Proceedings of the World Congress on Engineering and Computer Science 2007
WCECS 2007, October 24-26, 2007, San Francisco, USA

PID Controller Design with Guaranteed Stability Margin for MIMO Systems

T. S. Chang and A. N. Gündeş

Abstract— **Closed-loop stabilization with guaranteed stability margin using Proportional+Integral+Derivative (PID) controllers is investigated for a class of linear multi-input multi-output plants. A sufficient condition for existence of such PID-controllers is derived. A systematic synthesis procedure to obtain such PID-controllers is presented with numerical examples.**

Keywords– Simultaneous stabilization and tracking, PID control, integral action, stability margin.

I. INTRODUCTION

Proportional+Integral+Derivative (PID) controllers are the simplest integral-action controllers that achieve asymptotic tracking of step-input references [1]. Although the simplicity of PID-controllers is desirable due to easy implementation and from a tuning point-of-view, it also presents a major restriction that only certain classes of plants can be controlled by using PID-controllers. Rigorous PID synthesis methods based on modern control theory are explored recently in e.g., [2], [3], [4], [5], [6]. Sufficient conditions for PID stabilizability of multi-input multi-output (MIMO) plants were given in [6] and several plant classes that admit PID-controllers were identified.

The systematic controller design method given in [6] allows freedom in several of the design parameters. Although these parameters may be chosen appropriately to achieve various performance goals, these issues were not explored.

The authors are with the Department of Electrical and Computer Engineering, University of California, Davis, CA 95616. (emails: chang@ece.ucdavis.edu, angundes@ucdavis.edu)

The goal of this paper is to study closed-loop stabilization with guaranteed stability margins using PID-controllers. A sufficient condition is presented for existence of PID-controllers that stabilize linear, time-invariant, MIMO stable plants, where the closed-loop poles are guaranteed to have real-parts less than a pre-specified $-h$. A systematic design procedure is proposed and illustrated with several numerical examples. The choice of the free parameters can be optimized with a chosen cost function. Although stability margin can be considered as an important performance measure, there are other factors effecting the performance of the system and hence, "good" choice for the design parameters for overall performance is case-specific and cannot be generalized.

The paper is organized as follows: Section II shows the main result, where a sufficient condition for stabilizability using a PID-controller with guaranteed stability margin is given. Section III presents a systematic procedure to synthesize PID controllers and gives several illustrative examples. Section IV gives a short discussion, concluding remarks and some future directions.

II. MAIN RESULTS

Notation: Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ denote complex, real, positive real numbers. The extended closed right-half complex plane is $\mathcal{U} = \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I_n is the $n \times n$ identity matrix. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We drop (s) in transfer-matrices such as $G(s)$ wherever this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R_p}^{n_y \times n_u}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF), where $X, Y \in \mathcal{M}(\mathbf{S})$, $\det Y(\infty) \neq 0$.

Consider the linear time-invariant (LTI) MIMO unity-feedback system $Sys(G, C)$ shown in Fig. 1, where $G \in \mathbf{R_p}^{m \times m}$ is the plant's transfer-function and $C \in \mathbf{R_p}^{m \times m}$ is the controller's transfer-function. Assume that $Sys(G, C)$ is well-posed, G and C have no unstable hidden-modes, and $G \in \mathbf{R_p}^{m \times m}$ is full (normal) rank. We consider the realizable form of proper PID-controllers given by (1), where K_p, K_i, $K_d \in \mathbb{R}^{m \times m}$ are the proportional, integral, derivative constants, respectively, and $\tau \in \mathbb{R}_+$ [7]:

$$C_{pid} = K_p + \frac{K_i}{s} + \frac{K_d\, s}{\tau s + 1} \; . \tag{1}$$

For implementation, a (typically fast) pole is added to the derivative term so that C_{pid} in (1) is proper. The integral-action in C_{pid} is present when $K_i \neq 0$. The subsets of PID-controllers obtained by setting one or two of the three constants equal to zero are denoted as follows: (1) becomes a PI-controller C_{pi} when $K_d = 0$, an ID-controller C_{id} when $K_p = 0$, a PD-controller C_{pd} when $K_i = 0$, a P-controller C_p when $K_d = K_i = 0$, an I-controller C_i when $K_p = K_d = 0$, a D-controller C_d when $K_p = K_i = 0$.

