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Sensitivity Analysis of Motor PWM Control

Wenbo Zhang and Wei Zhan

***Abstract*—This paper discusses the sensitivity of Pulse Width Modulation (PWM) speed control for DC permanent magnetic motors with respect to the motor control system parameters. An analytic form for the steady state average motor speed is derived based on a first principle model. Compared to previous result based on a Simulink model, this analytic form for the steady state average motor speed greatly reduces the time required to conduct statistical analysis. As a result, five hundred sets of randomly generated motor parameters are used to determine the variation in the steady state average motor speed as a function of the variation in the motor parameters. This new approach allows for the sensitivity analysis and the design feasibility study for motor PWM control.**

***Index Terms*—Modeling and Simulation; Monte Carlo Analysis; Pulse Width Modulation; Sensitivity**

I. Introduction

Pulse Width Modulation (PWM) control is commonly used in industry [1, 4, 10]. Many automotive control systems use PWM to effectively regulate the battery voltage applied to the actuators. The simplicity and low costs, compared to hardware voltage regulator solutions, make it a very attractive choice for controlling actuators. However, it was found in [12] that the PWM method applied to DC permanent magnetic motors had large variation in the steady state average motor speed. The large variation of the PWM control is mainly due to the variation in the applied voltage, the PWM duty cycle, and the motor parameters. The robustness of the PWM control of motor speed was discussed in [12], using a first principle model presented in [11]. The baseline performance of PWM control was established using Monte Carlo analysis [2,5] and the Response Surface [7] was drawn based on the data from simulation using a Simulink model [6]. A new robust design for PWM motor speed control was proposed. Such statistics-based optimization technique has been used successfully by others to solve engineering problems [3]. However, the analysis based on the Simulink model simulation proposed in [12] turned out to be extremely time-consuming, and impractical for further analysis to improve the design of PWM motor speed control. In this paper, analytical form of the steady state average motor speed is derived and compared with results in [12]. The analytical form of the steady state average speed provides a much more efficient way to analyze the sensitivity of the system parameters. The time it takes to conduct a simulation related to the analysis of PWM motor speed control is greatly reduced. This allows one to conduct a thorough sensitivity analysis for each parameter in the system. Instead of a Pareto Chart derived in [12], a precise relationship between the system parameters and the average motor speed can be characterized.

This paper is organized as follows: In Section II, the analytical form for the average motor speed is derived based on the first principle model developed in [11]. In Section III, the simulation results based on the analytical form is compared to those derived in [12]. In Section IV, various sensitivity analyses are conducted using the analytical form. Conclusions and future work are discussed in Section V.

II. Analytical form for the average motor speed

The motor PWM control uses the duty cycle to achieve the desired level of average motor speed. The profile of a 30% duty cycle PWM control command is shown in Fig. 1.

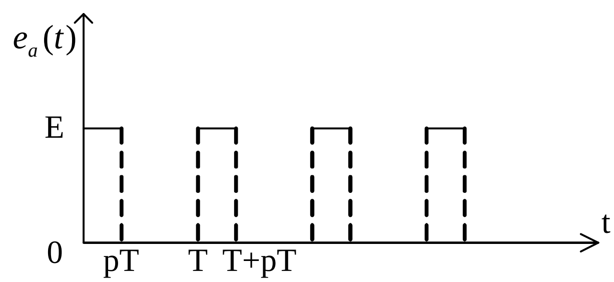

Figure 1. PWM control command

When a constant voltage source E is applied to a DC permanent magnetic motor, the control system can be modeled by the following Equations [11]

$$
\begin{aligned}
\frac{di(t)}{dt} &= \frac{1}{L}E - \frac{R}{L}i(t) - \frac{1}{L}e_b(t) \\
T_m(t) &= K_i i(t) \\
e_b(t) &= K_b \tfrac{d\theta(t)}{dt} \\
\frac{d^2\theta(t)}{dt^2} &= \frac{1}{J}T_m(t) - \frac{1}{J}T_l
\end{aligned}
\qquad (1)
$$

where $K_b = K_i$ (This can be easily derived based on the fact that energy going in equal to energy coming out); K_i is the torque constant, in *N-m/A*; K_b is the back-emf constant, in *V/(rad/sec)*; *i(t)* is the armature current, in *A*; *R* is the armature resistance, in Ω; $e_b(t)$ is the back emf, in *V*; T_l is the constant load torque, in *N-m*; $T_m(t)$ is the motor torque, in N-m; $\theta(t)$ is the rotor displacement, in *rad*; *L* is the armature

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Wenbo Zhang is with the Department of Industrial and Systems Engineering, Texas A&M University, College Station, TX 77843 USA.

