convergence criteria is satisfied.

12. Prescription of Wall Geometries.

In this paper the Dirichlet boundary conditions will be prescribed on the wall boundaries so that it is the radii values, y that are given as a function of ϕ on the boundaries. The function chosen to give a y distribution is based on the hyperbolic tangent, choosing $y(\phi) = Ctanh(a\phi + b) + k$ where C, a, b and k are constants, applying the conditions that $y=y_u$ at $\phi=0$ and $y=y_d$ at $\phi=\Phi$ taking $a\Phi + b = 3$ (arbitrary) and $b = -3$, so that $tanh(a\Phi + b) \approx 1$ and $tanh(b) \approx -1$, then it follows that

$$y(\phi) = \left(\frac{y_d - y_u}{2} \right) tanh(a\phi + b) + \left(\frac{y_d + y_u}{2} \right) \quad (21)$$

replacing ϕ , by x in equation (21) gives a $y(x)$ distribution. The inner radius is prescribed to be given by a hyperbolic tangent function in this paper. The geometries produced are shown in figures 12.1, 12.2 and 12.3 respectively.

13. Alternative Solution using an Integral formula based on Green's theorem.

Here a second method of solution is derived using an integral formula. Commencing with the generalized form of Green's theorem for the self adjoint elliptic operator $E(t)$ in normal form given by:

$$\iint_R vE(t) - tE^{(A)}(v) d\phi d\psi = \oint_C t \frac{\partial v}{\partial n} - v \frac{\partial t}{\partial n} ds$$

where $t = y^2$, $E(u) = E^{(A)}(u)$ where $E^{(A)}(u)$ is the adjoint of E and v is the fundamental solution to the adjoint equation. In this case the adjoint equation is given by $\nabla^2 = 0$ and $E(t) = g$ as defined by equation (6). The contour C bounding the surface R is traversed in the counter clockwise sense. For a doubly connected region introducing a singularity at the point (ϕ_0, ψ_0) (inside or on the contour C) and assuming

$$v(\phi, \psi) = F(\phi, \psi) \log_e |r| \text{ so that the distance } r$$

$$\text{is given by: } r = \left((\phi - \phi_0)^2 + (\psi - \psi_0)^2 \right)^{1/2}$$

with $F(\phi, \psi)$ analytic, then it can be shown that

$$\begin{aligned} m\pi t(\phi_0, \psi_0) F(\phi_0, \psi_0) &= \oint_C t \frac{\partial v}{\partial n} - v \frac{\partial t}{\partial n} ds \\ &- \iint_R v g \left(\frac{\partial^2 t}{\partial \phi^2}, \frac{\partial^2 L}{\partial \psi^2} \right) d\phi d\psi, \quad m = 1, 2 \end{aligned}$$

with $L = \log_e t$. Now $m = 2$ if (ϕ_0, ψ_0) is within C and $m=1$ if (ϕ_0, ψ_0) is on C (the $m=1$ case can be shown using the appropriate Plemelj formulae or by indenting the contour at (ϕ_0, ψ_0)). For the Dirichlet case of boundary condition of $t(\phi, \psi)$ the requirement is that $v(\phi, \psi) = 0$ on C in addition to $v(\phi, \psi)$ being harmonic and for the Neumann

conditions on t , the requirement is that $\frac{\partial v}{\partial n} = 0$ and $v(\varphi, \psi)$ once again satisfying Laplace's equation. Much literature is available for the Green's function for the Laplace equation (see Williams [6]) and need not be mentioned here. Hence for the Dirichlet problem without loss of generality setting $F(\varphi, \psi) = 1 \forall \varphi, \psi$ and for interior points:

$$2\pi t(\varphi_0, \psi_0) = -\oint_C t \frac{\partial v_D}{\partial n} ds - \iint_R v_D g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) d\varphi d\psi \quad (22)$$

i.e., the Green's function v_D satisfies Laplace's equation on $\partial\Omega$ where $\partial\Omega$ is defined by: $\varphi_0 \leq \varphi \leq \varphi_{M+1}$, $\psi_0 \leq \psi \leq \psi_{N+1}$ and vanishes on C . For the Neumann problem

$$2\pi t(\varphi_0, \psi_0) = -\oint_C v_N \frac{\partial t}{\partial n} ds - \iint_R v_N g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) d\varphi d\psi$$