Definition 2.1: a) $Sys(G, C)$ is said to be stable iff the transfer-function from (r, v) to (y, w) is stable. *b)* C is said to stabilize G iff C is proper and $Sys(G, C)$ is stable. △

The problem addressed here is the following: Suppose that $h \in \mathbb{R}_+$ is a given constant. Can we find a PID-controller C_{pid} that stabilizes the system $Sys(G, C_{pid})$ with a guaranteed stability margin, i.e., with real parts of the closed-loop poles of the system $Sys(G, C_{pid})$ less or equal to $-h$? It is clear that this goal is not achievable for some plants. Furthermore, even when it is achievable, it may be possible to place the closed-loop poles to the left of a shifted-axis that goes through $-h$ only for certain $h \in \mathbb{R}_+$. We start our investigation of plant classes for which we can achieve our goal by considering stable plants. The class of plants under consideration, denoted by $\mathcal{G}_h$, is described as follows:

Let $G \in \mathcal{G}_h \subset \mathbf{S}^{m \times m}$, i.e., let the given plant be stable. Furthermore, let G have no poles with real parts in $[-h, 0]$. Assume that $G(s)$ has no transmission-zeros (or blocking-zeros) at $s = 0$, i.e., $G(0)$ is invertible (note that this condition is necessary for existence of PID-controllers with nonzero integral-constant K_i [6]). The plant G may have transmission-zeros (or blocking-zeros) elsewhere in $\mathcal{U}$ but not at $s = 0$.

Now define

$$\hat{s} := s + h, \quad \text{or} \quad s =: \hat{s} - h \tag{2}$$

and

$$\hat{G}(\hat{s}) \;:=\; G(\hat{s} - h) \; ; \tag{3}$$

then $\hat{G}(\hat{s})$ has no poles in the closed right $\hat{s}$-plane. Similarly, define $\hat{C}_{pid}$ as

$$\hat{C}_{pid}(\hat{s}) := K_p + \frac{K_i}{\hat{s} - h} + \frac{K_d\,(\hat{s} - h)}{\tau(\hat{s} - h) + 1} \, . \tag{4}$$

Let $\mathcal{S}_h(G)$ denote the set of all PID-controllers that stabilize $G \in \mathcal{G}_h$, with real parts of the closed-loop poles of the system $Sys(G, C_{pid})$ less or equal to $-h$; i.e.,

$$\mathcal{S}_h(G) := \{\, C_{pid} \,|\, \hat{C}_{pid} \text{ stabilizes } \hat{G}(\hat{s}) \,\} \, . \tag{5}$$

Proposition 2.1: (*A sufficient condition):*

Let $h \in \mathbb{R}_+$ and $G \in \mathcal{G}_h$ be given. If for some $\hat{K}_p \in \mathbb{R}^{m \times m}$, $\hat{K}_d \in \mathbb{R}^{m \times m}$ and $\tau < 1/h$, the given $h \in \mathbb{R}_+$ satisfies

$$h < \frac{1}{2} \gamma(h, \hat{K}_p, \hat{K}_d), \tag{6}$$

where $\gamma = \gamma(h, \hat{K}_p, \hat{K}_d)$ is defined as

$$\gamma(h, \hat{K}_p, \hat{K}_d)$$

$$:= \|\hat{G}(\hat{s})(\hat{K}_p+\frac{\hat{K}_d(\hat{s}-h)}{\tau(\hat{s}-h)+1})+\frac{\hat{G}(\hat{s})G(0)^{-1}-I}{\hat{s}-h}\|^{-1}, \quad (7)$$

