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Study of Fold Bifurcation in a Discrete Recurrent Neural Network

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Abstract— **A simple two-neuron model of discrete recurrent neural network is analyzed. The local stability is analyzed with the associated characteristic model. In order to study the dynamic behavior, it is considerate the Fold bifurcation. In the case of two neurons, one necessary condition that produces the Fold bifurcation is found. In addition to this, the stability and direction of the fold bifurcation are determined by applying the normal form theory and the center manifold theorem.**

Index Terms— **Nonlinear System, Neural Networks, Fold Bifurcation, Fixed Points.**

I. Introduction

The purpose of this work is to present some results on the analysis of the dynamics of a discrete recurrent neural network. The particular model of network in which we are interested is the Williams-Zipser network, also known as Discrete-Time Recurrent Neural Network (DTRNN) in [1] Its state evolution equation is

$$x_i(k+1) = f(\sum_{n=1}^{N} w_{in} x_n(k) + \sum_{m=1}^{M} w'_{im} u_m(k) + w''_i) \tag{1}$$

where

$x_i(k)$ is the *i*th neuron output.

$u_m(k)$ is the *m*th input of the network.

w_{in}, w'_{im} are the weight factors of the neuron outputs, network inputs and w''_i is a bias weight.

N is the number of neuron.

M is the number of input.

$f(\cdot)$ is a continuous, bounded, monotonically increasing function such as the hyperbolic tangent.

The neural network have presents different classes of equivalent dynamics. A system will be equivalent to another if its trajectories have the same qualitative behaviour. This is made precise mathematically in the definition of topological equivalence [2. The simplest trajectories are those who are points of equilibrium or fixed points that do not change in time. Their character or stability is given by the local behaviour of nearby trajectories. A fixed point can attract (sink), repel (source) or have directions of attraction and repulsion (saddle) of close trajectories [3]. Next in complexity are periodic trajectories, quasiperiodic trajectories or even chaotic sets, each with its own stability characterization. All this features are similar in a class of topologically equivalent systems. When a system parameter is varied the system can reach a critical point in which it is no longer equivalent. This is called a bifurcation, and the system will exhibit new behaviours. The study of how these changes can be carried out will be another powerful tool in the analysis.

With respect to discrete recurrent neural networks as systems, several results about their dynamics are available in the literature. The most general result is derived using the Lyapunov stability theorem in [4] and it stablishes that for a symmetric weight matrix there are only stable equilibrium states are fixed points and period two limit cycles and also gives the conditions under which there are only fixed-point attractors. More recently Cao [5] have proposed other condition less restrictive and more complex. In [6], chaos is found even in a simple two-neuron network in a specific weight configuration by demonstrating its equivalence with a 1-dimension chaotic system (the logistic map). We will derive necessary conditions for the onset of chaos in a general configuration for this simple model. In [7], the same author describes another interesting type of trajectories, the quasi-periodic orbits. These are closed orbits with irrational periods that appear in complex phenomena like frequency-locking and synchronisation typical of biological networks. In the same paper, conditions for the stability of these orbits are given that can be simplified as we shall show.

Passeman [8] obtains some experimental results such as the coexisting of the periodic, quasi-periodic and chaotic attractors. In other hand, In [9] give the position, number and stability types of fixed points of a two-neuron discrete recurrent network with nonzero weights are investigated.

There are some works that analyze the hopfield continous neural networks [10, 11] like [12, 13, 14, 15], in this paper shown the stability of hopf-bifurcation with two delays.

We attempt first the determination of number and stability-type characterization of the fixed points. The next subject is the analysis of fold bifurcation. Finally, the simulations are shown and conclusions are given.

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II. DETERMINATION OF FIXED POINTS

For simplicity, we have studied the two-neuron network. This allows visualizing easily the problem. In this model, we have considered zero inputs to isolate the dynamics from the input action. Secondly, and without loss of generality with respect to dynamics, we have taken zero bias weights. The activation function is the hyperbolic tangent.

With these conditions, the network mapping function is

$$\begin{aligned} x_1(k+1) &= \tanh(w_{11}x_1(k) + w_{12}x_2(k)) \\ x_2(k+1) &= \tanh(w_{21}x_1(k) + w_{22}x_2(k)) \end{aligned} \quad (1)$$

where $x(k)$ and $y(k)$ are the neural output of the step k.