Which give integral formulae for the square of the radius t , from which the radius y can be determined., above Green's function v_N satisfies the Laplace equation on $\partial\Omega$ with $\frac{\partial v_N}{\partial n}$ vanishing

on C . Knowledge of the derivatives $\frac{\partial t}{\partial \varphi}$ and

$\frac{\partial t}{\partial \psi}$ are also required for the determination of the

speed q given by equation (2) hence differentiating under the integral sign above with respect to φ

and ψ gives integral formulae for both $\frac{\partial t}{\partial \varphi}$ and

$\frac{\partial t}{\partial \psi}$, such that:

$$2\pi \frac{\partial}{\partial \psi} t(\varphi_0, \psi_0) = \oint_C \frac{\partial t}{\partial \psi} \frac{\partial v_D}{\partial n} + t \frac{\partial^2 v_D}{\partial \psi \partial n} ds - \iint_R v_D \frac{\partial}{\partial \psi} \left(g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) \right) + \frac{\partial v_D}{\partial \psi} \cdot g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) d\varphi d\psi$$

and similarly for $\frac{\partial t}{\partial \varphi}$

$$2\pi \frac{\partial}{\partial \varphi} t(\varphi_0, \psi_0) = \oint_C \frac{\partial t}{\partial \varphi} \frac{\partial v_D}{\partial n} + t \frac{\partial^2 v_D}{\partial \varphi \partial n} ds - \iint_R v_D \frac{\partial}{\partial \varphi} \left(g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) \right) + \frac{\partial v_D}{\partial \varphi} \cdot g\left(\frac{\partial^2 t}{\partial \varphi^2}, \frac{\partial^2 L}{\partial \varphi^2}\right) d\varphi d\psi$$

14. Iterative solution

To convert formula (22) to a system of linear algebraic equations the point $t(\varphi_0, \psi_0)$ inside C is related to its boundary values on C . To obtain the first iterates $t_i^{(1)}(\varphi_0, \psi_0)$, $g_i^{(0)}$ is set equal to zero, so that

$$2\pi t_i^{(1)} = \sum_j^{2N+2M+4} \left(\frac{\partial v_D}{\partial n} \right)_j t_j \Delta s_j \quad i = 0, 1, 2, \dots, 2N + 2M + 4$$

Using the trapezoidal rule

$$\pi t_i^{(1)} = \sum_j^{2N+2M+4} \frac{1}{4} \left(\frac{\partial v_D}{\partial n} \right)_j (s_{j+1} - s_{j-1}) t_j$$

$$i = 0, 1, 2, \dots, 2N + 2M + 4$$

$$\Rightarrow t_i^{(1)} = \sum_j^{2N+2M+4} K(v_D, s) t_j, \quad i = 0, 1, 2, \dots, 2N + 2M + 4$$

$$\text{Where } K(v_D, s) = \frac{1}{4\pi} \left(\frac{\partial v_D}{\partial n} \right) (s_{j+1} - s_{j-1})$$

Using this method there is a simple self-consistency check. i.e. the t_j are known upstream and downstream for $j=0, 1, 2, \dots, N+1$ and $j=N+M+3, N+M+4, \dots, 2N+M+3$, hence the first iteration may be written as:

$$\begin{bmatrix} 1-K_{N+2} & -K_{N+3} & \cdot & \cdot & -K_{2N+2M+4} \\ -K_{N+2} & 1-K_{N+3} & -K_{N+4} & \cdot & -K_{2N+2M+4} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -K_{N+2} & \cdot & \cdot & \cdot & -K_{2N+2M+4} \end{bmatrix} \begin{bmatrix} t_{N+2} \\ t_{N+3} \\ \cdot \\ \cdot \\ t_{2N+2M+4} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_j K_j t_j \\ \sum_j K_j t_j \\ \cdot \\ \cdot \\ \sum_j K_j t_j \end{bmatrix}$$

