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Analytical Solution to Optimal Control by Orthogonal Polynomial Expansion

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***Abstract*— In this paper the use of orthogonal polynomials for obtaining an analytical approximate solution to optimal control problems with a weighed quadratic cost function, is proposed. The method consists of using the Orthogonal Polynomials for the expansion of the state variables and the control signal. This expansion results in a set of linear equations, from which the solution is obtained. A numerical example is provided to demonstrate the applicability and effectiveness of the proposed method..**

***Index Terms*— Optimal control, Orthogonal Polynomials, Spectral Method, Legendre Polynomials, Riccati Method.**

I. Introduction

The goal of an optimal controller is the determination of the control signal such that a specified performance criterion is optimized, while at the same time specific physical constraints are satisfied. Many different methods have been introduced to solve such a problem for a system with given state equations. The most popular is the Riccati method for quadratic cost functions however this method results in a set of usually complicated differential equations which must be solved recursively [1]. In the last few decades orthogonal functions have been extensively used in obtaining an approximate solution of problems described by differential equations [2-4]. The approach, also known as the spectral method [5], is based on converting the differential equations into an integral equation through integration. The state and/or control involved in the equation are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. The form of the operational matrix of integration depends on the particular choice of the orthogonal functions like Walsh functions [6], block-pulse functions [7], Laguerre series [8], Jacobi series [9-10], Fourier series [11], Bessel series [12], Taylor series [13], shifted Legendre [14], Chebyshev polynomials [15], and Hermite polynomials [16] and Wavelet functions [17]. In this paper apart from the shifted Legendre Polynomials, new set of Orthogonal Polynomials are considered based on the requirement of the problem. This method proves to be fairly precise from simulation results and may be expanded to a vast range of cost functions.

Since only linear systems are considered in this paper, the state space equations and the cost function are considered in the following formats:

$$\dot{X}(t) = AX(t) + Bu(t) \tag{1}$$

In which A and B are constant matrices. $X(t)$ is the state vector and $u(t)$ is the control signal.

And the cost function:

$$J = \frac{1}{2}\int_0^{t_f} [t^k X^T(t) Q X(t) + r u^2(t)] dt \tag{2}$$

In which Q is a positive definite matrix, and r and k are constant values. t_f is the final time and specified.

As it can be seen in equation (2) the finite horizon cost function consists of the weighting function t^k for the state vector. In the presented method here, $X(t)$ and $u(t)$ are expanded based on orthogonal polynomials .The main reason for the use of such an expansion is that it results in the simplification of the cost function J , this is due to the fact that the integral of the multiplication of non-identical orthogonal terms is zero. One reasonable approach is expansion based on shifted Legendre polynomials, however because of the presence of the term t^k in J new weighed orthogonal polynomials must be obtained in order to be useful for solving such a problem.

In this paper first some useful properties of orthogonal polynomials and means of obtaining them based on these properties are presented in section 2. In section 3, a method of obtaining an analytical approximate answer to the optimal control problem defined by equations 1 and 2 is presented based on the spectral method. The results of the presented method are provided for a numerical example and compared with those of the classical Riccati method in section 4 and finally a conclusion of the overall work is given in section 5.

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II. SOLVING OPTIMAL CONTROL PROBLEMS BY THE USE OF ORTHOGONAL POLYNOMIALS

In the presented method, $u(t)$ and $X(t)$ must be expanded based on orthogonal polynomials in order to solve the optimal control problem. However due to the presence of the term t^k in J , $X(t)$ may not be presented by the shifted Legendre Polynomials. In order to solve this problem, weighed orthogonal polynomials with the weighting function t^k , are defined which will be represented with $\varphi_i(t)$. Then $X(t)$ is expanded based on these new orthogonal polynomials $\{\varphi_i(t)\}$. Some of the characteristics of orthogonal functions are recapitulated next.

A. Orthogonal polynomials

The definition of orthogonal polynomials $\psi_n(t)$ and some of their features are presented below:

$$\int_a^b W(t)\psi_i(t)\psi_j(t)dt = \begin{cases} \chi_i & i=j \\ 0 & i \neq j \end{cases} \tag{3}$$

In which $W(t)$ is the weight function.

