

# Simulation Study on DOA Estimation using ESPRIT Algorithm

Jose Bermudez, Ridwan C. Chin, Payam Davoodian, Alfred Tsz Yin Lok,  
Zekeriya Aliyazicioglu, H. K. Hwang

**Abstract**— An array antenna system with innovative signal processing can enhance the resolution of a signal direction of arrival (DOA) estimation. The performance of DOA using estimation signal parameter via a rotational invariant technique is investigated in this paper. The DOA angles are derived from the auto-correlation and cross-correlation matrices. Three matrix estimation methods, (1) temporal averaging, (2) spatial smoothing, (3) temporal averaging and spatial smoothing are used to evaluate the performance. Extensive computer simulations are used to demonstrate the performance of the processing algorithms.

**Index Terms**—DOA estimation, array antenna, advanced signal processing.

## I. INTRODUCTION

Accurately estimating the direction of arrival (DOA) has many important applications in communication and radar systems. Using the conventional fixed antenna, the resolution of DOA is limited by the antenna mainlobe beamwidth. Using the array antenna and advanced signal processing techniques, the DOA estimation variance can be greatly reduced.

Two important classes of signal processing techniques are the model based approach and the eigen-analysis method[1]. The model based method assumes that the received data is modeled as the output of a linear shift invariant system. The DOA information can be obtained indirectly from the estimated model parameters. Several eigen-analysis methods such as multiple signal classification (MUSIC)[2], root MUSIC[3,4], polynomial root intersection for multi-dimensional estimation (PRIME)[5,6] have been investigated by many authors. This paper studies DOA finding using estimation signal parameter via a rotational invariant technique (ESPRIT)[7]. Two different array antennas are used in this simulation study. In this paper DOA performance is discussed as a function of signal to noise ratio

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Z. Aliyazicioglu is with the Electrical and Computer Engineering Department, California State Polytechnic University, Pomona, CA 91768 USA, phone: 909-869-3667; fax: 909-869-4687; (e-mail: zaliyazici@csupomona.edu).

H. K. Hwang is with the Electrical and Computer Engineering Department, California State Polytechnic University, Pomona, CA 91768 USA phone: 909-869-2539; fax: 909-869-4687; (e-mail: hkhwangr@csupomona.edu).

Jose Bermudez, Ridwan C. Chin, Payam Davoodian, and Alfred Tsz Yin Lok are graduate students at the Electrical and Computer Engineering Department, California State Polytechnic University, Pomona, CA 91768.

(SNR), number of snapshots and the effect of spatial smoothing.

## II. ESPRIT ALGORITHM

Two different array antennas are considered in this paper, a square array with 9 elements and a honeycomb array with 19 elements. Array elements are uniformly placed on an x-y plane as shown in Figure 1. The inter-element spacing  $d$  equals half of the signal wavelength.

