Numerical Solution to a Free Boundary Problem Arising from Mortgage Pricing

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Abstract—**This paper studies a borrower's optimal decision, when he has the choice to make early payment, to close a fixed rate mortgage. Mathematically we use several integral identities to design an effective Newton algorithm to solve for the optimal prepayment curve iteratively. Numerical evidence shows that the algorithm is fast and stable.**

Keywords: mortgage, prepayment, free boundary

1 Introduction

Many option pricing problems are formulated as free boundary problems [7]. The classical example is the valuation of American put option. These free boundary problems usually don't have closed form solutions. Rather, efforts have been focused on finding a fast and effective numerical scheme as well as the asymptotic expansions of the free boundary [1]. Here we consider a mortgage contract with a given duration T (years) and a fixed mortgage interest rate c (year^{-1}). At any time t during the term of the mortgage, the outstanding balance owed, $M(t)$, is decreased in the time interval $[t, t+dt)$ by

$$dM(t) = cM(t)dt - mdt \quad \forall t \le T$$

where $cM(t)dt$ is the interest accrued on the balance and mdt is the payment resulting from a constant continuous rate of payment of m (\$/year). Since the contract expires at $t = T$, the condition $M(T) = 0$ applies. Solving this ODE we have

$$M(t) = \frac{m}{c}\Big\{1 - e^{c(t-T)}\Big\}.$$

In this contract, the borrower is allowed to terminate the contract at any time t ($t < T$) of his choice by paying off the outstanding balance of $M(t)$ to the contract issuer.

We assume that the borrower always has sufficient funds to pay back the outstanding balance at any time. Then at any moment while the contact is in effect, the decision of the borrower on whether to terminate the contract depends on the rate of (short term) return that an investment can yield on the financial market. In this paper, we shall use the Vasicek model [6] for this short term market return rate, r_t, described by the stochastic differential equation

$$dr_t = k(\theta - r_t)dt + \sigma\, dW_t \tag{1}$$

where k, θ, and σ are assumed to be positive known constants and W_t is the standard Brownian motion. Here the units for k, θ, σ, and W_t are year^{-1}, year^{-1}, year$^{-3/2}$ and year$^{1/2}$ respectively.

Intuitively if an overall market return rate is expected to be low (relative to c) for a certain amount of time, one should choose to terminate the contract early. On the other hand, if the market return rate is strictly larger than c or if an overall market return rate is expected to be higher than c for a certain amount of time, one should choose to defer the closing date by an investment in the market of the capital $M(t)$ less the obligatory payment of m per unit time.

To find such a strategy, we introduce a function $V(r,t)$ being the (expected) value of the contract at time t and current market return rate $r_t = r$. This value can be regarded as an asset that the contract issuer (the mortgage company) possesses, or a fair price that a buyer would offer to the contract issuer in taking over the contract, say, in an issuer's restructuring or liquidation process. The value V is calculated according to the borrower's optimal decision; and the optimal decision for the borrower is to terminate the mortgage contract at the first time that the short term market return rate r_t is below $R(t)$, the unknown optimal prepayment boundary. Since the borrower can terminate the contract by paying $M(t)$ at any time t, we have

$$0 \le V(r,t) \le M(t) \qquad \forall r \in R,\ t \le T.$$

This automatically implies that $V(r,T) = 0$ for all r.

Similar problems have been discussed from option-pricing viewpoint by Buser & Hendershott [2], Epperson, Kau, Keenan, & Muller [3], etc. To address the fact that the Vasicek model is not sufficient to describe the whole term structure, here we assume for simplicity that in this model the market price of risk has been incorporated into the drift $k(\theta - r)$; that is to say, the probability associated with the Brownian motion $\{W_t\}$ is the risk-neutral probability. According to the standard mathematical finance theory [7], the problem can be formulated as a free

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boundary problem. Assuming the necessary regularity of V, what we need to find is a (R, V) such that

$$\begin{cases} V_t + \frac{\sigma^2}{2}V_{rr} + k(\theta - r)V_r - rV + m = 0, \\ \qquad \text{if } r > R(t), t < T, \\ V(r,t) < M(t), \\ \qquad \text{if } r > R(t), t < T, \\ V((R(t),t) = M(t), \quad \forall t \leq T, \\ V_r(R(t),t) = 0, \quad \forall t \leq T, \\ V(r,T) = 0, \quad \forall\, r \geq R(T) = c. \end{cases} \tag{2}$$