The expansion of an arbitrary function $f(t)$ on the $[0,t_f]$ region is as follows:

$$f(t) = \sum_{i=0}^{N} C_i \psi_i(t) \tag{4}$$

In which:

$$C_i = \frac{1}{\chi_i}\int_0^{t_f} t^k f(t)\psi_i(t)dt \tag{5}$$

One property of orthogonal polynomials is [19]:

$$\int \psi(t) = \kappa\psi(t) \tag{6}$$

Now for a weight function $W(t)=1$, we have the shifted Legendre polynomials:

$$\{P_i(t)\} = \{\psi_i(t)\} \tag{7}$$

And:

$$\int_0^{t_f} PP^T dt = \gamma \tag{8}$$

In which:

$$P(t) = [P_0(t), P_1(t), \ldots, P_n(t)]^T \tag{9}$$

$$\gamma = diag[\gamma_0, \gamma_1, \ldots, \gamma_n] \tag{10}$$

$$\gamma_i = \int_0^{t_f} P_i^2(t)dt \tag{11}$$

Now for a weight function $W(t)=t^k$, we define the polynomials $\{\varphi_i(t)\}$ shown below:

$$\{\varphi_i(t)\} = \{\psi_i(t)\}$$

$$\int_0^{t_f} t^k \varphi\varphi^T dt = \alpha \tag{12}$$

In which:

$$\varphi = [\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t)]^T \tag{13}$$

$$\alpha = diag[\alpha_0, \alpha_1, \ldots, \alpha_n] \tag{14}$$

$$\alpha_i = \int_0^{t_f} t^k \varphi_i^{\ 2}(t)dt \tag{15}$$

The integral expansion of the weighed orthogonal polynomials $\varphi_i(t)$ based on the set $\{\varphi_i(t)\}$ is as follows:

$$\int_0^t \varphi dt \cong D\varphi \tag{16}$$

The method for obtaining D is explained in the appendix.

It is worth noting that when dealing with functions and their derivatives, the property mentioned in equation (16) is of major importance.

B. Obtaining orthogonal polynomials

Different methods may be used to obtain orthogonal polynomials, namely, most commonly, the Graham-Schmidt method [18].However this method is computationally cumbersome for large sets and may produce inaccurate results. Here, another method is introduced which is based on the properties of orthogonal polynomials, for $W(t)=t^k$. The presented method is computationally effective and precise compared to the Graham Schmidt method due to the fact that approximations in numerical integration needed for the Graham Schmidt method are not required for the presented method. It is assumed that:

$$\varphi = ST \tag{17}$$

Hence:

$$\int_0^{t_f} t^k \varphi\varphi^T dt = \int_0^{t_f} t^k STT^T dt = S\left(\int_0^{t_f} t^k TT^T dt\right)S^T = SYS^T \tag{18}$$

In which:

$$T = [t^0, t^1, \ldots, t^n]^T$$

and

$$Y=\begin{bmatrix}\frac{t_f^{K+1}}{K+1} & \frac{t_f^{K+2}}{K+2} & \cdots & \frac{t_f^{K+n+1}}{K+n+1}\\ \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ \frac{t_f^{K+n+1}}{K+n+1} & \frac{t_f^{K+n+2}}{K+n+2} & \cdots & \frac{t_f^{K+2n+1}}{K+2n+1}\end{bmatrix}_{(n+1)\times(n+1)} \quad (19)$$

Now because Y is real symmetrical and positive definite it can be transformed to the form below by the Cholesky method:

$$Y=LL^T \quad (20)$$

Now based on the definition of orthogonal polynomials, we can assume that :

$$SYS^T=I \quad (21)$$

In which I is an $(n+1)\times(n+1)$ identity matrix. This would mean that:

$$S=L^{-1} \quad (22)$$

So we can now obtain S , and finally $\varphi=ST$.