Wei Zhan, is with the Department of Engineering Technology & Industrial Distribution, Texas A&M University, College Station, TX 77843-3367, USA. Tel. (979) 862-4342, Fax: (979) 847-9396 (e-mail: wei.zhan@tamu.edu).

inductance, in *H*; *E* is the applied motor voltage, in *V*; *J* is the rotor inertia, in $kg\text{-}m^2$.

When the motor is disconnected from the voltage source, according to Newton's Second Law, it can be modeled as

$$\frac{d^2\theta(t)}{dt^2} = -\frac{1}{J}T_l \qquad (2)$$

Therefore, during PWM control, the system can be modeled by Equations (1) and (2) with (1) characterizing the "motor on" stage and (2) characterizing the "motor off" stage. There is a very short period of transition between the on and off stages. But for all practical purposes, the effect of the transient on the average motor speed is negligible and hence not considered in this paper.

Define the following state variables for the motor PWM control system characterized by Equations (1) and (2): $x_1(t) = i(t), x_2(t) = \frac{d\theta(t)}{dt}$, one can rewrite (1) and (2) in the following state space equations:

$$\begin{cases} \dot{x} = Ax + Bu, & nT \le t \le nT + pT \\ \dot{x}_2(t) = -\frac{1}{J}T_l & nT + pT < t < (n+1)T \end{cases} \quad n = 0,1,2,.... \qquad (3)$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{k}{L} \\ \frac{k}{J} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{-1}{J} \end{bmatrix}, \quad C = [0 \quad 1],$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u(\tau) = \begin{bmatrix} E \\ T_l \end{bmatrix}.$$

Note that by definition of the state variables and the output, *y(t)* is the motor speed. The steady state average motor speed is the average motor speed over each PWM cycle after sufficiently long time has passed. The system in (3) is a switched linear system. In each stage, the system is a linear system. Therefore, for a given initial condition, the formula for the output can be explicitly derived for a PWM cycle. Since we are interested in the average motor speed after the system reaches steady state, the following equation holds

$$y(nT) = y(nT + T) \qquad (4)$$

Assuming that after significant amount of time t_0, the output of the systems has reached its steady state. Thus, the motor speed is oscillating with a period of *T*. Denote $y_0 = y(t_0)$, and assume t_0 is the beginning of the PWM cycle, i.e., t_0 = *nT*. Then, for $nT \le t \le nT + pT$,

$$y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Ce^{A(t-t_0)}x_0 \qquad (5)$$

where $x_0 = [0 \quad y_0]^T$. For $t = nT + pT$, the output reaches its maximum value, denoted by y_1:

$$y_1 = \int_{t_0}^{t_0+pT} Ce^{A(nT+pT-\tau)}Bu(\tau)d\tau + Ce^{ApT}x_0 \qquad (6)$$

For $nT + pT \le t \le (n+1)T$, simple integration results in

$$y(t) = -\frac{1}{J}T_l(t - nT - pT) + y_1 \qquad (7)$$

When $t = (n+1)T$, the output returns to y_0 according to Equation (4). Thus,

$$y_0 = y_1 - \frac{1}{J}T_l(T - pT) \qquad (8)$$

From Equations (5), (6), and (8), one can solve for y_0. The derivation was done using Matlab symbolic calculation. The detail of the derivation omitted and the formula for y_0 is given by the following formula

$$y_0 = \frac{\frac{(1-e^{S_1pT})}{2S_1}\left(\frac{T_lR-2kE}{\Delta}+\frac{T_l}{J}\right)+\frac{(1-e^{S_2pT})}{2S_2}\left(\frac{2kE-T_lR}{\Delta}+\frac{T_l}{J}\right)-\frac{1}{J}T_l(T-pT)}{1+\frac{1}{2\Delta}\left(-\Delta e^{S_2Tp}-JRe^{S_1Tp}+JRe^{S_2Tp}-\Delta e^{S_1Tp}\right)} \qquad (9)$$

where $\Delta = \sqrt{\frac{R^2}{L^2} - \frac{4k^2}{JL}}$, $S_1 = \frac{1}{2}\left(-\frac{R}{L} + \Delta\right), S_2 = \frac{1}{2}\left(-\frac{R}{L} - \Delta\right)$.