17. Figures

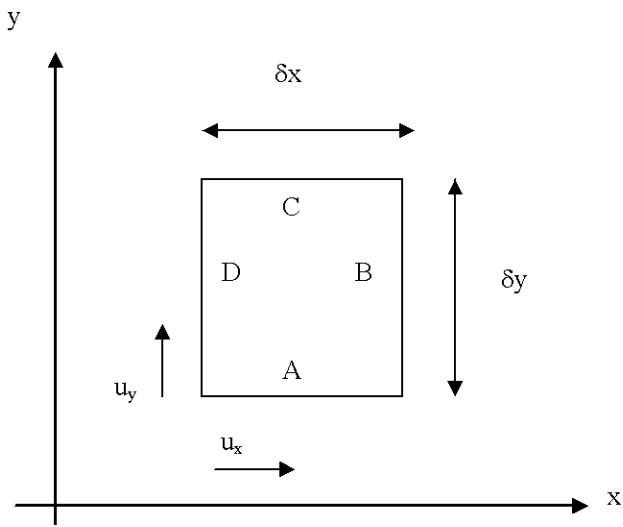


Fig 7.1. A Fluid element

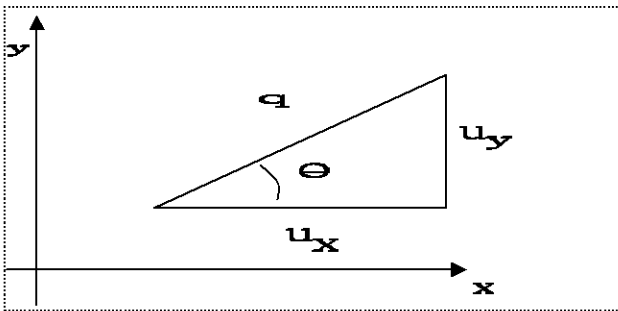


Fig 9.1. The meridional plane.