The mathematical formulation for problem and the well-posedness (2) have been carried out by Bian, Jiang, and Yi [5]. Here we are interested in the numerical solution to the problem. Because the original system is rather complicated, we first transform the PDE into a standard heat equation, hence the solution of the problem can be easily written as a function of the well-known heat kernel. We then derive several integral identities. Based on these integral identities, we are able to design a fast and effective Newton scheme to solve the free boundary iteratively. Numerical examples and the performance of our numerical program has been provided.

2 Integration Equations

The main purpose of this section is to make certain transformations to simplify the mathematical analysis of the equation for V; namely, we transfer the original Black–Scholes type equation into a heat equation. After the transformation, one can devise an effective Newton scheme to solve the optimal prepayment curve iteratively.

We shall use the following new variables

$$\tau := T - t, \tag{3}$$

$$\psi(r,\tau) := \frac{c}{m}\Big\{M(t) - V(r,t)\Big\}. \tag{4}$$

First note that τ is the time to expiry. This is convenient because we always know the value of the contract, early terminated or not, must have value zero at expiry. Secondly, note that ψ is a dimensionless quantity measuring the advantage of deferring termination. $M(t) - V(r,t)$ represents the amount of premium loss if the contract is closed at the current time t and market rate r and if, according to our theoretical result, it is actually not optimal to do so. Multiplying by the ratio c/m is nonessential in terms of financial interpretation, but is convenient for mathematical analysis.

In the new variables, (2) is equivalent to

$$\begin{cases} \psi_\tau - \sigma^2 2\psi_{rr} - k(\theta - r)\psi_r + r\psi = (r-c)(1 - e^{-c\tau}), \\ \qquad \text{if } \psi(r,\tau) > 0, \tau > 0, \\ \\ 0 \leq \psi(r,\tau) \leq 1 - e^{-c\tau}, \quad \forall\, \tau \geq 0, r \in R. \end{cases}$$

We remark that the constraint $\psi(r,\tau)1 - e^{-c\tau}$ on the upper bound, which corresponds to the original constraint $V0$, is not needed, since one can show that $1 - e^{-c\tau}$ is a super-solution so that by comparison

$$\psi(r,\tau) < 1 - e^{-c\tau} \qquad \forall\, r \in R, \tau > 0.$$

Also, differentiating in r one sees that

$$\Big\{\frac{\partial}{\partial\tau} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - k(\theta - r)\frac{\partial}{\partial r} + (r+k)\Big\}\psi_r = 1 - e^{-c\tau} - \psi$$

is greater or equal 0 for $\psi > 0$. The maximum principle then implies that $\psi_r(r,t) \geq 0$ for all $r \in R, \tau \geq 0$. Therefore, there exists a function $R : (-\infty, T) \to [-\infty, \infty)$ such that for each $\tau > 0$,

$$\psi(r,\tau) > 0 \qquad \Longleftrightarrow \qquad r > R(T - \tau).$$

Let $h = h(r,\tau)$ be a function to be determined shortly. We make the change of variables for the unknown function ψ by

$$\phi(r,\tau) := e^{-h(r,\tau)}\psi(r,\tau).$$

Then the constraint for ψ becomes the constraint $\phi \geq 0$ for ϕ. When $\phi > 0$ we have $\psi > 0$ and the differential equation for ψ is transformed to the following differential equation for ϕ:

$$\begin{aligned} \phi_\tau - \sigma^2 2\phi_{rr} - [\sigma^2 h_r + k(\theta - r)]\phi_r + q\phi \\ = (r-c)(1 - e^{-c\tau})e^{-h} \end{aligned}$$

where

$$q := h_\tau - \sigma^2 2h_{rr} - h_r\{\sigma^2 2h_r + k(\theta - r)\} + r.$$

We want to find a special h such that $q \equiv 0$. To this end we choose

$$h(r,\tau) \quad = \quad \frac{k}{\sigma^2}\Big(r + \frac{\sigma^2}{2k^2} - \theta\Big)^2 + \Big(k + \frac{\sigma^2}{2k^2} - \theta\Big)\tau.$$

