Computer Algebra and Mechanized Reasoning in Mathematical Epidemiology

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Abstract—**We are concerned by imminent future problems caused by biological dangers, here we think of a way to solve them. One of them is analyzing endemic models, for this we make a study supported by Computer Algebra Systems (CAS) and Mechanized Reasoning (MR). Also we show the advantages of the use of "CAS" and "MR" to obtain in that case, an epidemic threshold theorem. We prove a previously obtained theorem for S^nIR endemic model. Moreover using "CAS+MR" we obtain a new epidemic threshold theorem for the S^nI^mR epidemic model and finally we discuss the relevance of the theorems and some future applications.**

Index Terms—**Basic reproductive number, Computer algebra and mechanized reasoning, Differential susceptibility, Epidemic thresholds, SIR model.**

I. INTRODUCTION

At the moment, we are at the edge of a possible biological problem. Some people say that the 19th century was the century of chemistry, the 20th was the century of physics, and they say that the 21st will be the century of biology. If we think, the advances in the biological field in the recent years have been incredible, and like the physics and its atomic bomb, with biology could create global epidemics diseases. Also the climate change could produce a new virus better than the existing virus, creating an atmosphere of panic. For these reasons and others, we think in a solution using mathematical models with computer algebra and mechanized reasoning. Specifically we consider the SIR (Susceptible-Infective-Removed) model, with differential susceptibility and multiple kinds of infected individuals. The objective is to derive two epidemic threshold theorems by using the algorithm MKNW given in [1] and a little bit of mechanized reasoning.

Briefly the MKNW runs on: Initially we have a system of ordinary non-lineal differential equations S, whose coefficients are polynomial. We start setting all derivates to zero for finding equilibrium; we solve the system finding the equilibrium point T. Then we compute the Jacobian Jb for the system S and replace T in S. We compute the eigenvalues for Jb; from the eigenvalues we obtain the stability conditions when each eigenvalue is less than zero. Finally we obtain the reproductive number for the system S in the particular cases. Using deductive reasoning we obtain some theorems based on the particular cases.

The MKNW algorithm is not sufficient to prove the threshold theorems that will be considered here and for this reason, it is necessary to use some form of mechanized reasoning, specifically some strategy of mechanized induction.

The threshold theorem that we probe in section 2 was originally presented in [2] using only pen and paper and human intelligence. A first contribution of this paper is a mechanized derivation of such theorem using CAS.

The threshold theorem to be proved in section 3 is original and some particular cases of this theorem were previously considered via CAS in [3,4] and without CAS in [5].

II. CA AND MR APPLIED TO THE S^NIR EPIDEMIC MODEL

We introduce the system for the S^nIR epidemic model, with n-groups of susceptible individuals which is described by next equations [2]:

$$\frac{d}{dt}X_i(t) = \mu\,(p_i\,X_0 - X_i(t)) - \lambda_i\,X_i(t) \tag{1}$$

$$\frac{d}{dt}\mathrm{Y}(t) = \left(\sum_{k=1}^{n}\lambda_k\,X_k(t)\right) - (\mu+\gamma+\delta)\,\mathrm{Y}(t) \tag{2}$$

$$\frac{d}{dt}\mathrm{Z}(t) = \gamma\,\mathrm{Y}(t) - (\mu+\varepsilon)\,\mathrm{Z}(t) \tag{3}$$

with,

$$\lambda_i = \alpha_i\,\beta\,\eta\,\mathrm{Y}(t) \tag{4}$$

we define p_i as:

$$\sum_{i=1}^{n} p_i = 1 \tag{5}$$

That is a system with (n+2) equations and each constant is defined as follow:

μ : is the natural death rate.
γ : is the rate at which infectives are removed or become immune.
δ : is the disease-induced mortality rate for the infectives.
ε : is the disease-induced mortality rate for removed individuals.
α_i : is the susceptibility of susceptible individuals.
β : is the infectious rate of infected individuals.
η : is the average number of contacts per individual.