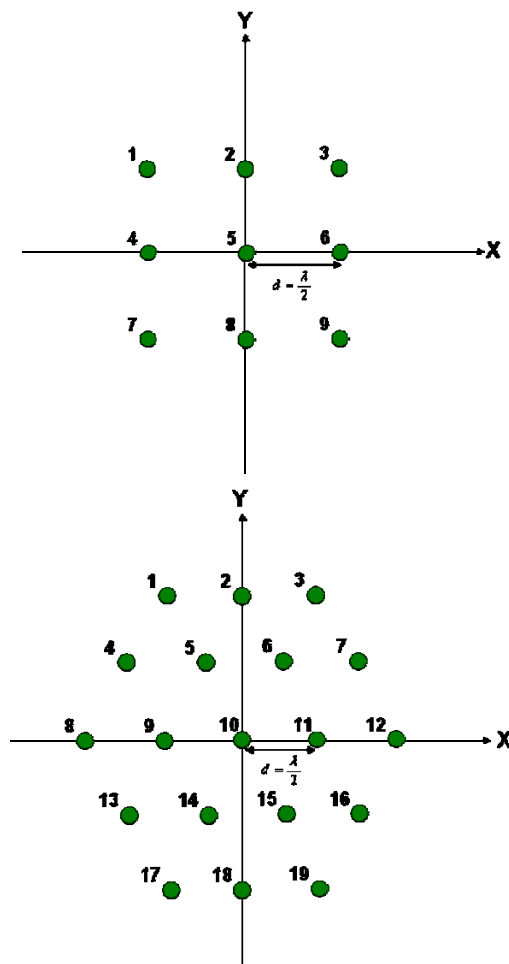


Figure 1 Two Dimensional Arrays with 9 and 19 Elements

Assume a narrowband signal impinging on the array from an elevation angle  $\theta$  and azimuth angle  $\phi$  as shown in Figure 2. Using the signal received by the center element  $s_c(t)$  as the reference, the signal received by the  $i^{\text{th}}$  element  $s_i(t)$  is

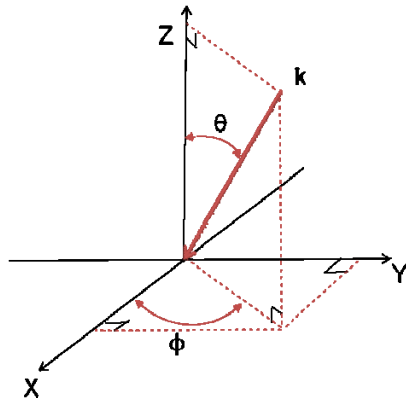


Figure 2 Coordinate of array system and signal direction

$$s_i(t) = s_c(t) e^{j\beta_i} \quad (1)$$

where the electrical angle of the  $i^{\text{th}}$  element  $\beta_i$  is

$$\beta_i = \frac{2\pi}{\lambda} \sin\theta(x_i \cos\phi + y_i \sin\phi) \quad (2)$$

where  $(x_i, y_i)$  are the coordinates of the  $i^{\text{th}}$  element.

Equation (2) shows that the signal DOA angles  $(\theta, \phi)$  are related to the electrical angle  $\beta$ . The ESPRIT algorithm derives the DOA angles from the phase factor  $\beta$ . Determining two angles  $(\theta, \phi)$  requires two different phase factors. Two independent phase factors can be derived from two independent position shifts. A brief description of ESPRIT using a 9 element square array is as follows:

The waveform  $\mathbf{y}$  received by the subset consists of elements (1, 2, 4, 5) and can be expressed as:

$$\mathbf{y}(n) = s_5(n)\mathbf{s} + [w_1(n), w_2(n), w_4(n), w_5(n)]^T \quad (3)$$

where  $s_5(n)$  is the signal received by center element,  $\mathbf{s} = [e^{j\beta_1}, e^{j\beta_2}, e^{j\beta_4}, 1]^T$ , and  $w_k(n)$   $k = 1, 2, 4, 5$  are the white noise received by the elements of this subset. The correlation matrix  $\mathbf{R}_{yy}$  of this subset is

$$\mathbf{R}_{yy} = E[\mathbf{y}\mathbf{y}^H] = \sigma_s^2 \mathbf{s}\mathbf{s}^H + \sigma_w^2 \mathbf{I} \quad (4)$$

where  $\sigma_s^2$  and  $\sigma_w^2$  are the variance of signal and noise respectively.

Shifting this subset horizontally to the right forms a new subset consisting of elements (2, 3, 5, 6). The received waveform of this new subset  $\mathbf{z}(n)$  is

$$\mathbf{z}(n) = e^{j\beta_6} s_5(n) [e^{j\beta_1}, e^{j\beta_2}, e^{j\beta_4}, 1]^T + [w_2(n), w_3(n), w_5(n), w_6(n)]^T \quad (5)$$

The cross correlation matrix  $\mathbf{R}_{yz}$  is

$$\mathbf{R}_{yz} = E[\mathbf{y}\mathbf{z}^H] = \sigma_s^2 e^{j\beta_6} \mathbf{s}\mathbf{s}^H + \sigma_w^2 \mathbf{Q} \quad (6)$$

$$\text{where } \mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Arranging the eigenvalues of matrix  $\mathbf{R}_{yy}$   $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  in descending order, the noise variance  $\sigma_w^2$  can be estimated by the following equation.