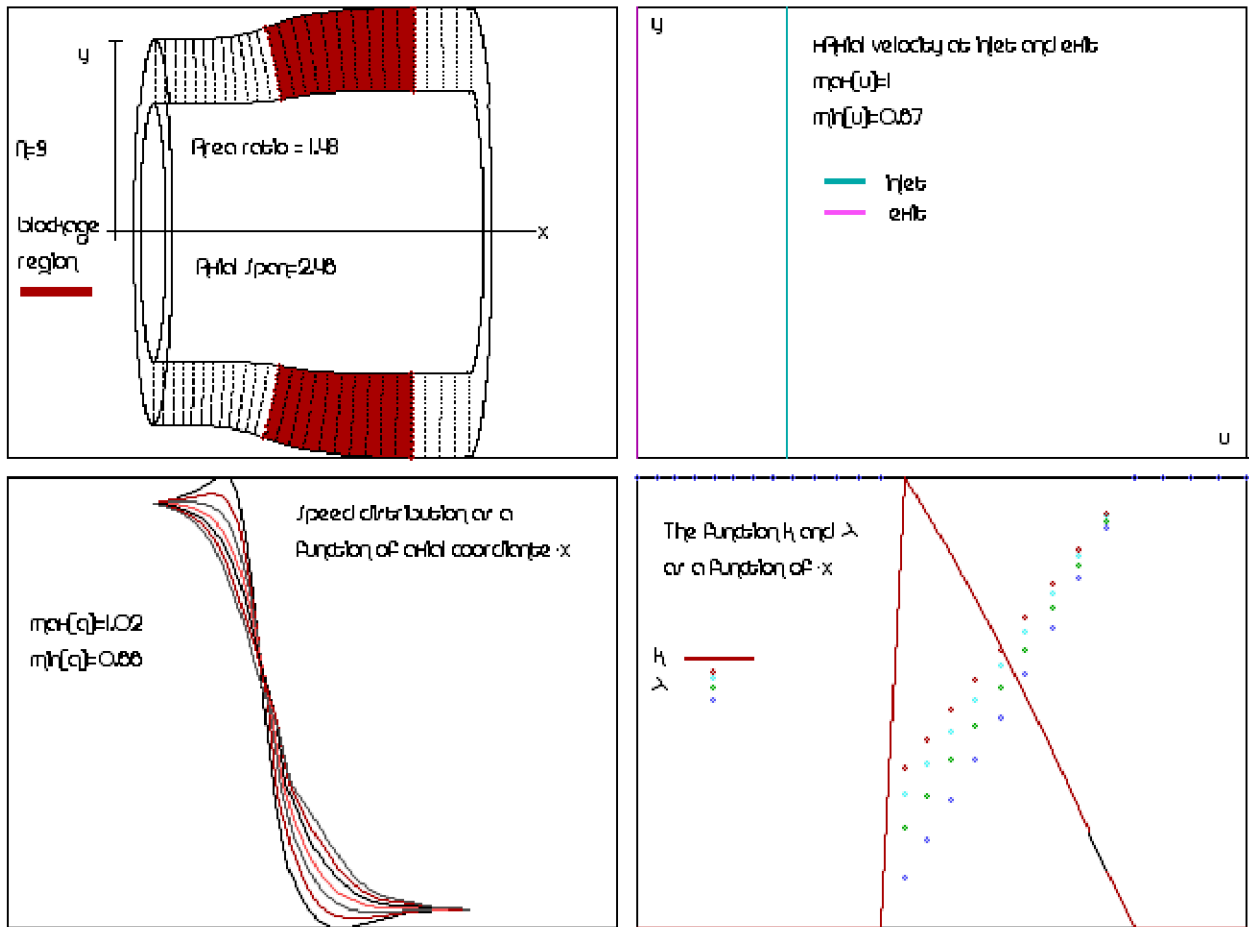


Fig 12.1. The geometry and speed distribution produced with $N=3$.

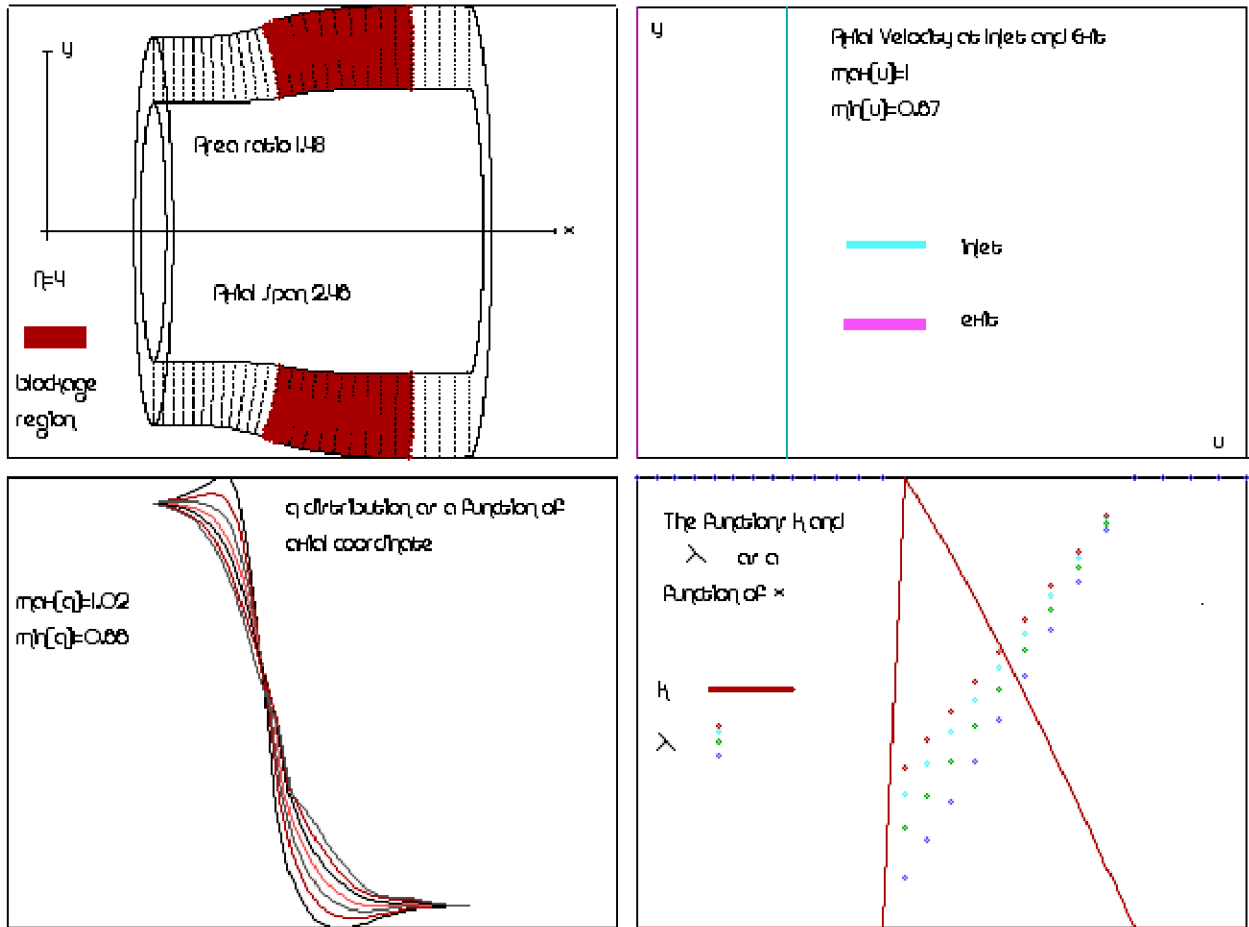


Fig 12.2. The geometry and speed distribution produced with $N=4$.