Each variable is defined as follow:

$X_i(t)$: are the n groups of susceptible in the time equal t,

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Y(t) : is the group of infectives in the time equal t.
Z(t) : is the group of removed in the time equal t.

A. The Standard SIR Model

As a particular case we analyze the SIR model with one group of susceptible [6]. It is described as follow:

$$\frac{d}{dt}X_1(t) = \mu\,(p_1 X_0 - X_1(t)) - \lambda_1 X_1(t) \quad (6)$$

$$\frac{d}{dt}\mathrm{Y}(t) = \lambda_1 X_1(t) - \mathrm{Y}(t)\,\mu - \gamma\,\mathrm{Y}(t) - \mathrm{Y}(t)\,\delta \quad (7)$$

$$\frac{d}{dt}\mathrm{Z}(t) = \gamma\,\mathrm{Y}(t) - (\mu + \varepsilon)\,\mathrm{Z}(t) \quad (8)$$

The infection-free equilibrium solution for the previous system, it's given by:

$$\{X_1 = p_1 X_0,\ Y = 0\} \quad (9)$$

We generate the Jacobian matrix for the equations system.

$$\begin{bmatrix} -\mu - \alpha_1\,\eta\,\beta\,Y & -\alpha_1\,\eta\,\beta\,X_1 \\ \alpha_1\,\eta\,\beta\,Y & \alpha_1\,\eta\,\beta\,X_1 - \mu - \gamma - \delta \end{bmatrix} \quad (10)$$

and substituting the infection-free equilibrium point in the Jacobian:

$$\begin{bmatrix} -\mu & -\alpha_1\,\eta\,p_1\,X_0\,\beta \\ 0 & \alpha_1\,\eta\,p_1\,X_0\,\beta - \mu - \gamma - \delta \end{bmatrix} \quad (11)$$

We find the eingenvalues for the previous Jacobian.

$$-\mu,\ \alpha_1\,\eta\,p_1\,X_0\,\beta - \mu - \gamma - \delta\ ; \quad (12)$$

and the corresponding stability condition is:

$$\alpha_1\,\eta\,p_1\,X_0\,\beta - \mu - \gamma - \delta < 0\ , \quad (13)$$

this can be rewritten as:

$$\frac{\alpha_1\,\eta\,p_1\,X_0\,\beta}{\mu + \gamma + \delta} < 1\ ; \quad (14)$$

also it can be written as:

$$R_0 < 1\ , \quad (15)$$

where,

$$R_0 = \frac{\alpha_1\,\eta\,p_1\,X_0\,\beta}{\mu + \gamma + \delta}\ , \quad (16)$$

this is known as the basic reproductive number.

B. The S^2IR Model

As another particular case we analyze the S2IR model where there are two groups of susceptibles. The equations for this system are:

$$\frac{d}{dt}X_1(t) = \mu\,(p_1 X_0 - X_1(t)) - \lambda_1 X_1(t) \quad (17)$$

$$\frac{d}{dt}X_2(t) = \mu\,(p_2 X_0 - X_2(t)) - \lambda_2 X_2(t) \quad (18)$$

$$\frac{d}{dt}\mathrm{Y}(t) = \lambda_1 X_1(t) + \lambda_2 X_2(t) - \mathrm{Y}(t)\,\mu - \gamma\,\mathrm{Y}(t) - \mathrm{Y}(t)\,\delta \quad (19)$$

$$\frac{d}{dt}\mathrm{Z}(t) = \gamma\,\mathrm{Y}(t) - (\mu + \varepsilon)\,\mathrm{Z}(t) \quad (20)$$

The infection-free equilibrium solution for the previous system, it's given by:

$$[X_1 = p_1 X_0,\ X_2 = p_2 X_0,\ Y = 0] \quad (21)$$

We generate the Jacobian matrix for the equations system.