$$\sigma_w^2 = (\lambda_2 + \lambda_3 + \lambda_4)/3 \quad (7)$$

Define matrices  $\mathbf{C}_{yy}$  and  $\mathbf{C}_{yz}$  as

$$\mathbf{C}_{yy} = \mathbf{R}_{yy} - \sigma_w^2 \mathbf{I} = \sigma_s^2 \mathbf{s}\mathbf{s}^H \quad (8)$$

$$\mathbf{C}_{yz} = \mathbf{R}_{yz} - \sigma_w^2 \mathbf{Q} = \sigma_s^2 e^{j\beta_6} \mathbf{s}\mathbf{s}^H \quad (9)$$

Then  $\mathbf{C}_{yy} - \lambda \mathbf{C}_{yz} = \sigma_s^2 (1 - \lambda e^{j\beta_6}) \mathbf{s}\mathbf{s}^H$ , thus  $\lambda = e^{-j\beta_6}$  is one of the roots of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yz})$ . One phase factor  $\beta_6$  can be obtained by finding the root of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yz})$  closest to the unit circle.

Forming subset (4, 5, 7, 8) by shifting the subset (1, 2, 4, 5) down by  $d$ , the second independent phase factor can be obtained. The received data vector of this subset  $\mathbf{v}$  is:

$$\mathbf{v}(n) = e^{j\beta_8} s_5(n) [e^{j\beta_1}, e^{j\beta_2}, e^{j\beta_4}, 1]^T + [w_4(n), w_5(n), w_7(n), w_8(n)]^T \quad (10)$$

The cross correlation matrix  $\mathbf{R}_{yv}$  is

$$\mathbf{R}_{yv} = E[\mathbf{y}\mathbf{v}^H] = \sigma_s^2 e^{j\beta_8} \mathbf{s}\mathbf{s}^H + \sigma_w^2 \mathbf{Q}_1 \quad (11)$$

$$\text{where } \mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Define matrices  $\mathbf{C}_{yy}$  and  $\mathbf{C}_{yv}$  as:

$$\mathbf{C}_{yy} = \mathbf{R}_{yy} - \sigma_w^2 \mathbf{I} = \sigma_s^2 \mathbf{s}\mathbf{s}^H \quad (12)$$

$$\mathbf{C}_{yv} = \mathbf{R}_{yv} - \sigma_w^2 \mathbf{Q}_1 = \sigma_s^2 e^{j\beta_8} \mathbf{s}\mathbf{s}^H \quad (13)$$

Then  $\mathbf{C}_{yy} - \lambda \mathbf{C}_{yv} = \sigma_s^2 (1 - \lambda e^{j\beta_8}) \mathbf{s}\mathbf{s}^H$ , thus  $\lambda = e^{-j\beta_8}$  is one of the roots of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yv})$ . The second phase factor  $\beta_8$  can be obtained by finding the root of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yv})$  closest to the unit circle.

Since  $\beta_6 = \pi \sin\theta \cos\phi$ ,  $\beta_8 = -\pi \sin\theta \sin\phi$ , DOA information can be obtained by solving  $e^{j\pi \sin\theta \cos\phi} = r_1 = e^{j\alpha_1}$  where  $r_1$  is

the root of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yz})$  closest to unit circle and  $e^{-j\pi\sin\theta\sin\phi} = r_2 = e^{j\alpha_2}$  where  $r_2$  is the root of  $\det(\mathbf{C}_{yy} - \lambda \mathbf{C}_{yv})$  closest to unit circle.

$$\pi\sin\theta\cos\phi = \alpha_1 \quad (14)$$

$$-\pi\sin\theta\sin\phi = \alpha_2 \quad (15)$$

The azimuth and elevation angles can be found from the following equations.

$$\phi = \tan^{-1}(-\alpha_2/\alpha_1) \quad (16)$$

$$\theta = \sin^{-1}(\alpha_1/\pi\cos\phi) \quad (17)$$

DOA information using the 19 element honeycomb array can be obtained in similar manner.