Therefore, according to Equation (5), for $nT \le t \le nT + pT$,

$$y(t) = \left[\frac{kE - T_lR}{2\Delta S_2} + \frac{T_l}{2JS_2} + \frac{T_lR - 2kE}{2\Delta S_1} + \frac{T_l}{2JS_1}\right] +$$
$$\frac{e^{S_1(t-t_0)}}{2S_1}\left(\frac{2kE - T_lR}{2\Delta} - \frac{T_l}{J}\right) + \frac{e^{S_2(t-t_0)}}{2S_2}\left(\frac{T_lR - 2kE}{\Delta} - \frac{T_l}{J}\right) \qquad (10)$$
$$-\frac{y_0}{2\Delta}\left(JRe^{S_2(t-t_0)} - \Delta e^{S_1(t-t_0)} - JRe^{S_1(t-t_0)} - \Delta e^{S_2(t-t_0)}\right)$$

For $nT + pT < t < (n+1)T$, according to Equation (7), one can get

$$y(t) = -\frac{1}{J}T_l(t - nT - pT) + y_1 \qquad (11)$$

where y_1 can be solved from Equation (8)

$$y_1 = y_0 + \frac{1}{J}T_l(T - pT) \qquad (12)$$

With the analytical form for the steady state motor speed derived in Equations (10-12), the steady state average speed can be found by integrating the output over a PWM cycle then divided by the period. This process is illustrated in Fig. 2.

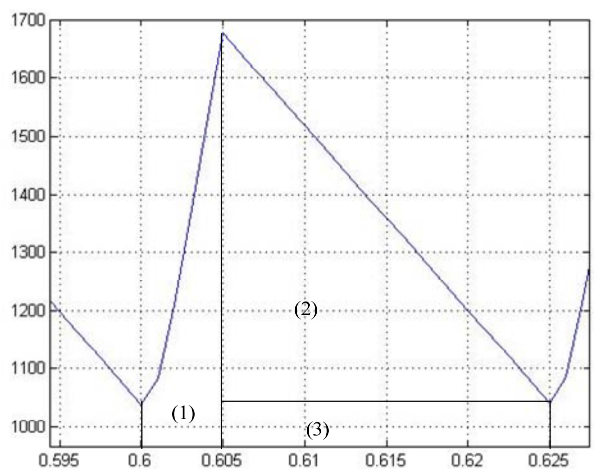

Figure 2. Calculating the average steady state speed

Area 1 is calculated by integrating *y(t)* in Equation (10) over the interval $[t_0, t_0 + pT]$.

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$$A_1 = \int_{t_0}^{t_0+pT} y(t)dt = \frac{pT}{2}\left[\frac{2kE - T_l R}{\Delta S_2} + \frac{T_l}{JS_2} + \frac{T_l R - 2kE}{\Delta S_1} + \frac{T_l}{JS_1}\right]$$
$$+ \frac{\left(e^{S_1 pT} - 1\right)}{2S_1}\left(\frac{2kE - T_l R}{\Delta} - \frac{T_l}{J}\right) + \frac{\left(e^{S_1 pT} - 1\right)}{2S_1}\left(\frac{JRy_0}{\Delta} + y_0\right) \quad (13)$$
$$\frac{\left(e^{S_2 (t-t_0)} - 1\right)}{2S_2}\left(\frac{T_l R - 2kE - JRy_0}{\Delta} - \frac{T_l}{2J} + y_0\right)$$

Area 2 and Area 3 are calculated by

$$A_2 = \tfrac{1}{2}(y_1 - y_0)T(1-p) \quad (14)$$

$$A_3 = y_0 T(1-p) \quad (15)$$

Finally, the steady state average motor speed is given by

$$\bar{y}_{ss} = (A_1 + A_2 + A_3)/T \quad (16)$$

The steady state average motor speed can be calculated by evaluating functions specified in Equations (9), (12-16).