then there exists a PID-controller C_{pid} of the form in (1) that stabilizes $G \in \mathcal{G}_h$, with real parts of the closed-loop poles of the system $Sys(G, C_{pid})$ less or equal to $-h$. Furthermore, a PID-controller $C_{pid} \in \mathcal{S}_h(G)$ is given by

$$C_{pid} = (\alpha+h)\hat{K}_p + \frac{(\alpha+h)G(0)^{-1}}{s} + \frac{(\alpha+h)\hat{K}_d\ s}{\tau s+1}\ , \quad (8)$$

where $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{m\times m}$ are arbitrary, $\tau < 1/h$, and $\alpha \in \mathbb{R}_+$ satisfies

$$h < \alpha < \gamma(h, \hat{K}_p, \hat{K}_d) - h\ . \quad (9)$$

△

Remark:

Condition (6) is obviously satisfied if $h = 0$, i.e., there exists a PID-controller C_{pid} of the form in (1) that stabilizes a given stable plant G, where the closed-loop poles of the system $Sys(G, C_{pid})$ may be anywhere in the open left-half complex plane [6].

Proof of Proposition 2.1:

Write G and C_{pid} given by (8) as

$$G = I^{-1}G\ , \quad (10)$$

$$C_{pid} = (\frac{s}{s+\alpha}C_{pid})(\frac{s}{s+\alpha}I)^{-1}\ . \quad (11)$$

Then C_{pid} stabilizes G if and only if

$$M := \frac{s}{s+\alpha}I + G(\frac{s}{s+\alpha}C_{pid}) \quad (12)$$

is unimodular. Similarly, substitute $\hat{s} = s - h$ as in (2), (3), (4) and write $\hat{G}(\hat{s})$, $\hat{C}_{pid}(\hat{s})$ as

$$\hat{G} = I^{-1}\hat{G}\ , \quad (13)$$

$$\hat{C}_{pid} = (\frac{\hat{s}-h}{\hat{s}+\alpha}\hat{C}_{pid})(\frac{\hat{s}-h}{\hat{s}+\alpha}I)^{-1}\ . \quad (14)$$

Then $\hat{C}_{pid}$ stabilizes $\hat{G}$ if and only if

$$\hat{M} = \frac{\hat{s}-h}{\hat{s}+\alpha}I + \hat{G}(\frac{\hat{s}-h}{\hat{s}+\alpha}\hat{C}_{pid}) \quad (15)$$

is unimodular. Write $\hat{M}$ as

$$\hat{M} = I - \frac{\alpha+h}{\hat{s}+\alpha}I + (\frac{\hat{s}-h}{\hat{s}+\alpha}\hat{G}\hat{C}_{pid}) =: I + \frac{\hat{s}-h}{\hat{s}+\alpha}W, \quad (16)$$

where $K_p = (\alpha+h)\hat{K}_p$, $K_d = (\alpha+h)\hat{K}_d$, $K_i = (\alpha+h)G(0)^{-1} = (\alpha+h)\hat{G}(h)^{-1}$ and

$$W := -\frac{(\alpha+h)}{\hat{s}-h}I + \hat{G}\hat{C}_{pid}$$
$$= -\frac{(\alpha+h)}{\hat{s}-h}I + \hat{G}(K_p + \frac{K_i}{\hat{s}-h} + \frac{K_d(\hat{s}-h)}{\tau(\hat{s}-h)+1})$$
$$= (\alpha+h)[\hat{G}(\hat{K}_p + \frac{\hat{K}_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}) + \frac{\hat{G}(\hat{s})G(0)^{-1}-I}{\hat{s}-h}]. \quad (17)$$