### III. MATRIX ESTIMATION

Section 2 shows that the DOA angles are derived from the auto-correlation and cross-correlation matrices. DOA performance depends on the accurate estimation of matrices  $\mathbf{R}_{yy}$ ,  $\mathbf{R}_{yz}$ ,  $\mathbf{R}_{yv}$ . Elements of matrices are estimated from the received data  $y(n) = s(n) + w(n)$  where  $s(n)$  and  $w(n)$  are the signal and white noise of the received data. Three matrix estimation methods, (1) temporal averaging, (2) spatial smoothing, (3) temporal averaging and spatial smoothing, are described in this section[8].

#### A. Temporal Averaging Method

This method estimates the matrix element  $r_{ij}$  by averaging the products of data received by the  $i^{\text{th}}$  and  $j^{\text{th}}$  elements over  $N$  snapshots according to the following equation:

$$r_{ij} = \frac{1}{N} \sum_{n=1}^N y_i(n)y_j^*(n) \quad (18)$$

#### B. Spatial Smoothing Method

Since the number of elements in the array is larger than the size of the subset, instead of discarding the data from elements outside of the subset, those data can be used to improve the estimation of  $r_{ij}$ . For example, elements of the correlation matrix of square array for subset (1, 2, 4, 5) are computed from the following equations.

$$r_{11}(n) = r_{22} = r_{44} = r_{55} = \frac{1}{9} \sum_{i=1}^9 y_i(n)y_i^*(n) \quad (19)$$

$$r_{12}(n) = \frac{1}{6} [y_1(n)y_2^*(n) + y_2(n)y_3^*(n) + y_4(n)y_5^*(n) + y_5(n)y_6^*(n) + y_7(n)y_8^*(n) + y_8(n)y_9^*(n)] \quad (20)$$

$$r_{14}(n) = \frac{1}{6} [y_1(n)y_4^*(n) + y_2(n)y_5^*(n) + y_3(n)y_6^*(n) + y_4(n)y_7^*(n) + y_5(n)y_8^*(n) + y_6(n)y_9^*(n)] \quad (21)$$

$$r_{15}(n) = \frac{1}{4} [y_1(n)y_5^*(n) + y_2(n)y_6^*(n) + y_4(n)y_8^*(n) + y_5(n)y_9^*(n)] \quad (22)$$

$$r_{24}(n) = \frac{1}{4} [y_2(n)y_4^*(n) + y_3(n)y_5^*(n) + y_5(n)y_7^*(n) + y_6(n)y_8^*(n)] \quad (23)$$

$$r_{25}(n) = r_{14}(n) \quad (24)$$

$$r_{45}(n) = r_{12}(n) \quad (25)$$

Elements of the cross-correlation matrices can be computed in a similar manner.

#### C. Temporal Averaging and Spatial Smoothing Method

This method combines spatial smoothing and temporal averaging. After estimating the matrix elements  $r_{ij}(n)$  from spatial smoothing, an estimated  $r_{ij}$  is obtained by further averaging over  $N$  snapshots according to the following equation.

$$r_{ij} = \frac{1}{N} \sum_{n=1}^N r_{ij}(n) \quad (26)$$

### IV. SIMULATION RESULTS

Assume a tone signal impinging the array from azimuth angle  $\phi = 60^\circ$ , and elevation angle  $\theta = 30^\circ$ . The signal to noise ratio SNR = 10 dB. Using 4 element subset, Figure 3(a) shows the scatter plot based on 1000 independent simulations using the 9 element square array. The estimated  $4 \times 4$  matrices are obtained by temporal averaging over only 32 snapshots. Most of the data points are centered on the true signal direction ( $30^\circ, 60^\circ$ ). There are also points scattered over the other angles. With temporal averaging and spatial smoothing, Figure 3(b) shows an improved scatter plot where most of the data points are concentrated on ( $30^\circ, 60^\circ$ ). The averaged estimated angle errors for (a) and (b) are  $11.84^\circ$  and  $2.19^\circ$ , respectively. The estimated angle error  $\varepsilon$  is computed by the following equation.

$$\varepsilon = \cos^{-1}(\mathbf{k} \cdot \hat{\mathbf{k}}) \quad (27)$$

where  $\mathbf{k}$  is the signal direction vector and  $\hat{\mathbf{k}}$  is the direction vector of the estimated DOA.

