Proceedings of the World Congress on Engineering and Computer Science 2008
WCECS 2008, October 22 - 24, 2008, San Francisco, USA

An Algorithm for the Pricing of Path-Dependent American Options using Malliavin Calculus

Henry Schellhorn and Hedley Morris *

Abstract–**We propose a recursive scheme to calculate backward the values of conditional expectations of functions of path values of Brownian motion. This schemes is based on the Clark-Ocone formula in discrete time. We construct an algorithm based our scheme to effectively calculate the price of American options on securities with path-dependent payoffs. Our algorithm remedies the decrease of performance experienced by regression-based Monte Carlo when the dimensionality of the necessary regressands becomes large due to path-dependence.**

Keywords: Malliavin calculus, Monte Carlo methods, optimal stopping.

1 Introduction

We consider a finite horizon optimal stopping problem. The uncertainty is described by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\{\mathcal{F}_t\}$ is the filtration generated by one-dimensional[1] Brownian motion W, and P is the risk-neutral measure. An American (more accurately a Bermudan) option can be exercised at time steps $\Delta, ..., M\Delta$. When exercised at time $m\Delta$, the option holder receives the $\mathcal{F}_{m\Delta}$-adapted payoff $h(m\Delta)$ which is a function of values taken by Brownian motion along its path:

$$h(m\Delta) = F(W(\Delta), ..., W(m\Delta), m) \qquad (1)$$

Without loss of generality we assume the interest rate is zero. By standard arguments, the price at time $m\Delta$ of the option, which we call $V(m\Delta)$ is obtained by backward induction:

$$\begin{aligned} V(M\Delta) &= h(M\Delta) && (2)\\ V(m\Delta) &= \max\{h(m\Delta), E[V((m+1)\Delta)|\mathcal{F}_{m\Delta}]\} && (3)\\ 0 &\leq m \leq M-1 \end{aligned}$$

This simple overall strategy has been plagued numerically by what many authors call the "curse of dimensionality", namely that in most models, the dimension of the state variables generating the information is too large to calculate an approximation $\hat{C}$ of the continuation value:

$$\hat{C}(m\Delta) \simeq E[V((m+1)\Delta)|\mathcal{F}_{m\Delta}]$$

Optimal stopping problems are usually analyzed as Markov decision problems. In that formulation, the value of the state variables at time t is sufficient to determine the continuation value of the option at time t. By augmenting the state space with lagged values of the state variables if necessary, the information contained in the (augmented) state variables at time t contains all the previous information, thus resulting often in a very large state space. This curse of dimensionality affects not only Markov chain implementations but also direct applications ("Monte Carlo on Monte Carlo") of the Monte Carlo method to calculate conditional expectations. Thus several techniques have been devised to improve Monte Carlo. Currently, practitioners seem to favor "regression-based" methods, such as the Tsitsiklis and Van Roy [3] algorithm (TVR) or the Longstaff and Schwartz [4] algorithm. Apparently, regression algorithms solve the "curse of dimensionality" problem: the number of scenarios does not increase exponentially with the number of time steps as in the "Monte Carlo on Monte Carlo" method. However, like in other methods, it re-enters the scene in disguise because the number of approximating functions required increases with the dimensionality of the state, as has been documented by Glasserman and Yu [5] and Egloff [6].

In [7] Schellhorn developed a "backward Taylor expansion" using the Clark-Ocone formula, which allowed us to calculate backward recursively conditional expectations of functionals based on the expected value of the Malliavin derivatives of all orders of the same functional at the next time step. We propose in this article a numerical scheme that allows to calculate the latter without having an explicit analytical representation. We still need to add scenarios to numerically calculate derivatives. However the key observation is that our algorithm bypasses the "curse of dimensionality" because of its use of derivation instead of integration. More specifically we need $(J+1)M$ times (M=number of time steps,J=order) as many random numbers. This is an improvement as running $(J+1)M\Omega$ scenarios beats running Ω^M scenarios (the benchmark in all these algorithms), especially since as we shall see these scenarios are almost identical, and almost no calculation is needed in the scenarios [2].

*Henry Schellhorn (Henry.Schellhorn@cgu.edu) and Hedley Morris (hedley.morris@cgu.edu)are with the School of Mathematical Sciences, Claremont Graduate University, CA 91711,USA.