Note that $\frac{\hat{G}(\hat{s})G(0)^{-1}-I}{\hat{s}-h} = \frac{\hat{G}(\hat{s})\hat{G}(h)^{-1}-I}{\hat{s}-h} \in \mathcal{M}(\mathbf{S})$. If (6) and (9) hold, then $h < \alpha$ and $\alpha + h < \gamma(h, \hat{K}_p, \hat{K}_d)$ imply

$$\|\frac{(\hat{s}-h)}{\hat{s}+\alpha}W\| \le \|W\| = \frac{\alpha+h}{\gamma(h, \hat{K}_p, \hat{K}_d)} < 1$$

and hence, $\hat{M}$ in (16) is unimodular by the "small-gain theorem" [8]. Therefore, $\hat{C}_{pid}$ stabilizes $\hat{G}$ and hence, $C_{pid} \in \mathcal{S}_h(G)$. △

III. PID CONTROLLER SYNTHESIS

From the sufficient condition in Proposition 2.1, the following systematic procedure to synthesize a PID controller is obtained: Given $h \in \mathbb{R}_+$ and $G \in \mathcal{G}_h$, define

$$\beta \triangleq max\{x | p = x + jy,\ \ where\ p\ is\ a\ pole\ of\ G(s)\}; \quad (18)$$

then $-h > \beta$. Choose any $\hat{K}_p$ and $\hat{K}_d$ and compute $\gamma(h, \hat{K}_p, \hat{K}_d)$ given by (7). If $\gamma(h, \hat{K}_p, \hat{K}_d) > 2h$ as in condition (6), then it is possible to find $\alpha \in \mathbb{R}_+$ satisfying (9). The PID-controller $C_{pid} \in \mathcal{S}_h(G)$ is then given by (8). If (6) is not satisfied, the process can be repeated for a smaller h value.

The following examples illustrate the PID-controller synthesis procedure and some of its properties.

Example 3.1: Consider the plant transfer-function

$$G(s) = \frac{(s+5)(s^2+8s+32)}{(s+2)(s+8)(s^2+12s+40)} \quad (19)$$

By (18), $\beta = -2$. Suppose that $h = 1$. Fig. 2 shows the constant contour of $\gamma(\hat{K}_p, \hat{K}_d)$, where the solid line represents $\gamma = 2h$ as the upper-bound for condition (6).

ISBN:978-988-98671-6-4

Each contour is evaluated in one point denoted by $*$, which is given in Table 1.

Table 1: Evaluated points for contours in Example 3.1

x	y	γ	x	y	γ	x	y	γ
-2.5	-3.5	0.40	-1	-2	0.70	0	-1	1.41
1	0	2.09	2	0.1	7.80	2.5	0	3.83

Note that any $(\hat{K}_p, \hat{K}_d)$ inside the solid boundary can be chosen. Suppose that we choose $(\hat{K}_p = 2.5,\ \hat{K}_d = 0.2)$ and $\tau = 0.05$. We compute $\gamma = 4.7 > 2h = 2$, and set $\alpha = 0.5\gamma$. The closed-loop poles are -1.79, -2.66, $-4.93 \pm j2.53i$, -6.87, -42.58, which all have real-parts less than $-h = -1$.

For a given $(\hat{K}_p, \hat{K}_d)$, there may exist a maximum value h_{max} such that condition (6) is violated, as indicated by the intersection point about $h_{max} = 1.81$ in Fig. 3. The solid line represents the γ curve in terms h for the selected $(\hat{K}_p, \hat{K}_d)$, and the dash-dotted line represents the straight line $2h$. △

Example 3.2: Consider the same transfer-function as in (19), except the real zero is now in the right-half complex plane, i.e.,

$$G(s) = \frac{(s-5)(s^2+8s+32)}{(s+2)(s+8)(s^2+12s+40)} \ . \tag{20}$$

Let $h = 1$ as in Example 3.1. Fig. 4 shows the constant contour of $\gamma(\hat{K}_p, \hat{K}_d)$. Clearly, the feasible region in this case is very different from the previous one in Example 3.1.