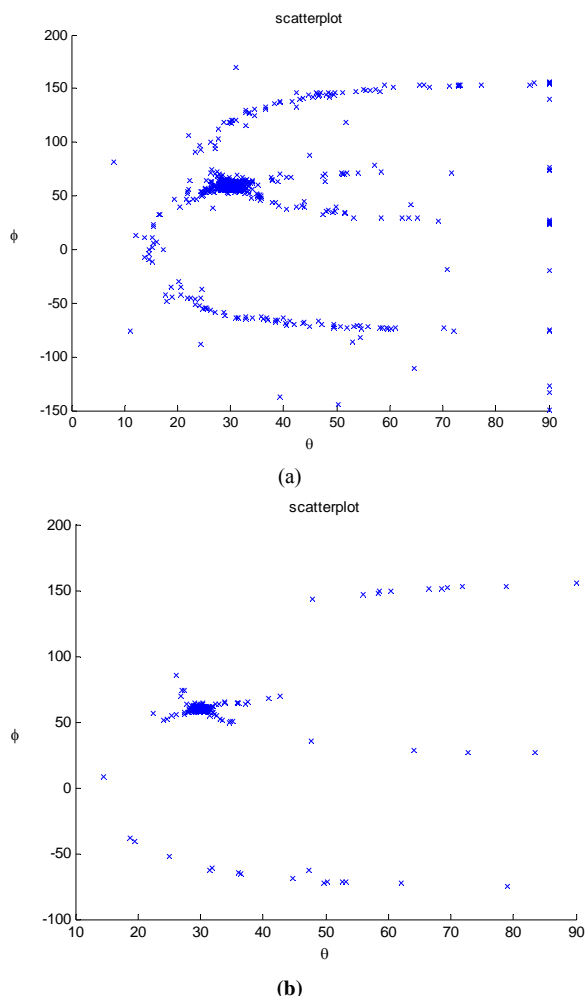


Figure 3 Scatter Plots Based on 1000 Independent Simulations

Figure 4 shows the scatter plots using a 19 element honeycomb array and  $4 \times 4$  matrices. Figure 4(a) shows the scatter plot of 1000 independent simulations assuming SNR = 10 dB using temporal averaging over 32 snapshots. The result of the combination of temporal averaging and spatial smoothing is shown in Figure 4(b). The estimated angle errors are  $12.98^\circ$  and  $0.97^\circ$  for (a) and (b) respectively. Comparing Figures 3 and 4, the 19 element honeycomb array provides much better performance with special smoothing.

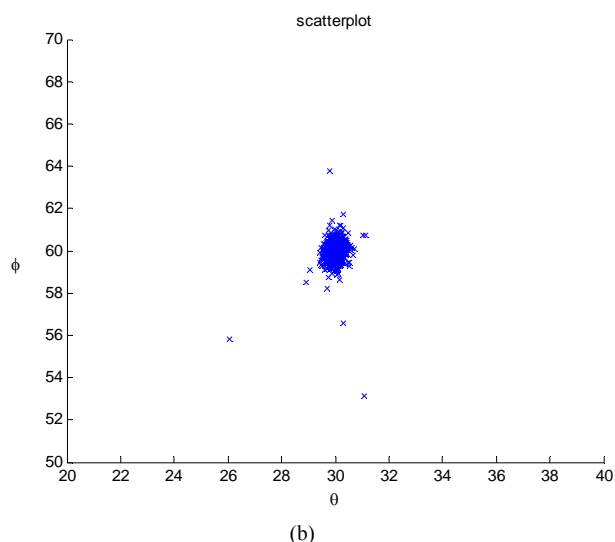
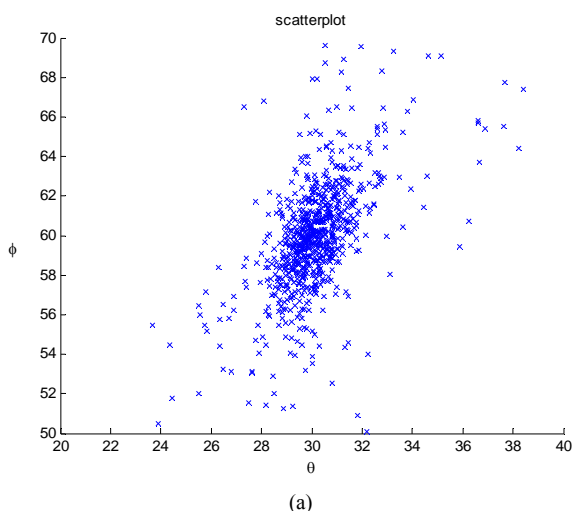