[1] All the results carry to higher dimension, with higher notational cost.

[2] To calculate all the (first) derivatives of a function of 3 variables, one needs only 4 point evaluations and not 8. Likewise, to calculate all the first derivatives of a function of M variables, one

Note that if the maximum function in (4) were infinitely differentiable, the optimal stopping algorithm would need only one scenario, i.e., $\Omega = 1$. The formula above for the complexity of the algorithm suggests that there is an optimal tradeoff between the number of scenarios and the order of the (truncated) Taylor expansion.

The first scheme was already proposed in Schellhorn [7], with some errors that rendered its application difficult. The second scheme bypasses the use of differentiating the approximate value of the option.

Fournie et al., [8, 9] show how to use the Malliavin integration by parts formula to numerically calculate this conditional expectation as the ratio of two well-behaved unconditional expectations. Bally, Caramellino and Zanette [1] extended these results, and incorporated them in an algorithm to price and hedge American options. Fujiwara and Kijima [2] extend their work to American options with a mild path-dependency. We found it difficult to generalize their result to heavily path-dependent options. Our approach uses different tools from theirs.

2 Recursive Calculation of Conditional Expectations

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space, where $\mathcal{F}_t$ is the filtration generated by Brownian motion W_t. The problem is to calculate the conditional expectation at time $i\Delta$ of a $\mathcal{F}^d_{(m+1)\Delta}$- measurable random variable F, where $m \geq i$, and the filtration $\{\mathcal{F}^d_t\}$ is the "discrete" subfiltration of $\{\mathcal{F}_t\}$, obtained by observing only increments of Brownian motion over time-steps of length Δ. We write $D^j_t F$ for the Malliavin derivative at time t of a functional F.

Example:

Let $\Delta = 1$ and

$$F = (W(1) + W^3(1))W^2(2)(1 + W(3) + W^4(3))$$

Calculating the expectation $E[F]$ at time zero can be done analytically by the law of iterated expectations but it is very laborious. Instead, our method involves only the calculation of the derivatives of F of all orders (in the example case at most 4) with respect to $W(1), W(2)$ and $W(3)$ for only **one** path. This is to be contrasted with Monte carlo simulation, which generates only an approximate solution after simulating **many** paths.

Proposition 2.1 *(Backward Taylor expansion) Let F be a functional for which the Malliavin derivative is well-defined. Then, for $i < m$:*

$$\begin{aligned} E[F|\mathcal{F}^d_{i\Delta}] &= E[F|\mathcal{F}^d_{(m+1)\Delta}] \\ &\quad - \sum_{k=i}^{m}\sum_{j=1}^{\infty}(-1)^{j+1} g_{k,j} E[D^j_{(k+1)\Delta}F|\mathcal{F}^d_{(k+1)\Delta}] \end{aligned} \tag{4}$$

where $g_{k,j}$ are functions which depend only on $W_{(k+1)\Delta} - W_{k\Delta}$ and deterministic terms. We have for instance:

$$\begin{aligned} g_{k,1} &= [W_{(k+1)\Delta} - W_{k\Delta}] \\ g_{k,2} &= \frac{1}{2}[W_{(k+1)\Delta} - W_{k\Delta}]^2 + \frac{1}{2}\Delta \\ g_{k,3} &= \frac{1}{6}[W_{(k+1)\Delta} - W_{k\Delta}]^3 + \frac{1}{2}[W_{(k+1)\Delta} - W_{k\Delta}]\Delta \end{aligned}$$

For simplicity we rewrite (4) in the form:

$$E[F|\mathcal{F}^d_{k\Delta}] = \sum_{j=0}^{\infty} \gamma_{k,j} E[D^j_{(k+1)\Delta}F|\mathcal{F}^d_{(k+1)\Delta}] \tag{5}$$

where:

$$\gamma_{k,j} = \begin{cases} 1 & \text{if } j = 0 \\ (-1)^{j+1} g_{k,j} & \phantom{\text{if }} j > 0 \end{cases}$$

Proposition 2.1 has two implementation issues. First, to be exact, the method is limited to polynomials. Second, while the Malliavin derivatives $D^j_{(k+1)\Delta}F$ can be calculated at all times $(k+1)\Delta$, the conditional expectations $E[D^j_{(k+1)\Delta}F|\mathcal{F}^d_{(k+1)\Delta}]$ cannot. However, using proposition 2.1 recursively, we obtain the following result.