The fixed points are solutions of the following equations

$$\begin{aligned} x_{1,p} &= \tanh(w_{11}x_{1,p} + w_{12}x_{2,p}) \\ x_{2,p} &= \tanh(w_{21}x_{1,p} + w_{22}x_{2,p}) \end{aligned} \quad (2)$$

The point $(0, 0)$ is always a fixed point for every value of the weights. The number of fixed points is odd because for every other fixed point $(x_{1,p}, x_{2,p})$, $(-x_{1,p},- x_{2,p})$ is also a fixed point.

To graphically determine the configuration of fixed points, we redefine the above equations as

$$\begin{aligned} x_{2,p} &= \frac{a\tanh(x_{1,p}) - w_{11}x_{1,p}}{w_{12}} = F(x_{1,p}, w_{11}, w_{12}) \\ x_{1,p} &= \frac{a\tanh(x_{2,p}) - w_{22}x_{2,p}}{w_{21}} = F(x_{2,p}, w_{22}, w_{21}) \end{aligned} \quad (3)$$

There are two qualitative behaviors function of the diagonal weights. We are going to determine the number of fixed points using the graphical representation of the above equations (3). First we can show that the graph of the F function has a maximum and a minimum if $w_{ii}>1$ or it is like the hyperbolic arctangent function with the opposite condition.

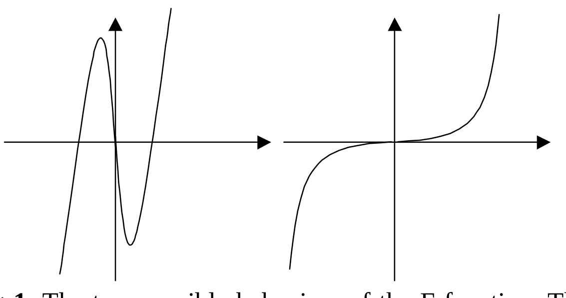

Fig 1. The two possible behaviors of the F function. The left figure corresponds to the respective diagonal weight lower than unity and on the right the opposite condition**.**

The combination of these two possibilities with another condition on the ratio of slopes in the origin of the two curves (3) gives the number of fixed points. The latter condition can be expressed as

$$|W| = w_{11} + w_{22} - 1$$

where $|W|$ is the weight matrix determinant.

If $w_{11}>1$, $w_{22}>1$ and $|W|>w_{11}+w_{22}-1$ then there can exist 9, 7 or 5 fixed points. When this condition fails there are 3 fixed points.

When a diagonal weight is less than one can be 3 or 1 fixed points.

III. LOCAL STABILITY ANALYSIS

In the development below two-neurons neural network are considerate. It is usual that the activation function is a sigmoid function or tangent hyperbolic function.

Considering the fixed point equation (2), the elements of the Jacobian matrix in the fixed point (x, y) are

$$J = \begin{bmatrix} w_{11}f'(x_1) & w_{12}f'(x_1) \\ w_{21}f'(x_2) & w_{22}f'(x_2) \end{bmatrix}$$

The associated characteristic equation of the linearized system evaluated in the fixed point is

$$\lambda^2 - [w_{11}f'(x_1) + w_{22}f'(x_2)]\lambda + |W| f'(x_1)f'(x_2) = 0 \quad (4)$$

where w_{11},w_{22} and $|W|$ are the diagonal elements and the determinant of the matrix weight, respectively.