III. Formulation of an Optimal Control Problem Using Weighed Orthogonal Polynomials

Now the optimal control problem described in equations (1) and (2) are formulized by the use of the orthogonal polynomials described in section 2. First $\dot{x}_i(t)$ will be expanded based on the set $\{\varphi_i(t)\}$ up to degree n :

$$\dot{x}_i=e_i^T\varphi\Rightarrow\dot{X}\cong E\varphi \quad (23)$$

In which:

$$E=\begin{bmatrix}e_{10} & e_{11} & \dots & e_{1n}\\ . & . & & .\\ . & . & \dots & .\\ . & . & & .\\ e_{m0} & e_{m1} & \dots & e_{mn}\end{bmatrix}_{m\times(n+1)},\ X=\begin{bmatrix}x_1(t)\\ .\\ .\\ .\\ x_m(t)\end{bmatrix}_{m\times 1} \quad (24)$$

Where m is the order of the system. It must be noted that E is unknown and will be obtained later on.

And $u(t)$ the control signal will be expanded based on the set $\{P_i(t)\}$ up to degree n :

$$u=\beta^TP \quad (25)$$

In which:

$$\beta=[\beta_0,\beta_1,\dots,\beta_n]^T_{1\times(n+1)}$$

Note that each of the functions $P_i(t)$ may be expanded based on the set $\{\varphi_i(t)\}$ or vice versa in equation (24). Therefore we have:

$$P=c^T\varphi \quad (26)$$

In which c is a $(n+1)\times(n+1)$ square matrix.

Substituting into equation (25) we have:

$$u=\beta^Tc^T\varphi \quad (27)$$

β is also unknown and will be obtained later on. Now by the use of equations (16) , (23) and (27) we can write:

$$X=ED\varphi+VP\Rightarrow X=(ED+Vc^T)\varphi \quad (28)$$

In which:

$$V=\begin{bmatrix}x_{10} & 0 & 0 & \dots & 0\\ . & . & . & \dots & .\\ . & . & . & \dots & .\\ . & . & . & \dots & .\\ x_{m0} & 0 & 0 & \dots & 0\end{bmatrix}_{m\times(n+1)} \quad (29)$$

Where x_{i0} in equation (29) is the initial condition for the state variable $x_i(t)$. By replacing equations (23), (27) and (28) in to the state equation (1) we have:

$$E\varphi=A(ED+Vc^T)\varphi+B\beta^Tc^T\varphi \quad (30)$$

Therefore the following equation holds for all values of t :

$$E-A(ED+Vc^T)-B\beta c^T=0 \quad (31)$$

And the matrix I may be defined as:

$$I\overset{\Delta}{=}Vec[(E-A(ED+Vc^T)-B\beta c^T)^T] \quad (32)$$

So $I=0$.

By replacing the expansions for u and X as formulated in equations (31) and (32) respectively, in the expression for J in equation (2) we have:

$$J=\frac{1}{2}\int_0^{t_f}t^k\varphi^T(ED+Vc^T)^TQ(ED+Vc^T)\varphi dt$$
$$+\frac{1}{2}r\int_0^{t_f}\beta^T\gamma\beta dt \quad (33)$$

By the use of the properties of orthogonal functions $P_i(t)$ and $\varphi_i(t)$, the functional J takes the simpler form of:

$$J=\frac{1}{2}\{trace[\alpha(ED+Vc^T]Q(ED+Vc^T)]$$
$$+r\beta^T\gamma\beta\} \quad (34)$$

For minimizing J with the restriction in equation (32) or (30) we can use the Lagrange coefficients and minimize the following expression instead:

$$\eta(E,\beta,\lambda)=J(E,\beta)+\lambda^T.I(E,\beta) \quad (35)$$

In which λ is the Lagrange coefficient and:

$$\lambda=[\lambda_0,\lambda_1,\dots,\lambda_{m\times n}]^T \quad (36)$$

Now $\eta(E,\beta,\lambda)$ must be minimized, which is done by solving the following equations:

$$\frac{\partial \eta}{\partial e_{ij}}=0 \ , \ \frac{\partial \eta}{\partial \beta_i}=0 \ , \ \frac{\partial \eta}{\partial \lambda_i}=0 \tag{37}$$

