On linear models for NP-problems

Sergey Gubin *

Abstract—**An argument in favor of the linear modeling of NP-problems.**

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1 Introduction

In his seminal work [1], Mihalis Yannakakis proved that "The TSP polytope cannot be expressed by a polynomial size symmetric LP", where symmetry of a linear program (LP) means that the polytope is an invariant under node relabeling. The theorem is often used as a major argument against linear modeling of NP-problems. This work presents an argument against such use of the theorem.

2 Symmetric linear models

Let's generalize the Subgraph Isomorphism Problem [2, 3, 4]: whether a given multi digraph g contains a subgraph which is isomorphic with another given multi digraph s.

Let n and m be powers of vertex-sets of g and s appropriately. Based on a node labeling/enumeration, let's construct adjacency matrices of these digraphs - matrices G and S appropriately. In terms of these matrices, the problem is a compatibility problem for the following quadratic inequality with unknown matrix X:

$$P_{mn}XGX^TP_{mn}^T \geq S,\ XX^T = U_{n\times n}, \tag{1}$$

- where $U_{n\times n}$ is $n \times n$ union matrix; and $P_{mn} = (U_{m\times m}0)_{m\times n}$ is truncation. Permutation matrix X presents relabeling of g.

Let's arbitrarily label/enumerate elements of G and S: if an element is equal a, then that element has a labels (zero-elements have no labels). Let's construct in-incidence matrix $I_G = (\alpha_{ij})$: $\alpha_{ij} = 1$ if i-th column of G contains j-th label; and $\alpha_{ij} = 0$, if otherwise. In the same way, let's construct in-incidence matrix I_S. Also, let's construct out-incidence matrices O_G and O_S but using rows instead of columns. Direct calculation proves the following decompositions:

$$G = O_GI_G^T,\ S = O_SI_S^T.$$

The incidence matrices are $(0,1)$-matrices with one and only one non-zero element per column. Let k be total of all elements of G; and l be total of all elements of S. Then matrices I_G and O_G are $n \times k$; and matrices I_S and O_S are $m \times l$. In digraph terms: k and l are powers of arc-sets of the given (multi) digraphs.

In terms of the incidence matrices, inequality (1) can be rewritten as a quadratic equivalence with additional unknown matrix Y:

$$\begin{cases} P_{mn}XO_GYP_{lk}^T = O_S \\ P_{mn}XI_GYP_{lk}^T = I_S \\ XX^T = U_{n\times n},\ YY^T = U_{k\times k} \end{cases}. \tag{2}$$

In digraph terms: X relabels nodes, and Y relabels arcs.

Let's arbitrarily enumerate all permutation matrices:

$$\{X \mid XX^T = U_{n\times n}\} = \{X_i \mid i = 1, 2, \ldots, n!\},$$
$$\{Y \mid YY^T = U_{k\times k}\} = \{Y_j \mid j = 1, 2, \ldots, k!\}.$$

Theorem 1: (2) is compatible iff the following system is compatible:

$$\begin{cases} \sum_{i,j} \lambda_{ij}P_{mn}X_iO_GY_jP_{lk}^T = O_S \\ \sum_{i,j} \lambda_{ij}P_{mn}X_iI_GY_jP_{lk}^T = I_S \\ \sum_{i,j} \lambda_{ij} = 1,\ \lambda_{ij} \geq 0 \end{cases}, \tag{3}$$

- where numbers λ_{ij} are unknown, $i = 1, 2, \ldots, n!$; $j = 1, 2, \ldots, k!$.

Proof: If X and Y are a solution of (2), then the following numbers are a solution of (3):

$$\lambda_{ij} = \begin{cases} 1, & (X_i = X) \wedge (Y_j = Y) = true \\ 0, & (X_i = X) \wedge (Y_j = Y) = false \end{cases}.$$

If λ_{ij} are a solution of (3), then there is $\lambda_{i_0j_0} > 0$. Because the incidence matrices are $(0,1)$-matrices with one and only one non-zero element per column, X_{i_0} and Y_{j_0} are a solution of (2). QED.