We can define new variables

$$\sigma_1 = \frac{w_{11}f'(x_1) + w_{22}f'(x_2)}{2}$$

$$\sigma_2 = |W| f'(x_1)f'(x_2)$$

The eigenvalues of the characteristic equation (4) are defined as

$$\lambda_{\pm} = \sigma_1 \pm \sqrt{\sigma_1^2 - \sigma_2}$$

The Fold bifurcation appears when two complex conjugate eigenvalues reach the unit circle. It is easy to show that the limit conditions are

$$\lambda = 1$$

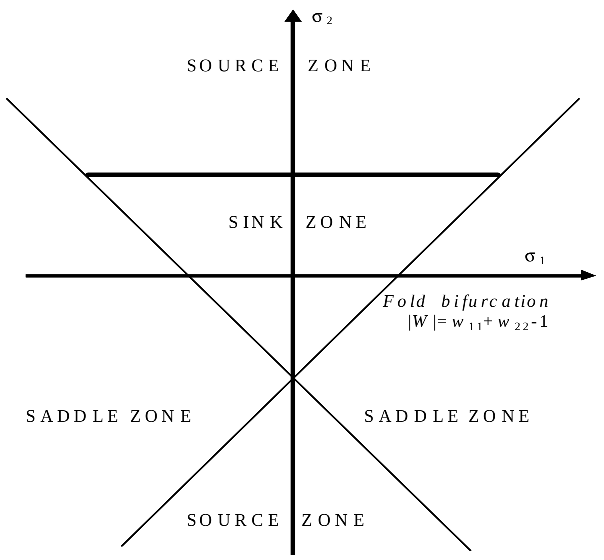

Fig. 2. The stability regions and the fold bifurcation line in the fixed point (*0, 0*).

The boundaries between the regions shown in Fig. 2 are the bifurcations, that is to say, the limit zones where the fixed point changes its character. The fold bifurcation represented by the line $|W|=w_{11}+w_{22}-1$ in Fig. 2.

IV. FOLD BIFURCATION DIRECTION

In order to determinate the direction and stability of fold bifurcation it is necessary to use the center manifold theory [2]. The center manifold theory demonstrate that the mapping behavior in the bifurcation is the complex mapping below

$$\tilde{u} = u + a(0)u^2 + c(0)u^3 + O(\|u\|^4) \qquad (5)$$

The parameters $a(0)$ and $b(0)$ are [2]

$$a(0) = \frac{1}{2}\langle p, B(q,q)\rangle \qquad (6)$$

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\langle p, B(q,(A-E)^{-1}B(q,q))\rangle \qquad (7)$$

where E is the identity matrix, B and C are the second and third derivative terms of the mapping Taylor development, respectively, J is the Jacobian matrix, the notation *INV* and <.,.> represents the inverse matrix and scalar product, respectively, and p, q are the eigenvector Jacobian matrix and its transpose, respectively. These vectors satisfy the normalization condition

$$\langle p, q\rangle = 1$$

The above coefficients are evaluated in the critical parameter of the system where produce the bifurcation takes place. In order to simplify the analytical calculation into account the quadiatric term in the equation (5), In case that $a(0)$ is zero will be necessary considerate the parameter $c(0)$ associate to third term of Taylor development. In this case, the fold bifurcation is rename [2] pitchfork bifurcation. The $a(0)$ sign determinate the bifurcation direction. When $a(0)$ is negative the stable fixed point disappear and appear two additional fixed points (saddle and source). In opposite case, $a(0)$ positive, a unstable fixed point disappear and appear two unstable fixed points.

In the neural network mapping, p and q are

$$q = \frac{d}{e+d}\left\{\frac{e}{w_{21}X_{2,0}}, -1\right\} \qquad (8)$$

$$p = \left\{\frac{e}{w_{12}X_{1,0}}, -1\right\} \qquad (9)$$

Where

$$d = w_{11}X_{1,0} - 1$$
$$e = w_{22}X_{2,0} - 1$$
$$X_{1,0} = 1 - x_{1,0}^2$$
$$X_{2,0} = 1 - x_{2,0}^2$$

$x_{1,0}$ and $x_{2.0}$ are the fixed point coordinates where the bifurcation is produced.

The Taylor development terms are

$$\begin{aligned} B_i(q,q) &= \sum_{j,k=1}^{2}\frac{\partial f_i}{\partial x_j \partial x_k} q_j q_k \\ &= f''(0)\sum_{j,k=1}^{2}\delta_{ik} w_{ij} w_{ik} q_j q_k \\ &= f''(0)\sum_{j=1}^{2} w_{ij}^2 q_j^2{}_j \end{aligned} \qquad (10)$$

$$\begin{aligned} C_i(q,q,q) &= \sum_{j,k,l=1}^{2}\frac{\partial f_i}{\partial x_j \partial x_k \partial x_l} q_j q_k q_l \\ &= f'''(0)\sum_{j,k,l=1}^{2}\delta_{ik}\delta_{il} w_{ij} w_{ik} w_{il} q_j q_k q_l \\ &= f'''(0)\sum_{j=1}^{2} w_{ij}^3 q_j^2 q_j \end{aligned} \qquad (11)$$

where δ_{ij} is the Kronecker delta.