III. Comparing Simulation Results Using the Analytical Form and the Simulink Model

In [12], the parameters in the motor PWM control system are assumed to be random variables. The variance of the steady state average speed was simulated using the Simulink model shown in Fig. 3. Design of Experiment (DOE) [8,9] was conducted to show that the PWM duty cycle and the applied voltage were the major contributing factors to the variance of the steady state average speed. Using the Response Surface Method (RSM), it was discovered that the variance of the steady state average speed could be greatly reduced if the PWM duty cycle was adjust as a function of the applied voltage. Further analysis of the sensitivity of the steady state average speed variance with respect to the variance of each parameter is difficult due to the lengthy simulation time using the Simulink model. With the analytical form for the steady state average speed derived in Section II, one expects to significantly reduce the simulation time needed for the same tasks. To compare the simulation results from using the Simulink model and the analytical form, one thousand sets of motor PWM control parameters are randomly generated. For each parameter set, the steady state average speed is simulated for a speed target of 3000 rpm using the Simulink model and the analytical form.

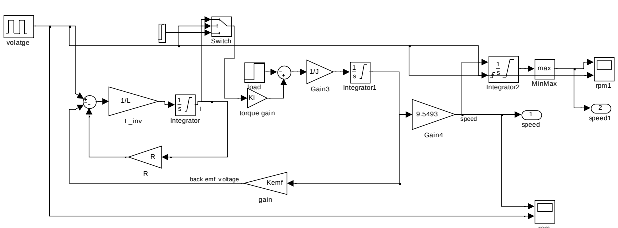

Figure 3. Simulink model

Figure 4 shows the steady state average speeds calculated using the analytical form specified in Equations (9), (12-16). The difference between the steady state average speeds using the two different methods is illustrated in Figure 5. It can be seen that the differences are less than 3 rpm, which is less than 0.1% of the 3000 rpm speed target. The difference can be from two sources: the rounding error when evaluating the formulas and the limited time step size for the Simulink model. The small difference between the simulation results using the two different methods validates the formulas derived in Section II.

During the simulations, the simulation times for both methods were recorded. Using the Simulink model the simulation time was 716.97 seconds. Using the analytical form for the steady state average speeds, the simulation time was 0.731 second. In other words, with similar simulation accuracy, the simulation time is reduced roughly by 1000 times. This allows one to further analyze the sensitivity of the motor PWM control system.