$$\begin{bmatrix} -\mu - \alpha_1\,\eta\,\beta\,Y & 0 & -\alpha_1\,\eta\,\beta\,X_1 \\ 0 & -\mu - \alpha_2\,\eta\,\beta\,Y & -\alpha_2\,\eta\,\beta\,X_2 \\ \alpha_1\,\eta\,\beta\,Y & \alpha_2\,\eta\,\beta\,Y & \alpha_1\,\eta\,\beta\,X_1 + \alpha_2\,\eta\,\beta\,X_2 - \mu - \gamma - \delta \end{bmatrix} \quad (22)$$

and substituting the infection-free equilibrium point in the Jacobian:

$$\begin{bmatrix} -\mu & 0 & -\alpha_1\,\eta\,p_1\,X_0\,\beta \\ 0 & -\mu & -\alpha_2\,\eta\,p_2\,X_0\,\beta \\ 0 & 0 & \alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta - \mu - \gamma - \delta \end{bmatrix} \quad (23)$$

We find the eingenvalues for the previous Jacobian.

$$-\mu,\ -\mu,\ \alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta - \mu - \gamma - \delta \quad (24)$$

and the corresponding stability condition is:

$$\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta - \mu - \gamma - \delta < 0 \quad (25)$$

this can be rewritten as:

$$\frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta}{\mu + \gamma + \delta} < 1 \quad (26)$$

also it can be written as:

$$R_0 < 1 \quad (27)$$

where,

$$R_0 = \frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta}{\mu + \gamma + \delta} \quad (28)$$

this is the basic reproductive number for S2IR model.

C. The S^3IR Model

As another particular case we analyze the S3IR model where there are three groups of susceptibles. The equations for this system are:

$$\frac{d}{dt}X_1(t) = \mu\,(p_1 X_0 - X_1(t)) - \lambda_1 X_1(t) \quad (29)$$

$$\frac{d}{dt}X_2(t) = \mu\,(p_2 X_0 - X_2(t)) - \lambda_2 X_2(t) \quad (30)$$

$$\frac{d}{dt}X_3(t) = \mu\,(p_3 X_0 - X_3(t)) - \lambda_3 X_3(t) \quad (31)$$

$$\frac{d}{dt}\mathrm{Y}(t) = \lambda_1 X_1(t) + \lambda_2 X_2(t) + \lambda_3 X_3(t) - \mathrm{Y}(t)\,\mu - \gamma\,\mathrm{Y}(t) - \mathrm{Y}(t)\,\delta \quad (32)$$

$$\frac{d}{dt}\mathrm{Z}(t) = \gamma\,\mathrm{Y}(t) - (\mu + \varepsilon)\,\mathrm{Z}(t) \quad (33)$$

The infection-free equilibrium solution for the previous system, it's given by:

$$[X_1 = p_1 X_0,\ X_2 = p_2 X_0,\ X_3 = p_3 X_0,\ Y = 0] \quad (34)$$

The corresponding stability condition obtained from the Jacobian for this system:

$$\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta + \alpha_3\,\eta\,\beta\,p_3\,X_0 - \mu - \gamma - \delta < 0 \quad (35)$$

this can be rewritten as:

$$\frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta + \alpha_3\,\eta\,\beta\,p_3\,X_0}{\mu + \gamma + \delta} < 1 \quad (36)$$

also it can be written as:

$$R_0 < 1 \quad (37)$$

where,

$$R_0 = \frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta + \alpha_3\,\eta\,\beta\,p_3\,X_0}{\mu + \gamma + \delta} \quad (38)$$

this is the basic reproductive number for S3IR model.

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D. The S^4IR and S^5IR Models

Here we show the S4IR and S5IR models where there are four and five groups of susceptibles, respectively. With these models we do the same process, so we show only the basic reproductive number.

$$R_0 = \frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta + \alpha_3\,\eta\,\beta\,p_3\,X_0 + \alpha_4\,\eta\,\beta\,p_4\,X_0}{\mu+\gamma+\delta} \quad (39)$$

this is the basic reproductive number for S^4IR model.