Suppose that we choose $(\hat{K}_p = -3,\ \hat{K}_d = -0.2)$ and $\tau = 0.05$. We compute $\gamma = 3.22 > 2h = 2$, and set $\alpha = 0.5\gamma$. The closed-loop poles are -1.32, $-2.66 \pm j3.31$, -7.62, $-4.73 \pm j12.69$, which all have real-parts less than $-h = -1$. The maximum value h_{max} can be similarly obtained, which is about 1.3 and is lower than that in Example 3.1. △

Example 3.3: Consider the quadruple-tank apparatus in [9], which consists of four interconnected water tanks and two pumps. The output variables are the water levels of the two lower tanks, and they are controlled by the currents that are manipulating two pumps. The transfer-matrix of the linearized model at some operating point is given by

$$G = \begin{bmatrix} \frac{3.7b_1}{62s+1} & \frac{3.7(1-b_2)}{(23s+1)(62s+1)} \\ \frac{4.7(1-b_1)}{(30s+1)(90s+1)} & \frac{4.7b_2}{90s+1} \end{bmatrix} \in \mathbf{S}^{2\times 2}. \tag{21}$$

One of the two transmission-zeros of the linearized system dynamics can be moved between the positive and negative real-axis by changing a valve. The adjustable transmission-zeros depends on parameters γ_1 and γ_2 (the proportions of water flow into the tanks adjusted by two valves). For the values of b_1, b_2 chosen as $b_1 = 0.43$ and $b_2 = 0.34$, the plant G has transmission-zeros at $z_1 = 0.0229 > 0$ and $z_2 = -0.0997$.

By (18) $\beta = -1/90 = -0.0111$. Suppose that $h = 0.004$, and choose

$$\hat{K}_p = \begin{bmatrix} -22.61 & 37.61 \\ 72.14 & -43.96 \end{bmatrix}, \tag{22}$$

$$\hat{K}_d = \begin{bmatrix} 5.28 & 6.21 \\ 6.53 & 7.84 \end{bmatrix}, \tag{23}$$

and $\tau = 0.05$. We can compute $\gamma = 0.0099 > 2h = 0.008$, and set $\alpha = 0.5\gamma$. The maximum of the real-parts of the closed poles can now be computed as -0.0059, which is less than $-h = -0.004$. Thus the requirement is fulfilled. In this example, h_{max} is very small as shown in Fig. 5, due to the fact that β is very close to the imaginary-axis. △

Example 3.4: The PID-synthesis procedure based on Proposition 2.1 involves free parameter choices. Consider the same transfer-function as in (19) of Example 3.1. Let $h = 1$, choose $\tau = 0.05$, and set $\alpha = 0.5\gamma$ as before. If we choose $(\hat{K}_p = 2.5,\ \hat{K}_d = 0.2)$, then the the dash line in Fig. 6 shows the closed-loop step response. However, if we choose $(\hat{K}_p = 2,\ \hat{K}_d = -0.1)$, then we obtain a completely different step response as shown with the dash-dotted line in Fig. 6. It is natural to ask then if the free parameters can be chosen optimally in some sense.

Consider a prototype second order model plant, with $\zeta = 0.7$ and $\omega_n = 6$; i.e.,

$$T_{model} = \frac{\omega_n^2}{s^2 + 2\zeta\omega + \omega_n^2} \tag{24}$$

We want the closed-loop step response $s_m(t)$ using the model plant T_{model} to be as close as possible to the actual step response $s_o(t)$. The step response using T_{model} is shown with the solid line in Fig. 6. Let us consider the cost function

$$error = \frac{1}{3}\int_0^3 (s_m(t) - s_o(t))^2 dt, \tag{25}$$

where $s_o(t)$ is the step response for any choice of $(\hat{K}_p, \hat{K}_d)$. By plotting the contour of the error in terms of $(\hat{K}_p, \hat{K}_d)$ in Fig. 7, we find the global minimum of the error to occur at $(\hat{K}_p = 1.47, \hat{K}_d = -0.15)$. The step response corresponding to this choice of $(\hat{K}_p, \hat{K}_d)$ is shown with the solid line with a circle in Fig. 6, which is closer to the model step response than the other two. △