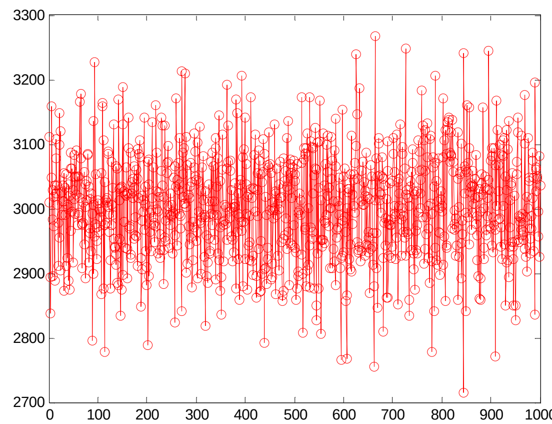

Figure 4. The steady state average speeds

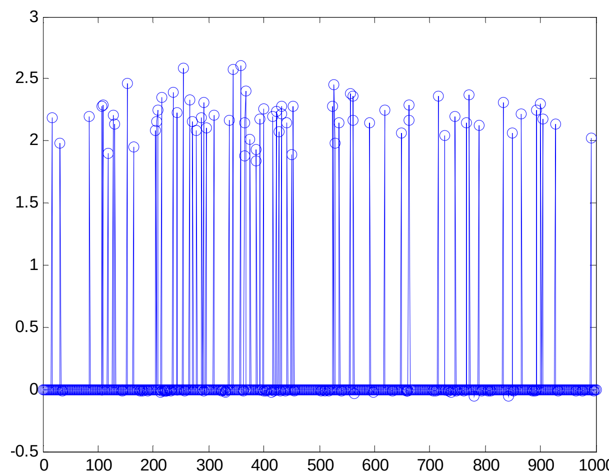

Figure 5. Difference between the simulation results using the two methods

IV. Sensitivity Analysis

With the significantly reduced simulation time using the analytical form for the steady state average speed, it is feasible to analyze the sensitivity of the steady state average speed with respect to each parameter in the system. Under the assumption that each parameter is a Gaussian random variable, the sensitivity of each parameter can be analyzed by varying the variance of this particular parameter with other parameter assuming the nominal values. For example, to study the sensitivity with respect to the motor coil resistance, all other parameters are assumed to be equal to their nominal values. The motor coil resistance is then assumed to have a Gaussian distribution with the mean equal to the nominal value and the variance equal to V_R. For each value of V_R, random values of the resistance are generated. The steady state average speed for each resistance value is calculated using the analytical form. Based on these steady state average speed values, the mean and the variance of the steady state average speed can be calculated. This process is repeated for different values of V_R. In the end, the variance of the steady state average speed can then be plotted as a function of the variance of the parameter being studied. This can be used to specify the tolerance band for the design parameters.

Since the analysis is based on the randomly generated values of the parameters, the sample size needs to be large enough to have consistent results. For example, if two sets of 20 randomly generated points are used for the resistance values, the steady state average speed may have two variances that are quite different. This difference can be significantly reduced if the number of randomly generated points is increased to 500. However, more points means longer simulation time. The analytical form derived in Section II makes this less of a problem. But considering the huge simulation runs needed for the sensitivity analysis, it is still important to find out what the optimal number of points is. To find the optimal number of points, the following simulation is carried out:

1. Randomly generate N sets of parameters with fixed means and variances with the assumption that they are all Gaussian;
2. For each set of the parameter values, calculate the steady state average speed;
3. Calculate the variance of the steady state average speed based on the results from the first two steps;
4. Repeat the above steps M times;
5. Repeat the above steps with different values of N. Record the simulation time for each N;
6. Plot the variance of the steady state average speed's mean and variance as a function of N. Plot the simulation time as a function of N.

For $M = 100$, the results are plotted in Figs. 6 and 7. Notice that for $N = 2{,}000$, the total simulation runs are $N \times M = 200{,}000$.

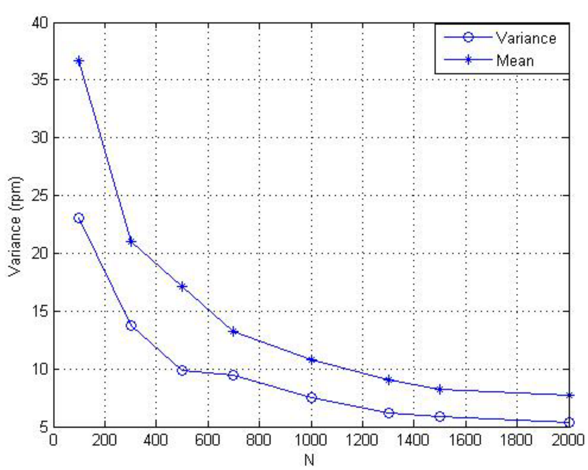

Figure 6. Variance of the steady state average speed's mean and variance as a function of N

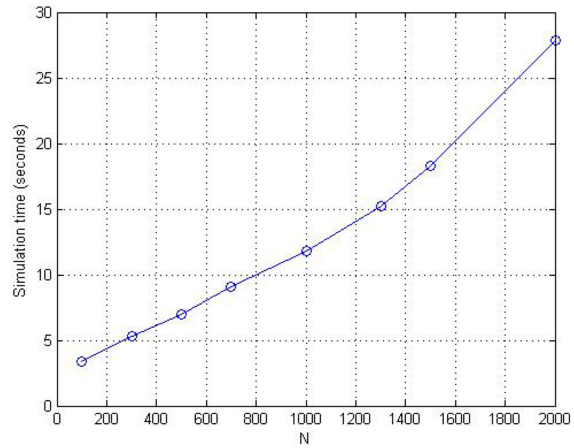

Figure 7. Simulation time as a function of N

Based on Fig. 6 and 7, the number of randomly generated points is chosen to be 500. Different values of M results in similar conclusion. Using $M = 100$ and $N = 500$, the sensitivity of the steady state average speed with respect to the resistance, inductance, rotor inertia, and applied voltage are plotted in Fig. 8. Notice that the PWM duty cycle is not analyzed for the new design in [12] defined the duty cycle as a function of the applied voltage to reduce the variation.