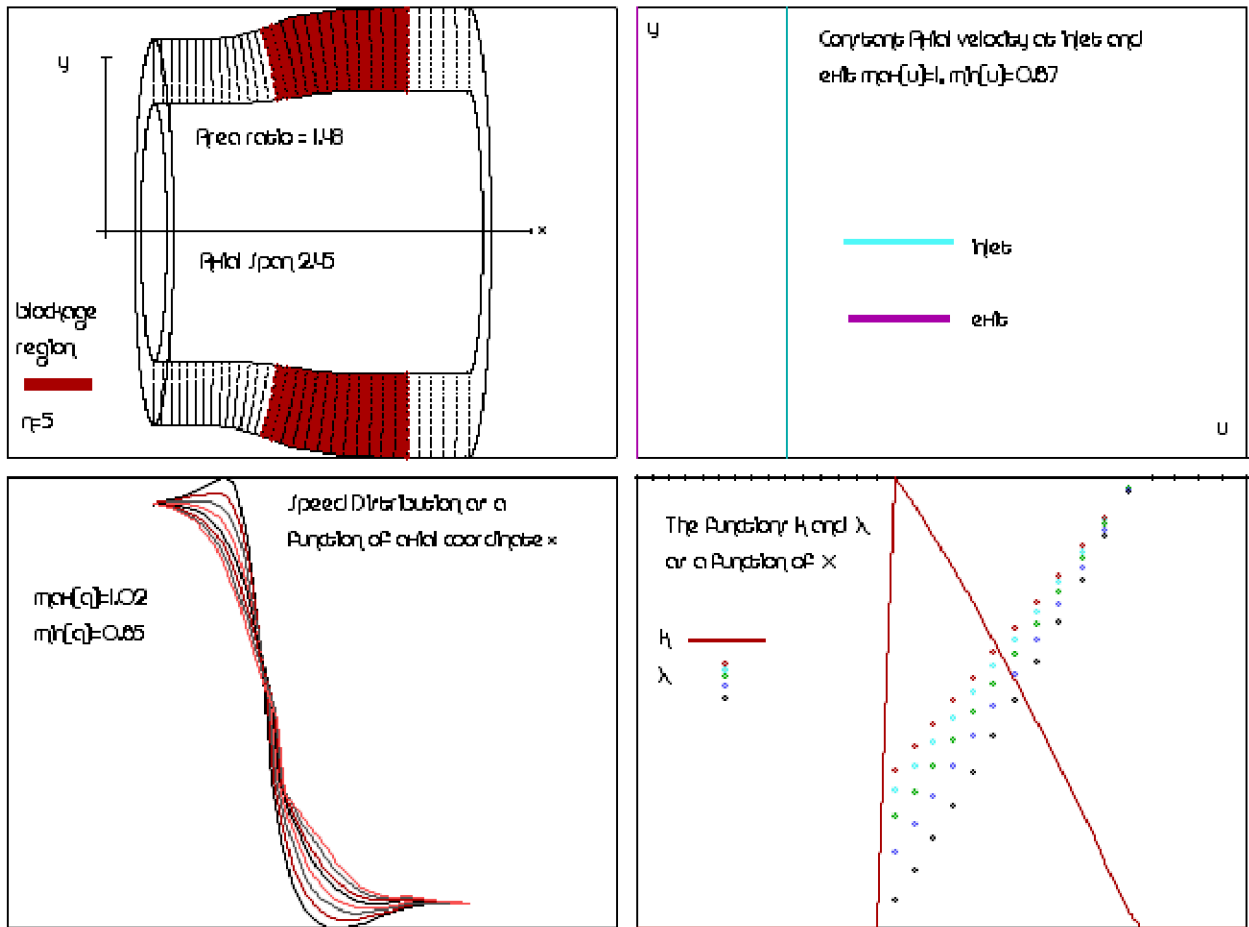


Fig 12.3. The geometry and speed distribution produced with $N=5$.