The equation for ϕ becomes

$$\begin{cases} \phi_\tau - \sigma^2 2\phi_{rr} - \{kr + \frac{\sigma^2}{k} - k\theta\}\phi_r = (r-c)(1 - e^{-c\tau})e^{-h}, \\ \qquad \text{if } \phi > 0, \\ \phi(r,\tau) \geq 0 = \phi(r,0) \quad \forall\, r \in R, \tau > 0. \end{cases}$$

Finally, we make the change of variables

$$\begin{aligned} x &= k^{1/2}e^{k\tau}[r + \frac{\sigma^2}{k^2} - \theta]/\sigma, \\ s &= e^{2k\tau}, \\ u(x,s) &= \frac{2\sqrt{\pi}k^{3/2}}{\sigma}\phi(r,\tau). \end{aligned}$$

Then the system for ϕ becomes

$$\begin{cases} u_s - 1/4u_{xx} = f(x,s), \\ \qquad \text{if } u(x,s) > 0, s > 1, \\ u(x,s) \geq 0 = u(x,1), \\ \qquad \forall\, s > 1, x \in R \end{cases} \tag{5}$$

where

$$f(x,s) = \sqrt{\pi}k^{1/2}(r-c)(1-e^{-c\tau})e^{-2k\tau-h}/\sigma.$$

Note that f can be written as

$$f(x,s) = \sqrt{\pi}(s^{\gamma}-1)s^{-\nu-1}(x-\beta\sqrt{s})e^{-(\frac{x}{\sqrt{s}}-\alpha)^2},$$

where α, β, γ, and ν are dimensionless constants given by

$$\alpha := \frac{\sigma}{2k^{3/2}}, \quad \gamma := \frac{c}{2k}, \quad \beta := \frac{\sqrt{k}}{\sigma}\Big(c-\theta+\frac{\sigma^2}{k^2}\Big),$$
$$\nu := 1+\frac{\sigma^2}{4k^3}+\frac{c-\theta}{2k}.$$

Once we find the free boundary $x = X(s)$ such that for each $s > 1$,

$$u(x,s) > 0 \quad \Longleftrightarrow \quad x > X(s),$$

the optimal boundary $r = R(t)$ for terminating the mortgage is given by

$$R(t) = c + \sigma\Big[\frac{X(s)}{\sqrt{s}} - \beta\Big]/\sqrt{k}. \tag{6}$$

Using a standard theory of variational inequalities (e.g. [4]), one can show (c.f. [5]) that there exists X such that

$$\begin{cases} u_s - 14u_{xx} = f(x,s)\,\mathbf{1}_{[X(s),\infty)}(x) \\ \text{ in } \;R\times(1,\infty), \\ u(x,s) > 0 \quad \forall x > X(s), s > 1, \\ u(x,1) = 0 \quad \forall x \in R, \\ u(x,s) = 0 \quad \forall s > 1, x \le X(s) \end{cases} \tag{7}$$

where

$$\mathbf{1}_{[z,\infty)}(x) = 1 \text{ if } x \ge z, \qquad \mathbf{1}_{[z,\infty)}(x) = 0 \quad \text{if} \quad x < z.$$

Here the differential equation for u is in the L^p sense, i.e., both u_s and u_{xx} are in $L^p_{loc}(R\times[0,\infty))$ for any $p \in (1,\infty)$. Denote by

$$\Gamma(x,s) := \frac{e^{-x^2/s}}{\sqrt{\pi s}}$$

the fundamental solution associated with the heat operator $\partial_s - 1/4\partial^2_{xx}$. Using Green's identity, the solution u to the differential equation in (7) can be expressed as

$$u(x,s) = \int_1^s d\varsigma \int_{X(\varsigma)}^{\infty} \Gamma(x-y, s-\varsigma)f(y,\varsigma)\,dy \tag{8}$$

for $\forall x \in R, s \ge 1$.