In order to determinate the parameters $a(0)$ and $c(0)$ is necessary calculate the second and third derivates give by equations (10) and (11), respectively.

$$\frac{\partial f_i}{\partial x_j \partial x_k} = -2x_i(1-x_i^2)w_{ij}w_{ik}$$

$$\frac{\partial f_i}{\partial x_j \partial x_k \partial x_l} = 2(1-x_i^2)(3x_i^2-1)w_{ij}w_{ik}w_{il}$$

then the Taylor development terms are

$$B_i(a,b) = -2\sum_{j,k=1}^{2} x_i(1-x_i^2)w_{ij}w_{ik}a_jb_k$$

$$C_i(a,b,c) = 2\sum_{j,k,l=1}^{2}(1-x_i^2)(3x_i^2-1)w_{ij}w_{ik}w_{il}a_jb_kc_l$$

Taking account the below equations and the q autovector equation (8) then

$$\mathbf{B}(\mathrm{q},\mathrm{q}) = -2\begin{bmatrix}\dfrac{x_{1,0}X_{1,0}w_{12}^2}{(e+d)^2}\\ \dfrac{y_{2,0}X_{2,0}w_{12}^2}{(e+d)^2}\end{bmatrix}$$

$$\mathbf{C}(\mathrm{q},\mathrm{q},\mathrm{q}) = 2\begin{bmatrix}\dfrac{2X_{1,0}(3x_{1.0}^2-1)w_{12}^3}{d^3}\\ \dfrac{(1-3x_{2,0}^2)}{X_{2,0}^2}\end{bmatrix}.$$

In the rest of the section it can differentiate between the Pitchfork bifurcation associate with the zero fixed point, and Fold bifurcation that it appears in the fixed points difference to the zero.

A. Pitchfork Bifurcation in zero fixed point

Firstly, it can show, in the equation (2), that the zero is always fixed point. In this case the B coefficient give by the expression (10) is always zero. The normal form [2] is redefine like Pitchfork bifurcation, this is

$$u(k+1) = u(k) + c(0)u(k)^3 + o(u(k)^4)$$

where $c(0)$ is redefine

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle$$

because

$$B(a,b) \equiv 0.$$

Replacing the expressions of $C(q,q,q)$, q and p given by the equations (11), (8) and (9), respectively, and evaluating them in the zero fixed point, the previous coefficient is

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle = \frac{w_{12}^2(w_{22}-1)+(w_{11}-1)^3}{3(1-w_{11})^2(2-w_{11}-w_{22})}$$

and

The expression previous is not defined in the following cases:

a) $w_{11} = 1$

b) $w_{11} + w_{22} = 2$

In this paper it considerate only the a) condition because the b) condition implies that appear another kind of bifurcation known like Neimark-Sacker and it was study in previous paper [16].

Taking account the a) condition and the parameter equation that determinate the bifurcation

$$|W| = w_{11} + w_{22} - 1$$

then

$$w_{12}w_{21} = 0.$$

In this particular case the eigenvalues match with element diagonal of weight matrix

$$\lambda_1 = w_{11}$$
$$\lambda_2 = w_{22}$$

With the new q and p eigenvector give as

$$q = \{1,0\}$$
$$p = \left\{1, -\frac{w_{12}}{w_{22}-1}\right\}$$

The $c(0)$ coefficient is

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle = -\frac{1}{3}w_{11}^3 = -\frac{1}{3}$$

Therefore, in this case particularly, the coefficient of the normal form c (0) is negative, a fixed point stable becomes a saddle fixed point while that appear two stable symmetrical fixed points.