Figure 4 Scatter Plots Based on 1000 Independent Simulations

Increasing the number of temporal averaging improves DOA performance. Figure 5 shows the estimated angle error as a function of number of snapshots  $N$  using a 19 element honeycomb array. Increasing the number of temporal averaging improves the matrix element estimation; consequently the estimated angle error is reduced. The 19 element array provides sufficient number of spatial smoothing in matrix element estimation. The estimated angle error after spatial smoothing is considerably smaller than the estimated angle error without spatial smoothing. After spatial smoothing, temporal averaging over 200 snapshots provides a very good estimation. Further increasing the number of snapshots does not significantly reduce the estimation error.

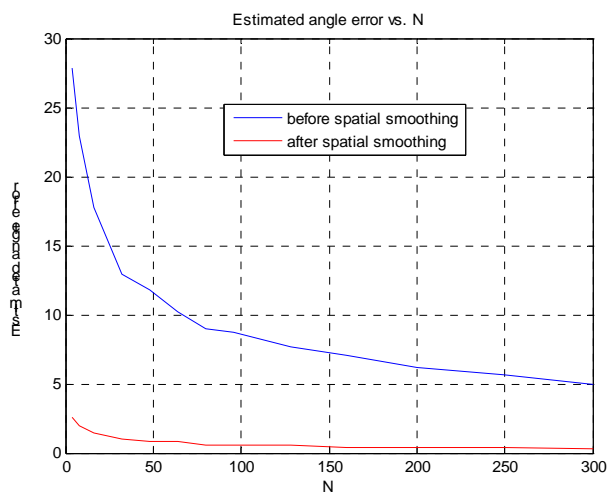


Figure 5 Estimated Angle Error as a Function of the Number of Temporal Averaging  $N$

Better SNR help improves the DOA performance. Figure 6 shows the estimated angle error using 19 element honeycomb array with  $N = 32$ .

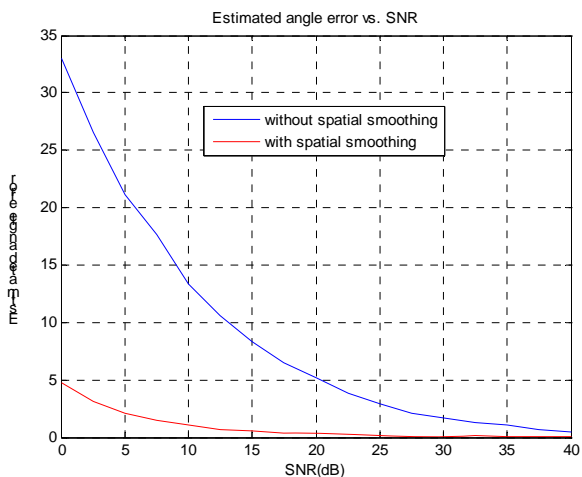


Figure 6 Estimated Angle Error as Function of SNR

With the elevation angle fixed at  $\theta = 30^\circ$ , Figure 7 shows that the estimation error is fairly independent of the azimuth angle. This result is based on using a 19 element array with SNR = 10 dB, temporal averaging over 32 snapshots; all matrices are  $4 \times 4$ . Figures 7 also indicate that estimation error can be reduced by an order of magnitude if the elements of matrix  $r_{ij}$  are estimated by spatial smoothing and temporal averaging methods.