Using the facts

$$u(X(s),s) = 0, \tag{9}$$

$$u_x(X(s),s) = 0, \tag{10}$$

$$u_{xx}(X(s)+,s) - u_{xx}(X(s)-,s) = -4f(X(s),s). \tag{11}$$

one can derive the following three integral identities for the unknown free boundary function $X(\cdot)$ defined on $(1,\infty)$:

$$0 = \int_1^s d\varsigma \int_{X(\varsigma)}^{\infty} \Gamma(X(s)-y, s-\varsigma)f(y,\varsigma)\,dy = 0, \tag{12}$$

$$0 = \int_1^s d\varsigma \int_{X(\varsigma)}^{\infty} \Gamma_x(X(s)-y, s-\varsigma)f(y,\varsigma)\,dy = 0, \tag{13}$$

$$2f(X(s),s) = -\int_1^s \Gamma_x(X(s)-X(\varsigma), s-\varsigma)f(X(\varsigma),\varsigma)d\varsigma$$
$$+\int_s^1 \int_{X(\varsigma)}^{\infty} \Gamma_x(X(s)-y, s-\varsigma)f_y(y,\varsigma)\,dyd\varsigma. \tag{14}$$

Once these integral identities are established, we can try to design a Newton scheme to solve for the free boundary iteratively. First of all, we can verify that $X(1) = \beta$. Financially this means that as time approaches to expiry date, the optimal prepayment boundary must approach to the mortgage rate c, otherwise an arbitrage opportunity will be possible. The initial value of the free boundary $X(1)$ is known, and at each moment $s > 1$, the value of the free boundary $X(s)$ must be chosen such that each of the above integral identities hold. This provides the theoretical foundation for our numerical schemes.

3 Numerical Algorithm and Results

Recall the original optimal prepayment curve R is transformed into X in the new system, we now seek to numerically solve X from the integral equation (13). For this, we define an operator Q from $\rho \in C^1((1,\infty))$ to $Q[\rho]$ by

$$Q[\rho](s) \quad := \quad \int_s^1 \int_{\rho(\varsigma)}^{\infty} \Gamma_x(\rho(s)-y, s-\varsigma)f(y,\varsigma)\,dyd\varsigma.$$

Thus, our problem is to find $X \in C([1,\infty)) \cap C^{\infty}((0,\infty))$ such that $Q[X] \equiv 0$. For this, we use Newton's method.

Now we use Newton's method to devise an iteration scheme for the unknown function X. Suppose we have already found X in $[1, s-\Delta s]$ and want to find X on $(s-\Delta s, s]$. Picking an initial guess $X^{old}(s)$, say $X^{old} \equiv X(s-\Delta s)$ on $[s-\Delta s, s]$. We can find an iterative update scheme from X^{old} to X^{new} according the following rationale. Let $\zeta = X(s) - X^{old}(s)$ be the amount of unknown correction needed. Then $X^{old} = X - \zeta$ and using $Q[X](s) = 0$ we have

$$Q[X^{old}](s) = Q[X-\zeta](s) - Q[X](s) \approx 2f(X(s),s)\zeta(s).$$

This gives us the approximation formula for the correction ζ in $X^{new} = X^{old} + \zeta$:

$$\zeta(s) \approx \frac{Q[X^{old}](s)}{2f(X(s), s)}.$$

Thus, we have the following Newton scheme, in a continuous setting,

$$X^{new}(\varsigma) = X^{old}(\varsigma) + \frac{Q[X^{old}](\varsigma)}{2f(X^{old}(\varsigma), \varsigma)} \quad \forall \varsigma \in (s - \Delta s, s].$$

We remark that in the interval $(1, 1+\Delta s]$, one could pick the very first initial guess $X^{old} \equiv \beta$.