Table 2: Evaluated points for contours in Example 3.4

x	y	$error$	x	y	$error$	x	y	$error$
0.9	-0.1	30.44	1.5	-0.1	3.37	1.9	-0.1	31.38
2	0	15.71	2.15	-0.1	6.90	2.2	-0.19	5.06
2.3	-0.2	6.13	2.5	0.2	14.15			

IV. CONCLUSIONS

For stable plants whose poles have negative real-parts less than a pre-specified $-h$, we obtained a sufficient condition for existence of PID-controllers that achieve integral-action and closed-loop poles with real-parts less than $-h$. We proposed a systematic design procedure, which allows freedom in the choice of parameters. We showed in an example how this freedom can be used to improve a single-input single-output system's performance. Extending the optimal parameter selection to MIMO systems would be a challenging goal.

These results are limited to stable plants. Future directions of this study will involve extension to certain classes of unstable MIMO plants. In addition, optimal parameter selections for the MIMO case will be explored.

REFERENCES

[1] K. J. Aström and T. Hagglund, *PID Controllers: Theory, Design, and Tuning, Second Edition*, Research Triangle Park, NC: Instrument Society of America, 1995.

[2] M. Morari, "Robust Stability of Systems with Integral Control," *IEEE Trans. Autom. Control*, 47: 6, pp. 574-577, 1985.

[3] M.-T. Ho, A. Datta, S. P. Bhattacharyya, "An extension of the generalized Hermite-Biehler theorem: relaxation of earlier assumptions," *Proc. 1998 American Contr. Conf.*, pp. 3206-3209, 1998.

[4] G. J. Silva, A. Datta, S. P. Bhattacharyya, *PID Controllers for Time-Delay Systems*, Birkhäuser, Boston, 2005.

[5] C.-A. Lin, A. N. Gündeş, "Multi-input multi-output PI controller design," *Proc. 39th IEEE Conf. Decision & Control*, pp. 3702-3707, 2000.

[6] A. N. Gündeş, A. B. Özgüler, "PID stabilization of MIMO plants," *IEEE Transactions on Automatic Control*, to appear.

[7] G. C. Goodwin, S. F. Graebe, M. E. Salgado, *Control System Design*, Prentice Hall, New Jersey, 2001.

[8] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

[9] K. H. Johansson, "The quadruple-tank process: A multivariable laboratory process with an adjustable zero," *IEEE Trans. Control Systems Technology*, 8, (3), pp. 456-465, 2000.

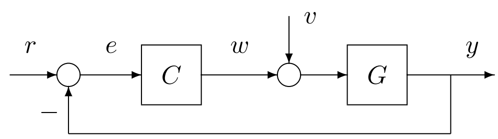

Fig. 1. Unity-Feedback System $Sys(G, C)$.