$$R_0 = \frac{\alpha_1\,\eta\,p_1\,X_0\,\beta + \alpha_2\,\eta\,p_2\,X_0\,\beta + \alpha_3\,\eta\,\beta\,p_3\,X_0 + \alpha_4\,\eta\,\beta\,p_4\,X_0 + \alpha_5\,\eta\,\beta\,p_5\,X_0}{\mu+\gamma+\delta} \quad (40)$$

(this is the basic reproductive number for S^5IR model. (

E. The S^nIR Model

Theorem. *For the equations system given by (1), (2) y (3). The infection-free equilibrium is locally stable if R0<1, and is unstable if R0>1, where:*

$$R_0 = \frac{X_0\,\beta\,\eta\left(\sum_{i=1}^{n}\alpha_i\,p_i\right)}{\mu+\gamma+\delta} \quad (41)$$

1) Proof:

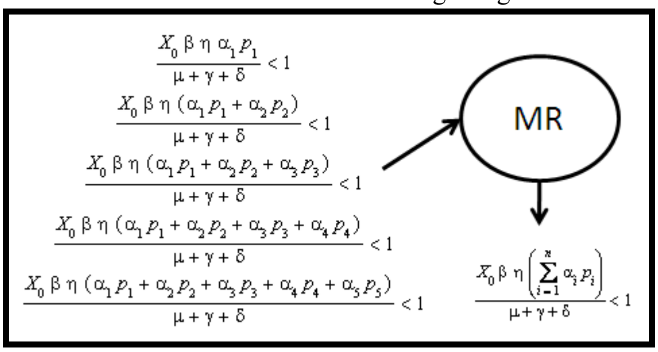
If we look the inequalities corresponding to stability conditions for each system previously considered, we have the list:

$$\left[\begin{array}{c} \frac{X_0\,\beta\,\eta\,\alpha_1\,p_1}{\mu+\gamma+\delta} < 1 \\ \frac{X_0\,\beta\,\eta\,(\alpha_1\,p_1+\alpha_2\,p_2)}{\mu+\gamma+\delta} < 1 \\ \frac{X_0\,\beta\,\eta\,(\alpha_1\,p_1+\alpha_2\,p_2+\alpha_3\,p_3)}{\mu+\gamma+\delta} < 1 \\ \frac{X_0\,\beta\,\eta\,(\alpha_1\,p_1+\alpha_2\,p_2+\alpha_3\,p_3+\alpha_4\,p_4)}{\mu+\gamma+\delta} < 1 \\ \frac{X_0\,\beta\,\eta\,(\alpha_1\,p_1+\alpha_2\,p_2+\alpha_3\,p_3+\alpha_4\,p_4+\alpha_5\,p_5)}{\mu+\gamma+\delta} < 1 \end{array}\right] \quad (42)$$

Using mechanized induction we obtain the general expression for the stability conditions for a system with (n+2) equations.

$$\frac{X_0\,\beta\,\eta\left(\sum_{i=1}^{n}\alpha_i\,p_i\right)}{\mu+\gamma+\delta} < 1 \quad (43)$$

A schematic of the deductive reasoning using "MR" is



Here we have an idea for the MR, it finds the similar components in each item and it has a viewer or a detector that find the sequential form for the dissimilar parts. It is just an idea, we believe this system have to be improved by the scientific community.

III. CA and MR Applied to the S^NI^MR Epidemic Model

We introduce the system for the S^nI^mR epidemic model, with n-groups of susceptible individuals and m-groups of infected people, which is described by next equations:

$$\frac{d}{dt}X_i(t) = \mu\,(p_i\,X_0 - X_i(t)) - \alpha_i\,\eta\left(\sum_{j=1}^{m}\beta_j\,Y_j(t)\right)X_i(t) \quad (44)$$

$$\frac{d}{dt}Y_j(t) = \beta_j\,\eta\,Y_j(t)\left(\sum_{i=1}^{n}\alpha_i\,X_i(t)\right) - Y_j(t)\,\mu - \gamma\,Y_j(t) - Y_j(t)\,\delta \quad (45)$$

$$\frac{d}{dt}Z(t) = \gamma\,Y_j(t) - (\mu+\varepsilon)\,Z(t) \quad (46)$$

with,

$$\lambda_{i,j} = \alpha_i\,\beta_j\,\eta\,Y_j(t) \quad (47)$$

we define pi as:

$$\sum_{i=1}^{n} p_i = 1 \quad (48)$$

That is a system with (n+1) equations and each constant is defined as follow:

μ : is the natural death rate.
γ : is the rate at which infectives are removed or become immune.
δ : is the disease-induced mortality rate for the infectives.
ε : is the disease-induced mortality rate for removed individuals.
α_i : is the susceptibility of susceptible individuals.
β_i : is the infectious rate of infected individuals.
η : is the average number of contacts per individual.

Each variable is defined as follow:

Xi(t) : are the groups of susceptible in the time equal t, with i from 1 to n.
Yj(t) : are the groups of infectives in the time equal t, with j from 1 to m.
Z(t) : is the group of removed in the time equal t.

A. The SI^2R Model

We analyze a particular case with one group of susceptibles and two groups of infectives. The following equations describe this case:

$$\frac{d}{dt}X_1(t) = \mu\,(p_1\,X_0 - X_1(t)) - \alpha_1\,\eta\left(\sum_{j=1}^{2}\beta_j\,Y_j(t)\right)X_1(t) \quad (49)$$

$$\frac{d}{dt}Y_1(t) = \beta_1\,\eta\,Y_1(t)\left(\sum_{i=1}^{1}\alpha_i\,X_i(t)\right) - Y_1(t)\,\mu - \gamma\,Y_1(t) - Y_1(t)\,\delta \quad (50)$$

$$\frac{d}{dt}Y_2(t) = \beta_2\,\eta\,Y_2(t)\left(\sum_{i=1}^{1}\alpha_i\,X_i(t)\right) - Y_2(t)\,\mu - \gamma\,Y_2(t) - Y_2(t)\,\delta \quad (51)$$

Solving the system for the infection-free equilibrium, we find:

$$\left\{ Y_1 = 0,\, Y_2 = 0,\, X_1 = \frac{\mu\, p_1\, X_0}{\mu + \alpha_1\, \eta \left(\sum_{j=1}^{2} \beta_j\, Y_j \right)} \right\} \tag{52}$$

We generate a Jacobian coming off the equations system:

$$\begin{bmatrix} \mu\, p_1\, X_0 - \mu\, X_1 - \alpha_1\, \eta\, \beta_1\, Y_1\, X_1 - \beta_2\, \eta\, Y_2\, \alpha_1\, X_1 \\ \alpha_1\, \eta\, \beta_1\, Y_1\, X_1 - Y_1\, \mu - \gamma\, Y_1 - Y_1\, \delta \\ \beta_2\, \eta\, Y_2\, \alpha_1\, X_1 - Y_2\, \mu - \gamma\, Y_2 - Y_2\, \delta \end{bmatrix} \tag{53}$$

substituting the infection-free equilibrium point:

$$\begin{bmatrix} -\mu & -\alpha_1\, \eta\, \beta_1\, p_1\, X_0 & -\alpha_1\, \eta\, \beta_2\, p_1\, X_0 \\ 0 & \alpha_1\, \eta\, \beta_1\, p_1\, X_0 - \mu - \gamma - \delta & 0 \\ 0 & 0 & \alpha_1\, \eta\, \beta_2\, p_1\, X_0 - \mu - \gamma - \delta \end{bmatrix} \tag{54}$$

We find the eigenvalues for the system,

$$-\mu,\, \alpha_1\, \eta\, \beta_1\, p_1\, X_0 - \mu - \gamma - \delta,\, \alpha_1\, \eta\, \beta_2\, p_1\, X_0 - \mu - \gamma - \delta \tag{55}$$

The stability condition shall satisfy,

$$\begin{bmatrix} \alpha_1\, \eta\, \beta_2\, p_1\, X_0 - \mu - \gamma - \delta < 0 \\ \alpha_1\, \eta\, \beta_1\, p_1\, X_0 - \mu - \gamma - \delta < 0 \end{bmatrix} \tag{56}$$