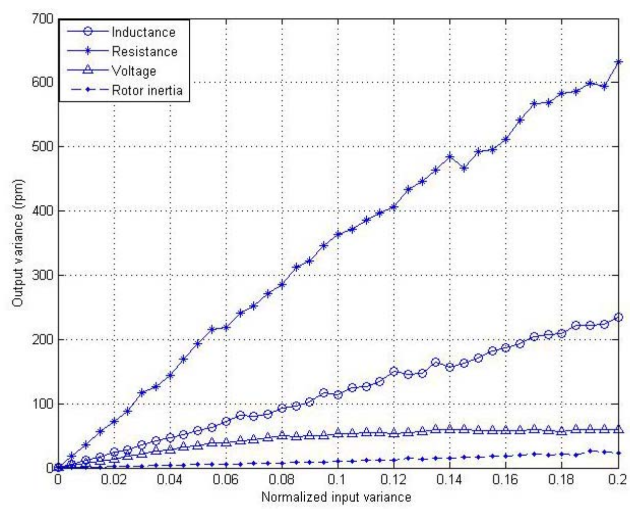

Figure 8. Variance of steady state average speed

From Fig. 8 it can be seen that the steady state average speed is most sensitive to the variance of the motor coil resistance and least sensitive to the variance of the rotor inertia. The sensitivity with respect to the rotor inertia is relatively small and negligible if the requirement on the steady state average speed variance is large, e.g., greater than 200 rpm. The sensitivity with respect to the applied voltage is also relatively small compared to those for the resistance and inductance. It increases in an exponential fashion. Because of the nonlinearity of the steady state average speed variance as a function of the variance in applied voltage, Fig. 8 provides one with more accurate information than a typical local sensitivity analysis method such as using the partial derivative as the sensitivity. The sensitivities with respect to the variances of the resistance and inductance are large and approximately linear. Fig. 8 provides an upper bound for each parameter, which can be used by the design engineers to

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determine the tolerance bands for the parameters and conduct trade-off studies. For example, if the requirement on the steady state average speed variance is 200 rpm or less, then from Fig. 8, the variance of the coil resistance should not be greater than 5% of the nominal resistance; the variance of the coil inductance should not be greater than 17% of the nominal inductance. Of course, these are necessary but not sufficient conditions for the steady state average speed variance to meet the requirement. With the simulation model established, one can find a tolerance bands for resistance and inductance that are close to 17% and 5% of their respective nominal values. It is worth noting that in Fig. 8 only one variable is changed for each sensitivity curve, and no interactions between variables are considered.

V. CONCLUSIONS AND DISCUSSION

An analytical form for the steady state average speed of motor PWM control is derived in this paper. Compared to the Simulink model used in [12], the analytical form significantly reduces the simulation time related to sensitivity analysis, which makes further sensitivity analysis feasible. In this paper, only the impact of each parameter on the steady state average speed is analyzed. The research work is still being carried out to fully analyze the sensitivity with all parameters and their interactions considered. The model-based analysis result can be used to predict the performance of PWM motor speed control for a given set of tolerance bands for the design parameters. It can also provide the design engineers with information to choose the tolerance band for each parameter and possible overall trade-off between the parameters and cost. For any given variance for the steady state average speed, a range for each parameter or certain constraint between the parameter can be found. Tens of thousand of tests can be conducted in the simulation environment instead of building so many actual systems. The benefit in time-to-market and cost savings can be significant. Even though the derivation of the analytical form may not apply to all other systems, tools for symbolic computations allows one to deploy the method used in this paper in many other similar problems. Future work also includes validation of the simulation results by conducting actual testing.

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