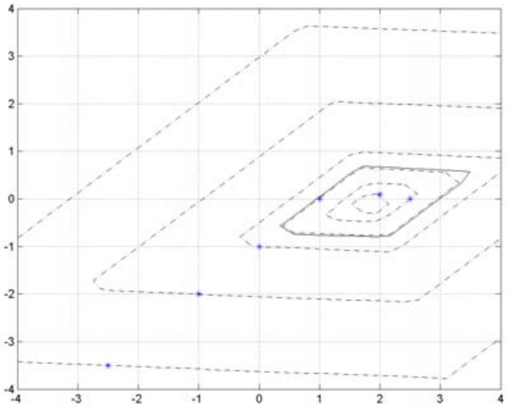

Fig. 2. Contour of $\gamma(\hat{K}_p, \hat{K}_d)$ for Example 3.1

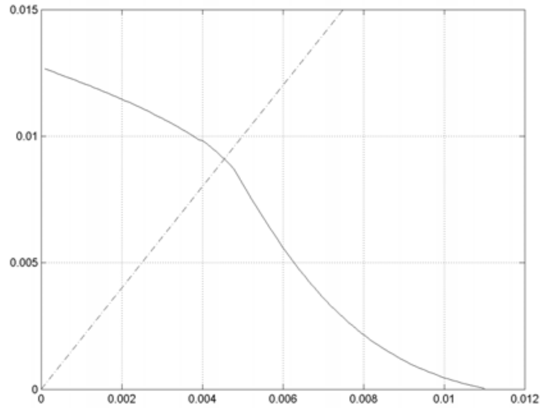

Fig. 5. Finding h_{max} for Example 3.3

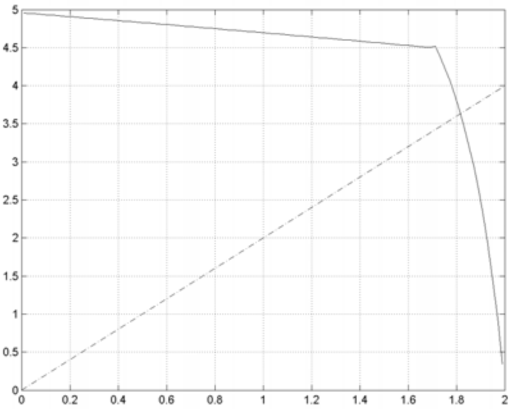

Fig. 3. Finding h_{max} for Example 3.1

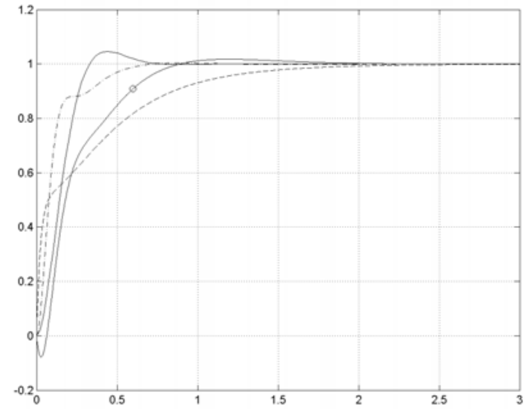

Fig. 6. Step responses for Example 3.4

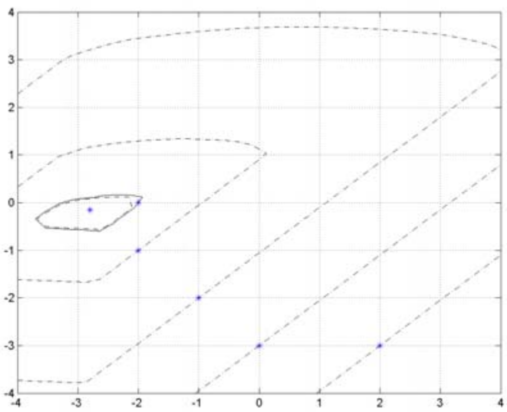

Fig. 4. Contour of $\gamma(\hat{K}_p, \hat{K}_d)$ for Example 3.2

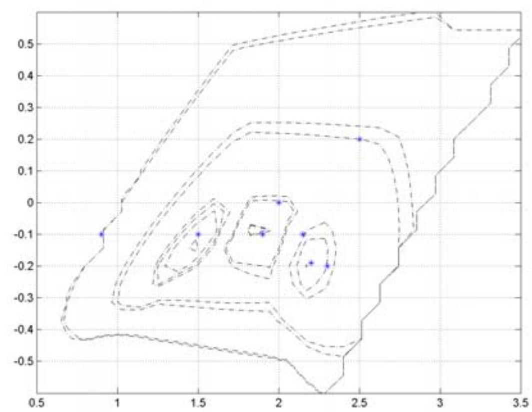

Fig. 7. Contour of $error(\hat{K}_p, \hat{K}_d)$ for Example 3.4

ISBN:978-988-98671-6-4