System (3) is an explicit exponential size linear model of NP-problems. It is symmetric in that sense that the convex hull at the left side of the system is an invariant under relabeling.

The explicitness of (3) can be exploited in the ellipsoid/separation design [1]; in the branch/bound-like designs to cut (3) down to a polynomial size; and others.

Let's call matrices G and S the model's pattern and input appropriately. The pattern's/input's conjugacy classes parametrize the whole NP zoo. Let's illustrate that with several examples [2, 3, 4, 5, 6]:

*Genesys Telecommunication Laboratories, 1255 Treat Blvd., Walnut Creek, CA USA 94596, Email: sgubin@genesyslab.com

(Sub)GI. Pattern and input are adjacency matrices of given (multi di-) graphs.

Clique. Pattern is an adjacency matrix of a given (multi di-) graph. Input is a matrix whose diagonal elements are 0 and remaining elements are 1.

HC/HP. Pattern is adjacency matrix of a given (multi di-) graph. Input is a circular permutation matrix except, in case of HP, one 1 is poked out.

ATSP. Let W be a given weight matrix. A symmetric exponential size LP:

$$(W, \sum_{i,j} \lambda_{ij} X_i O_G Y_j P) \to \min,$$

- under the constrains modeling HC. The matrix scalar product $(*, *)$ totals products of appropriate elements of its multiplicands.

3-SAT. Let's arbitrarily enumerate strings in truth tables of given clauses. By definition, two strings are compatible if they are consistent and equal *true*. The strings' compatibility matrix is the pattern. The input is a box matrix with 8×8 boxes: all elements in the box are 0 except $(1, 1)$-element which is 1.

2-SAT. The same as above, except the input's boxes are 4×4.

Matrix G was called "pattern" because the left side of (3) defines a convex hull. The hull is an invariant under relabeling. And the compatibility problem is: whether the incidence matrices of a given input are located in that convex hull. Regarding LP: when they are, they are an extreme point of the hull.
Obviously, the independence of node-/arc-labeling is excessive and (3) can be reduced, at least, to the following:

$$\begin{cases} \sum_{i=1}^{n!} \lambda_i P_{mn} X_i O_G P_{lk}^T = O_S \\ \sum_{i=1}^{n!} \lambda_i P_{mn} X_i I_G P_{lk}^T = I_S \\ \sum_{i=1}^{n!} \lambda_i = 1, \ \lambda_i \geq 0 \end{cases},$$

3 Asymmetric linear models

For a given NP-instance G and S, all happens in the $O(nmlk)$-dimensional vector space of matrices involved in (3). Thus, in accordance with the Carathodory's theorem (convex hull), the system can be replaced with its subsystem of $O(nmlk)$ linear equivalences:

$$\begin{cases} \sum_{i,j=1}^{O(nmlk)} \lambda_{ij} P_{mn} X_i O_G Y_j P_{lk}^T = O_S \\ \sum_{i,j=1}^{O(nmlk)} \lambda_{ij} P_{mn} X_i I_G Y_j P_{lk}^T = I_S \\ \sum_{i,j=1}^{O(nmlk)} \lambda_{ij} = 1, \ \lambda_{ij} \geq 0 \end{cases}, \quad (4)$$

- where λ_{ij} are unknown; and permutation matrices are enumerated appropriately. The system is asymmetric: relabeling will rotate the convex hull defined at the system's left side. The hull would rotate inside of the hull defined at the left side of (3). But, that rotation does not affect the right side of system (4), which is an extreme point of the hull defined by (3). That proves that the convex hull defined by (3) has $O(nmlk)$ extreme points only, or
Theorem 2: There are asymmetric polynomial size linear models for any NP-problem.
Let mention, a model can be asymmetric polynomial size LP

$$\sum_{i,j=1}^{O(nmlk)} \lambda_{ij} \to min,$$

- under constrains (4).
To find an asymmetric polynomial size linear model, all heuristics are good. That could be fixing of particular G and S; exclusion of unknown from (3); basis manipulations in the linear hull of permutation matrices; creating of appropriate mathematical tables; and others.

References

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