After performing the above differentiations and simplification [19], the following important equations are obtained:

$$\frac{\partial trace(\alpha D^T E^T QED)}{\partial vec(E^T)}=2Q\otimes(D\alpha D^T) \tag{38}$$

$$\frac{\partial \lambda^T .vec(D^T E^T A^T)}{\partial vec(E^T)}=(A^T\otimes D)\lambda \tag{39}$$

$$\frac{\partial trace(\alpha D^T E^T QVc^T)}{\partial vec(E^T)}=vec(D\alpha cV^T Q) \tag{40}$$

$$\frac{\partial \lambda^T .vec(D^T E^T A^T)}{\partial \lambda}=vec(D^T E^T A^T) \tag{41}$$

$$vec(D^T E^T A^T)=(A\otimes D^T).vec(E^T) \tag{42}$$

And the set of linear equations are finally obtained:

$$\begin{bmatrix} Q\otimes(D\alpha D^T) & 0 & I-A^T\otimes D \\ I-A\otimes D^T & -B\otimes c & 0 \\ 0 & r\gamma & -B^T\otimes c^T \end{bmatrix}\begin{bmatrix} Vec(E^T) \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} -Vec[D\alpha cV^T Q] \\ Vec[cV^T A^T] \\ 0 \end{bmatrix} \tag{43}$$

In which $\otimes$ is the symbol of Kronecker multiplication and by Vec (matrix) we mean placing the columns of the matrix in consecutive order in one vector.

The unknown variables λ_i , β_i and e_{ij} in (37) are of first order and hence are easily obtainable from solving the linear set of equations in (43). By Solving the set of linear equations in (43) the coefficients e_{ij} and β_i which were used in the expansions represented in equations (25) and (28) are obtained. Therefore we have now obtained the following solution to our original problem:

$$u=\beta^T P \qquad , \qquad X=ED\varphi+VP \tag{44}$$

IV. NUMERICAL EXAMPLE

In this section the result formulated in equation (43) is applied and compared with the result of the classical Riccati method for a specific system with the following specifications:

$$A=\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} ,Q(t)=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ r=0.5 \tag{45}$$

And we wish to minimize the cost function:

$$J=\frac{1}{2}\int_0^{t_f}[t^k X^T(t)Q(t)X(t)+ru^2(t)]dt \tag{46}$$

The problem is solved for $k=0$ and $k=3$, with $n=9$ .The results are shown in Figures 1 and 2 respectively. For $t_f=5$, $k=0$ and $n=9$ the following solutions are obtained:

$$\begin{aligned} X_1 &= -(0.227034\times10^{-4})t^9+(0.634407\times10^{-3})t^8 \\ &-(0.733284\times10^{-2})t^7+(0.462403\times10^{-1})t^6 \\ &-0.170695t^5+0.338268t^4-(0.806054\times10^{-1})t^3 \\ &-1.476029t^2+4.000000t-4.000000 \end{aligned}$$

$$\begin{aligned} X_2 &= -(0.257824\times10^{-6})t^9-(0.204330\times10^{-3})t^8 \\ &+(0.507526\times10^{-2})t^7-(0.513299\times10^{-1})t^6 \\ &+0.277442t^5-0.853478t^4+1.353073t^3 \\ &-0.241816t^2-2.952058t+4.000000 \end{aligned}$$

$$\begin{aligned} u &= (0.450041\times10^{-4})t^9-(0.147546\times10^{-2})t^8 \\ &+(0.181063\times10^{-1})t^7-0.108283t^6+0.310854t^5 \\ &-0.142803t^4-1.899631t^3+6.769460t^2 \\ &-11.43569t+9.047942 \end{aligned}$$

For $t_f=5$, $k=3$ and $n=9$ the following solutions are obtained:

$$\begin{aligned} X_1 &= (0.978057\times10^{-4})t^9-(0.234829\times10^{-2})t^8 \\ &+(0.228522\times10^{-1})t^7-0.112345t^6+0.271386t^5 \\ &-0.198410t^4-0.180791t^3-0.845372t^2 \\ &+4.000000t-4.000000 \end{aligned}$$

$$\begin{aligned} X_2 &= (0.865658\times10^{-8})t^9+(0.880251\times10^{-3})t^8 \\ &-(0.187863\times10^{-1})t^7+0.159966t^6-0.674073t^5 \\ &+1.356930t^4-0.793643t^3-0.542373t^2 \\ &-1.690745t+4.000000 \end{aligned}$$

$$\begin{aligned} u &= -(0.195626\times10^{-3})t^9+(0.557675\times10^{-2})t^8 \\ &-(0.574488\times10^{-1})t^7+0.253152t^6-0.257048t^5 \\ &-1.616614t^4+4.99565t^3-1.232559t^2 \\ &-10.77549t+10.30925 \end{aligned}$$

As it can be seen from the simulation of the results presented method and the classical Riccati method, The results of both have fallen upon one another when plotted as shown in Figure 1 and Figure 2.