Notation: let $\{s^0_\omega\}$ be the original scenarios for $\omega = 1..\Omega$. We will use the notation $\{s^{m,p}_\omega\}$ for $\omega = 1..\Omega$ and $m = 0..M$, $p = 1..J$ for the scenario consisting of:

$$W(i\Delta, s^{m,p}_\omega) = \begin{cases} W(i\Delta, s^0_\omega) & m = 0 \\ W(i\Delta, s^0_\omega) & \text{if } \; i \neq m; m > 0 \\ W(i\Delta, s^0_\omega) + p\varepsilon & i = m; m > 0 \end{cases} \tag{6}$$

Let

$$\begin{aligned} p_1(\frac{x}{\Delta}) &= \frac{x}{\Delta} \\ p_2(\frac{x}{\Delta}) &= \frac{x^2}{\Delta^2} - \frac{1}{\Delta} \\ p_3(\frac{x}{\Delta}) &= \frac{x^3}{\Delta^3} - \frac{3}{\Delta^2} \\ p_4(\frac{x}{\Delta}) &= \frac{x^4}{\Delta^4} - \frac{6x^2}{\Delta^3} + \frac{3}{\Delta^2} \end{aligned}$$

needs only $2M$ points and not 2^M.

ISBN: 978-988-98671-0-2

with the convention that $p_{-n}(\frac{x}{\Delta}) = 0$ for $n \geq 0$. We define also the coefficients of numerical differentiation:

$$\begin{aligned} l_{0,0} &= 1 \\ l_{1,0} &= -\frac{1}{\varepsilon} \quad l_{1,1} = \frac{1}{\varepsilon} \\ l_{2,0} &= \frac{1}{\varepsilon^2} \quad l_{2,1} = -\frac{2}{\varepsilon^2} \quad l_{2,2} = \frac{1}{\varepsilon^2} \end{aligned}$$

Let us write:

$$F = \varphi(W(\Delta), ..., W(I\Delta))$$

Suppose that we have access to point estimates of $E[F|\mathcal{F}^d_{(m+2)\Delta}]$ but we do not know φ, just that it is a polynomial. We have:

$$\begin{aligned} D_{(m+1)\Delta}F &= D_{(m+2)\Delta}F + \\ & \frac{\partial}{\partial W((m+1)\Delta)}\varphi(W(\Delta), ..., W(I\Delta)) \end{aligned} \tag{7}$$

Thus we can calculate:

$$\begin{aligned} &\frac{\partial}{\partial W((m+1)\Delta, s^0_\omega)}\varphi(W(\Delta, s^0_\omega), ..., W(I\Delta), s^0_\omega)) = \\ &\frac{1}{\varepsilon}[\varphi(W(\Delta, s^0_\omega), ..., W(I\Delta, s^0_\omega) - \\ &\varphi(W(\Delta, s^{m+1,1}_\omega), ..., W(I\Delta, s^{m+1,1}_\omega))] \end{aligned}$$

We can also numerically differentiate to higher order. Namely:

$$\begin{aligned} &\frac{\partial^j}{\partial W^j((m+1)\Delta)}\varphi(W(\Delta, s^0_\omega), ..., W(I\Delta, s^0_\omega)) \\ &= \sum_{p=0}^{j} l_{j,p}\varphi(W(\Delta, s^{m+1,p}_\omega), ..., W(I\Delta, s^{m+1,p}_\omega)) \end{aligned}$$

Lemma 2.1

$$\begin{aligned} E[F|\mathcal{F}^d_{m\Delta}] &= E[F|\mathcal{F}^d_{(m+1)\Delta}] - \sum_{j=1}^{\infty}(-1)^{j+1}g_{m,j}\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix} \\ &\times \sum_{p=0}^{r} l_{k,p}K(k-r, m+1, s^{m+1,p}_\omega) \end{aligned}$$

where

$$K(j,m) = E[Fp_j(W((m+1)\Delta) - W(m\Delta))|\mathcal{F}^d_{m\Delta}]$$

Proof of lemma 2.1:

Taking the expectation of (7) we have:

$$\begin{aligned} &E[D_{(m+1)\Delta}F|\mathcal{F}^d_{(m+1)\Delta}] \\ &= E[D_{(m+2)\Delta}F|\mathcal{F}^d_{(m+1)\Delta}] + E[\frac{\partial F}{\partial W((m+1)\Delta)}|\mathcal{F}^d_{(m+1)\Delta}] \end{aligned}$$

Likewise

$$\begin{aligned} E[D^2_{(m+1)\Delta}F|\mathcal{F}^d_{(m+1)\Delta}] &= E[D^2_{(m+2)\Delta}F|\mathcal{F}^d_{(m+1)\Delta}] \\ &+ 2E[\frac{D_{(m+2)\Delta}F}{\partial W((m+1)\Delta)}|\mathcal{F}^d_{(m+1)\Delta}] \\ &+ E[\frac{\partial^2 F}{\partial W^2((m+1)\Delta)}|\mathcal{F}^d_{(m+1)\Delta}] \end{aligned}$$

Using the basic integration by parts relation, we have:

$$\begin{aligned} &E[D^j_{(m+2)\Delta}F|\mathcal{F}^d_{(m+1)\Delta}] \\ &= E[Fp_j(\frac{W((m+2)\Delta) - W((m+1)\Delta)}{\Delta})|\mathcal{F}^d_{(m+1)\Delta}] \end{aligned}$$

Finally:

$$\begin{aligned} E[D_{(m+1)\Delta}F|t = (m+1)\Delta, s^0_\omega] &= K(1, m+1, s^0_\omega) + \\ \sum_{p=0}^{1} l_{1,p}K(0, m+1, s^{m+1,p}_\omega) \end{aligned}$$

$$\begin{aligned} &E[D^2_{(m+1)\Delta}F|t = (m+1)\Delta, s^0_\omega] = K(2, m+1, s^0_\omega) + \\ &2\sum_{p=0}^{1} l_{1,p}K(1, m+1, s^{m+1,p}_\omega) + \sum_{p=0}^{2} l_{2,p}K(0, m+1, s^{m+1,p}_\omega) \end{aligned}$$

The general result is:

$$\begin{aligned} D^k_{(m+1)\Delta}E[F|t = (m+1)\Delta, s^0_\omega] &= \\ \sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}\sum_{p=0}^{r} l_{r,p}K(k-r, m+1, s^{m+1,p}_\omega) \end{aligned} \tag{8}$$

Lemma 2.1 yields a recursive formula for the evaluation

$$E[F|t = m\Delta, s^0_\omega] \equiv K(0, m+1, s^0_\omega)$$

Lemma 2.2 generalizes this recursion to $K(n, m+1, s^0_\omega)$ for $n \neq 0$.

Lemma 2.2:

$$\begin{aligned} &K(n, m, s^0_\omega) = \\ &p_n(W((m+1)\Delta) - W(m\Delta))K(0, m+1, s^0_\omega) \\ &-\sum_{k=0}^{J+n} A(m, n, k, s^0_\omega)\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}\sum_{p=0}^{r} l_{r,p}K(k-r, m+1, s^{m+1,p}_\omega) \end{aligned}$$

with

$$\begin{aligned} A(m, n, k, s^0_\omega) &= \sum_{j=k}^{J+n}(-1)^{j+1}g_{m,j}\begin{bmatrix} j \\ k \end{bmatrix}p_{n-(j-k)} \\ &\times(W((m+1)\Delta, s^0_\omega) - W(m\Delta, s^0_\omega)) \end{aligned}$$

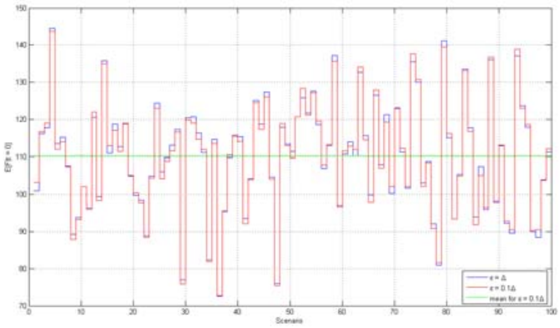

Figure 1: Computed values of $E[F|t=0]$ for two different values of ϵ. The mean value line is also shown.