Rewriting the previous equation:

$$\begin{bmatrix} \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0}{\mu + \gamma + \delta} < 1 \\ \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0}{\mu + \gamma + \delta} < 1 \end{bmatrix} \tag{57}$$

also it can be written as:

$$\begin{bmatrix} R_{0,1} < 1 \\ R_{0,2} < 1 \end{bmatrix} \tag{58}$$

where,

$$\begin{bmatrix} R_{0,2} = \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0}{\mu + \gamma + \delta} \\ R_{0,1} = \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0}{\mu + \gamma + \delta} \end{bmatrix} \tag{59}$$

Here, we have the two basic reproductive numbers for SI^2R model.

B. *The S^2I^2R Model*

We analyze a particular case with two groups of susceptibles and two groups of infectives. The following equations describe this case:

$$\frac{d}{dt} X_1(t) = \mu\, (p_1\, X_0 - X_1(t)) - \alpha_1\, \eta \left(\sum_{j=1}^{2} \beta_j\, Y_j(t) \right) X_1(t) \tag{60}$$

$$\frac{d}{dt} X_2(t) = \mu\, (p_2\, X_0 - X_2(t)) - \alpha_2\, \eta \left(\sum_{j=1}^{2} \beta_j\, Y_j(t) \right) X_2(t) \tag{61}$$

$$\frac{d}{dt} Y_1(t) = \beta_1\, \eta\, Y_1(t) \left(\sum_{i=1}^{2} \alpha_i\, X_i(t) \right) - Y_1(t)\, \mu - \gamma\, Y_1(t) - Y_1(t)\, \delta \tag{62}$$

$$\frac{d}{dt} Y_2(t) = \beta_2\, \eta\, Y_2(t) \left(\sum_{i=1}^{2} \alpha_i\, X_i(t) \right) - Y_2(t)\, \mu - \gamma\, Y_2(t) - Y_2(t)\, \delta \tag{63}$$

Solving the system for the infection-free equilibrium, we find:

$$\left\{ Y_1 = 0,\, Y_2 = 0,\, X_1 = \frac{\mu\, p_1\, X_0}{\mu + \alpha_1\, \eta \left(\sum_{j=1}^{2} \beta_j\, Y_j \right)},\, X_2 = \frac{\mu\, p_2\, X_0}{\mu + \alpha_2\, \eta \left(\sum_{j=1}^{2} \beta_j\, Y_j \right)} \right\} \tag{64}$$

We generate a Jacobian coming off the equations system:

$$\begin{bmatrix} \mu\, p_1\, X_0 - \mu\, X_1 - \alpha_1\, \eta\, \beta_1\, Y_1\, X_1 - \beta_2\, \eta\, Y_2\, \alpha_1\, X_1 \\ \mu\, p_2\, X_0 - \mu\, X_2 - \alpha_2\, \eta\, \beta_1\, Y_1\, X_2 - \alpha_2\, \eta\, X_2\, \beta_2\, Y_2 \\ \alpha_1\, \eta\, \beta_1\, Y_1\, X_1 + \alpha_2\, \eta\, \beta_1\, Y_1\, X_2 - Y_1\, \mu - \gamma\, Y_1 - Y_1\, \delta \\ \beta_2\, \eta\, Y_2\, \alpha_1\, X_1 + \alpha_2\, \eta\, X_2\, \beta_2\, Y_2 - Y_2\, \mu - \gamma\, Y_2 - Y_2\, \delta \end{bmatrix} \tag{65}$$

substituting the infection-free equilibrium point:

$$\begin{bmatrix} -\mu, & 0, & -\alpha_1\, \eta\, \beta_1\, p_1\, X_0, & -\alpha_1\, \eta\, \beta_2\, p_1\, X_0 \\ 0, & -\mu, & -\alpha_2\, \eta\, \beta_1\, p_2\, X_0, & -\alpha_2\, \eta\, \beta_2\, p_2\, X_0 \\ 0,\, 0,\, \alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0 - \mu - \gamma - \delta,\, 0 \\ 0,\, 0,\, 0,\, \alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0 - \mu - \gamma - \delta \end{bmatrix} \tag{66}$$