V. Conclusion

In this paper we have presented an alternative method for obtaining an analytical approximate solution to optimal control problems with time variant weight in the cost function. The presented method makes use of the properties of orthogonal polynomials and transforms the problem into a linear set of equations. The results of the presented method proved to be accurate by comparison with that of the classical Riccati method.

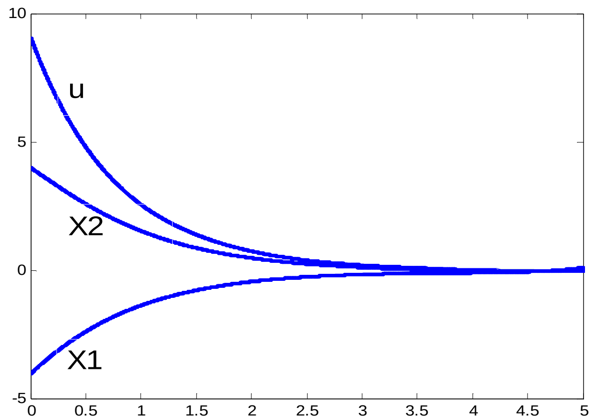

Fig.1. Depicted numerical results for k=0,n=9, of both the presented method and that of the Riccati method.

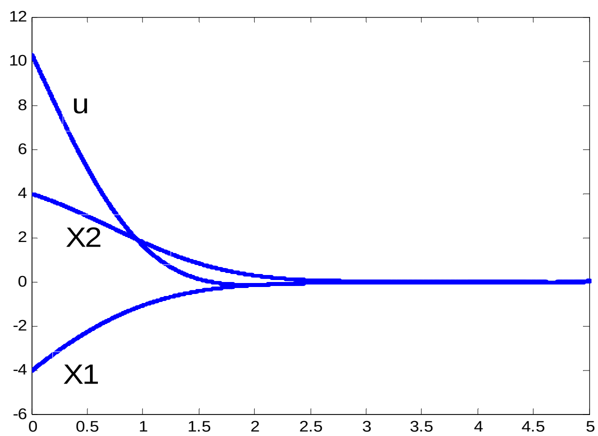

Fig.2. Depicted numerical results for k=3,n=9, of both the presented method and that of the Riccati method.

Appendix

In this Appendix a numerically efficient method is introduced for obtaining the D matrix required in equation 16.

$$\int_a^b W(t)\varphi_i(t)\varphi_j(t)dt = \begin{cases} \alpha_i & i=j \\ 0 & i \neq j \end{cases} \qquad \text{A-1}$$

$$\varphi_n(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 \qquad \text{A-2}$$

$$\int_0^t \varphi dt \cong D\varphi \qquad \text{A-3}$$

In which:

$$\varphi = ST \quad , \quad T = [t^0, t^1, \ldots, t^n]^T \qquad \text{A-4}$$

Where

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_{10} & 1 & 0 & \cdots & 0 \\ S_{20} & S_{21} & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ S_{N0} & \cdots & \cdots & \cdots & 1 \end{bmatrix} \qquad \text{A-5}$$

We know that:

$$\int_0^t \varphi dt = \int_0^t STdt \cong tS\lambda T = \ldots$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & S_{10} & 1 & 0 & \cdots & 0 \\ 0 & S_{20} & S_{21} & 1 & \cdots & \vdots \\ 0 & \vdots & \vdots & \vdots & & 1 \\ 0 & S_{N0} & \cdots & \cdots & \cdots & S_{N,N-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \cdots & \frac{1}{N} \end{bmatrix}$$

$$\cdot \begin{bmatrix} t^0 \\ t^1 \\ \vdots \\ t^N \end{bmatrix} = \hat{S}\lambda T \qquad \text{A-6}$$

And we know that:

$$T = S^{-1}\varphi \qquad \text{A-7}$$

Therefore:

$$\int_0^t \varphi dt \cong \hat{S}\lambda S^{-1}\varphi \Rightarrow D = \hat{S}\lambda S^{-1} \qquad \text{A-8}$$

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