Algorithm

(i) Simulate Ω independent paths $\{W(m\Delta, s^0_\omega)\}$ of Brownian motion, for $\omega = 1..\Omega$, $m = 1..M$.

(ii) Simulate paths $\{W(m\Delta, s^{i,p}_\omega)\}$ for $i = 1..M$ and $p = 1..J$ according to (6)

(iii) Set for all $s^{i,p}_\omega$ for $i = 1..M$ and $p = 1..J$

$$\begin{aligned} K(0, M, s^{i,p}_\omega) &= h(M\Delta, s^{i,p}_\omega) \\ K(n, M, s^{i,p}_\omega) &= 0 \quad \text{for } n > 0 \end{aligned}$$

(iv) Apply backward induction: for $m = M-1, .., 1$ for $\omega = 1..\Omega$

(a) for $n = 0..J$ and $i = 0..m$ and $p = 0..J$

$$K(n, m, s^{i,p}_\omega) = p_n(W((m+1)\Delta) - W(m\Delta))K(0, m+1, s^0_\omega)$$
$$-\sum_{k=0}^{J+n} A(m, n, k,^{i,p}_\omega) \sum_{r=0}^{k} \left[\begin{array}{c} k \\ r \end{array} \right] \sum_{p=0}^{r} l_{r,p} K(k-r, m+1, s^{m+1,p}_\omega)$$

(b) if $K(0, m, s^0_\omega) < h(m\Delta, s^0_\omega)$, then, for $n = 0..J$ and $i = 1..m$ and $p = 1..J$

$$\begin{aligned} K(0, m, s^{i,p}_\omega) &= h(M\Delta, s^{i,p}_\omega) \\ K(n, M, s^{i,p}_\omega) &= 0 \quad \text{for } n > 0 \end{aligned}$$

(iv) Set

$$\hat{V}(0) = \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} K(0, 1, s^0_\omega)$$

3 Numerics

The algorithm is fairly insensitive to the value of ϵ. Figure 1 shows stair plots of the value of $E[F|t=0]$ for a toy European Asian option for $\epsilon = \Delta$ and $\epsilon = 0.1\Delta$. The two curves are close and their means are similarly close.

References

[1] Bally, V., L. Caramellino, and A. Zanette (2005). Pricing and Hedging American Options by Monte Carlo Methods using a Malliavin Calculus Approach, Monte Carlo Methods and Applications, 11, pp. 97-133, 2005.

[2] Fujiwara, H., and H. Kijima (2007). Pricing of path-dependent American Options by Monte Carlo Simulation. *Journal of Economic Dynamics and Control*, 31, 11, 3478-3501.l

[3] Tsitsiklis, J. and B. Van Roy (1999). Optimal Stopping of Markov Processes: Hilbert Space Theory, Approximation Algorithms, and an Application to Pricing High-Dimensional Financial Derivatives. *IEEE Transactions on Automatic Control,* 44, 1850-1851.

[4] Longstaff, F. and E. Schwartz (2001). Valuing American Options by Simulation: a Simple Least-Squares Approach. *Review of Financial Studies*, 14, 113-147.

[5] Glasserman, P. and B. Yu (2001). Number of Paths versus Number of Basis Functions in American Option Pricing. *Finance and Stochastics* 5, 201-236.

[6] Egloff, D. (2005). Monte Carlo Algorithms for Optimal Stopping and Statistical Learning. *The Annals of Applied Probability*, 15, 2 1396-1432.

[7] Schellhorn, H. (2007). An Algorithm for Optimal Stopping with Path-Dependent Rewards Based on Regression and Malliavin Calculus. *American Institute of Physics Proceedings* 936, 503.

[8] Fournie, E., J.M. Lasry, J. Lebuchoux, P.-L. Lions and N. Touzi (1999). Some Applications of Malliavin Calculus to Monte Carlo Methods in Finance. *Finance and Stochastics* 3, 391-412.

[9] Fournie, E., J.-M. Lasry, J. Lebouchoux, and P.-L. Lions (2001). Applications of Malliavin Calculus to Monte Carlo Methods in Finance II. *Finance and Stochastics,* 5, 201-36.

ISBN: 978-988-98671-0-2