We find the eingenvalues for the system,

$$\begin{gathered} -\mu,\, -\mu,\, \alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0 - \mu - \gamma - \delta, \\ \alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0 - \mu - \gamma - \delta \end{gathered} \tag{67}$$

The stability condition shall satisfy:

$$\begin{bmatrix} \alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0 - \mu - \gamma - \delta < 0 \\ \alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0 - \mu - \gamma - \delta < 0 \end{bmatrix} \tag{68}$$

and rewriting the previous equation:

$$\begin{bmatrix} \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0}{\mu + \gamma + \delta} < 1 \\ \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0}{\mu + \gamma + \delta} < 1 \end{bmatrix} \tag{69}$$

Here, we obtain the two basic reproductive numbers for S^2I^2R model:

$$\begin{bmatrix} R_{0,2} = \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0}{\mu + \gamma + \delta} \\ R_{0,1} = \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0}{\mu + \gamma + \delta} \end{bmatrix} \tag{70}$$

C. *The S^nI^mR Model*

Theorem. *For the equations system given by (44), (45) y (46). The infection-free equilibriums are locally stable if the reproductive numbers of infection R0,j<1, and is unstable if R0,j>1, with j from 1 to m, where:*

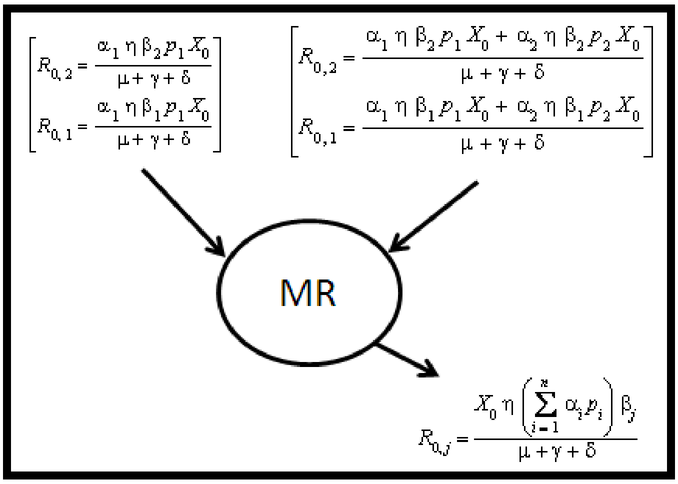
$$R_{0,j} = \frac{X_0\, \eta \left(\sum_{i=1}^{n} \alpha_i\, p_i \right) \beta_j}{\mu + \gamma + \delta} \tag{71}$$

To prove the theorem, we used mechanized induction starting from the particular results previously obtained:

$$\begin{bmatrix} R_{0,2} = \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0}{\mu + \gamma + \delta} \\ R_{0,1} = \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0}{\mu + \gamma + \delta} \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} R_{0,2} = \dfrac{\alpha_1\, \eta\, \beta_2\, p_1\, X_0 + \alpha_2\, \eta\, \beta_2\, p_2\, X_0}{\mu + \gamma + \delta} \\ R_{0,1} = \dfrac{\alpha_1\, \eta\, \beta_1\, p_1\, X_0 + \alpha_2\, \eta\, \beta_1\, p_2\, X_0}{\mu + \gamma + \delta} \end{bmatrix} \tag{73}$$

Finally, we find the general solution for the basic reproductive numbers for the S^nI^mR model according with:



IV. Conclusions

We finally obtain two theorems which can help us to demonstrate that CAS+MR are important tools for solving problems in every situation that mathematics could model. The theorems are useful to make strategies to fight against epidemic diseases in the future biological dangers.

Due to use CAS, in our case "Maple 11", we can proceed to solve the mathematical problem and we can obtain results very fast that without them could take us too much time.

The use of CAS+MR can help in teaching and learning the mathematics to engineering, whose don't have time and need to give quickly solutions. It can be implemented in engineer programs.

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