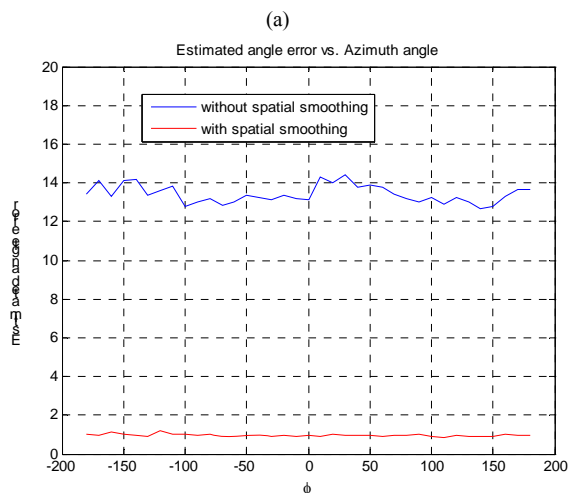


Figure 7 Estimated Angle Error as a Function of Elevation and Azimuth Angles

At high elevation angle, there are some special azimuth angles that yield a very large estimated angle error. The estimated angle error as a function of azimuth angles for 9 element arrays for signal impinging the array at high elevation angle ( $\theta = 89^\circ$ ) is shown in Figure 8. This result is based on SNR = 10 dB and matrix elements are estimated by temporal averaging over 32 snapshots and spatial smoothing method. If the signal impinging the 9 element array from azimuth angle of  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$  and  $270^\circ$ , the estimated angle error is very large. This is due to the fact that the received signal vectors of subset (1, 2, 4, 5) and subset (2, 3, 5, 6) are very close if the signal impinging the array from  $90^\circ$  or  $270^\circ$ . The received signal vector of subset (1, 2, 4, 5) and subset (4, 5, 7, 8) are very close if the signal impinging the array from  $0^\circ$  or  $180^\circ$ . The 9 element array produces very large

estimation error whenever signal impinging the array from those special azimuth angles. Similarly, if 19 element array is used to estimate signal's DOA, using the subset (1, 2, 4, 5) and shifting this subset to (2, 3, 5, 6) and (4, 5, 8, 9) would result a large estimated angle error the signal impinging the array from  $90^\circ$ ,  $270^\circ$ ,  $150^\circ$  and  $330^\circ$ .

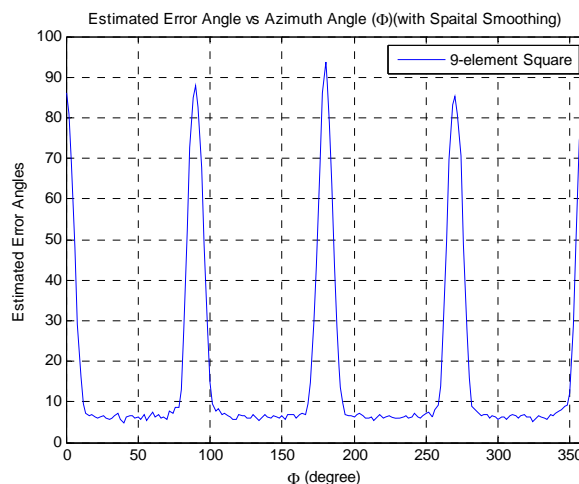


Figure 8 Estimated Angle Error vs Azimuth Angle for Elevation Angle  $\theta = 89^\circ$

## V. CONCLUSION

The conclusions based on the results of this simulation study are summarized as follows:

1. The ESPRIT method estimates signal DOA by finding the roots of two independent equations closest to the unit circle. This method does not require using a scan vector to scan over all possible directions like the MUSIC (Multiple Signal Classification) algorithm.
2. Estimation error is relatively independent of signal azimuth angle if the signal impinging the array from low elevation angle.
3. When the signal impinging the array from high elevation angle, there are some critical azimuths angles that yield a very large estimation error. This is due to the fact that at those critical azimuth angles, the received data vectors are very close. Thus there is not sufficient information to process the received data. To avoid large estimation error, we suggest to alternatively choosing a different subset and shifting the subset in different directions.
4. Estimation error can be reduced by (a) using an array containing a large number of elements, (b) increasing the number of temporal averaging in matrix element estimation.
5. Array element position may deviate from the ideal position. Position deviation will degrade DOA performance. Sensitivity analysis due to imprecise element position will be carried out in future study.

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