We propose the following scheme for the existence of a solution X to (13). Let $1 = s_0 < s_1 < s_2 < \cdots$ be mesh points in the sense that $\Delta s_n = s_n - s_{n-1}$ is not large (so that $o(1)$ is indeed small). Our objective is to find the a solution X to

$$Q[X] \equiv 0$$

starting with $X(0) = \beta$. We find iteratively the function X on $(s_{n-1}, s_n]$, for $n = 1, 2, \cdots$, via the following

$$\begin{cases} X^0(\varsigma) = X(s_{n-1}), \quad \forall \varsigma \in (s_{n-1}, s_n], \\ X^{q+1}(\varsigma) = X^q(\varsigma) + \frac{Q[X^q](\varsigma)}{2f(X^q(\varsigma),\varsigma)}, \\ \qquad \forall \varsigma \in (s_{n-1}, s_n], \ q = 0, 1, \cdots, \\ X(\varsigma) = \lim_{q\to\infty} X^q(\varsigma) \quad \forall \varsigma \in (s_{n-1}, s_n]. \end{cases} \qquad (15)$$

Setting $X_0 = \beta$ and a "ghost" value $X_{-1} = \beta + 0.334\sqrt{s_1 - 1}$, we can calculate $\{X_n\}$ iteratively for $n = 1, 2, \cdots$ by the following scheme

$$\begin{cases} z_0 &= X_{n-1} + \frac{X_{n-1}-X_{n-2}}{s_{n-1}-s_{n-2}}(s_n - s_{n-1}), \\ z_{q+1} &= z_q + \dfrac{\bar{Q}_n(z_q)}{2f(z_q, s_n)}, \quad q = 0, 1, 2, \cdots, \\ X_n &= z_{q+1} \quad \text{if} \ |z_{q+1} - z_q| \le \varepsilon. \end{cases} \qquad (16)$$

where ε is a given tolerance level.

When ε is set to be 5×10^{-7}, the average number of iteration needed is about 0.2, i.e., in most of the calculation, q in (16) is equal to 0. The rate of convergence is observed by numerical experimentation to be about $O((\Delta s)^{3/2})$:

$$X(s_n) - X_n = O(\Delta s)^{3/2}).$$

That is, when the mesh size Δs is halved, the error reduces by a factor $2\sqrt{2} = 2.8$.

The above claim is indeed supported by our numerical experiments. To make our numerical results convincing enough, we have tested the whole range of all the parameters appearing in the Vasicek model, which have been obtained via maximum likelihood method. Of course we cannot cover continuously all the values of the parameters, but we tried with reasonable discrete values. We did numerical experiments by changing only one parameter, say c, for instance, at one time, and kept the three others fixed. Since historically c varies from 0.01 to 0.08, we simulated all cases for c=0.01, 0.02, ..., 0.08. Then we did the same for θ, and so on. It is based on a large amount of numerical experimentation we made claims about our numerical convergence.

The following table is the convergence rates of our numerical method. Here we keep $\theta = 0.05, k = 0.15, \sigma = 0.015$, and we change the values of c. Similar results have been achieved for other possible cases. One can see that although the convergence rate may start at very different values, its stationary value is always around 2.83.

c	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
0.01	2.7705	2.8835	2.9062	2.8998
0.02	2.6989	2.7929	2.8253	2.8355
0.03	2.6111	2.7127	2.7620	2.7882
0.04	2.5097	2.6399	2.7105	2.7521
0.05	2.3260	2.5309	2.6419	2.7063
0.06	1.0619	1.9848	2.3489	2.5317
0.07	3.6402	3.5464	3.4403	3.3336
0.08	3.1452	3.0838	3.0249	2.9758
0.09	3.0215	2.9882	2.9522	2.9211
0.10	2.9568	2.9420	2.9187	2.8969

c	$N = 2048$	$N = 4096$	$N = 8192$
0.01	2.8869	2.8729	2.8521
0.02	2.8361	2.8377	2.8342
0.03	2.8017	2.8128	2.8164
0.04	2.7766	2.7948	2.8041
0.05	2.7475	2.7730	2.7899
0.06	2.6373	2.7002	2.7413
0.07	3.2331	3.1451	3.0704
0.08	2.9368	2.9073	2.8854
0.09	2.8964	2.8778	2.8639
0.10	2.8789	2.8652	2.8549

4 Conclusion

Assuming the short term rate of market return follows the Vasicek Model, we formulate the mortgage valuation problem as a free boundary problem. The main PDE is transformed into a heat equation after a series of change of variables. Several integral equations are derived and then used to design an effective and efficient numerical scheme for computing the free boundary. Our method is demonstrated to be stable and fast, and the convergence rate is about 3.

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