B. Fold Bifurcation

In the normal form (5) it supposed that fixed point is zero. In general the normal form to fixed point difference to zero is

$$\eta(k+1) = \beta + [1+\lambda]\eta(k) + a(0)\eta(k)^2 + o(\eta(k)^3)$$

where

$$\beta(w_{21}) = |a(0)|\frac{\partial u_0}{\partial w_{21}}(w_{21}-w_{21}^+) + O(|w_{21}-w_{21}^+|^2) \quad (12)$$

and

$$u_0 = \langle p, x_0\rangle = x_{1,0}p_1 + x_{2,0}p_2$$
$$\lambda = eigenvalue - 1$$

w_{21}^+ is the parameter that the bifurcation is produced.

$x_{1,0}$ and $x_{2,0}$ are the fixed point coordinates.

To properly determine the address the parameter variance β is necessary determinate the partial derivative that appear in equation (12)

$$\frac{\partial u_0}{\partial w_{21}} = -\frac{e x_0}{w_{12} w_{21} X_0}$$

The normal formal $a(0)$ give by the equation (6) is

$$a(0) = \frac{1}{2}\langle p, B(q,q)\rangle = \frac{\left(d^2 x_{2,0} - e w_{12} x_{1,0} X_{2,0}\right)}{2(e+d)^2 X_{2,0}}$$

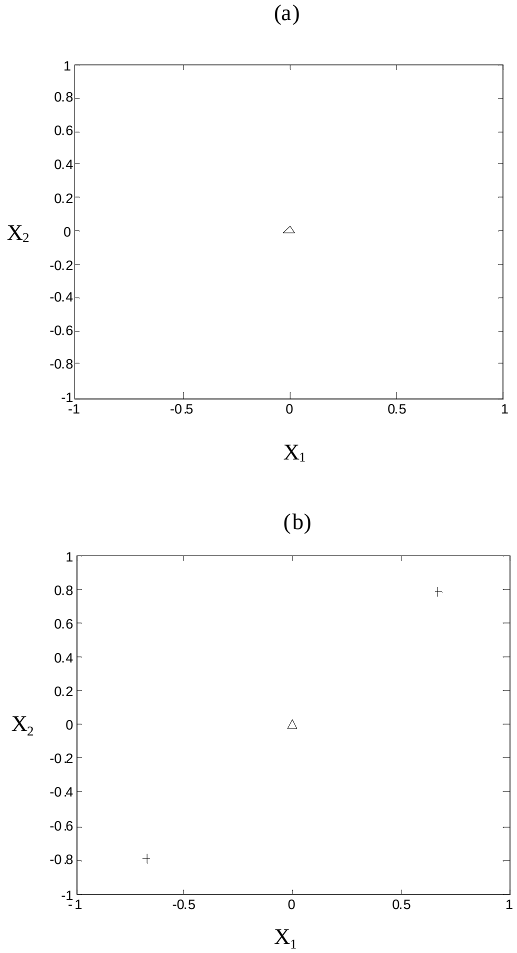

Fig. 3. The dynamical behavior when the Pitchfork bifurcation is produced. + and $\triangle$ is the saddle and source fixed point, respectively. (a): w_{11}=0.9, w_{12}=0.1, w_{21}=1 and w_{22}=0.5 ; (b): w_{11}=1.1, w_{12}=0.1, w_{21}=1 and w_{22}=0.5 .

V. SIMULATIONS

In order to show the result that has been obtained, two examples are considered. In the first simulation it is considerate the Pitchfork bifurcation, in the fig.3 is considered bifurcation the diagonal element matrix weight w_{11} like parameter that produce the Pitchfork. In the fig 3.a it can see the dynamical configuration before the bifurcation is produced, only exist one stable fixed point, in other hand when the bifurcation is produce fig 3.b appears two new stable fixed points and the zero fixed point is convert in unstable fixed point (the c bifurcation parameter is negative). In the second simulation fig 4 it show the Fold bifurcation and it takes the bifurcation parameter the non-diagonal weight element w_{12} is bifurcation parameter. In the fig 4.a. it can see the dynamical configuration before the bifurcation is produced, only exist one stable fixed point, in other hand when the bifurcation is produce fig 4.b. appears four new stable fixed points (two stable and two saddle) and the zero fixed point disappear (the a bifurcation parameter is negative).

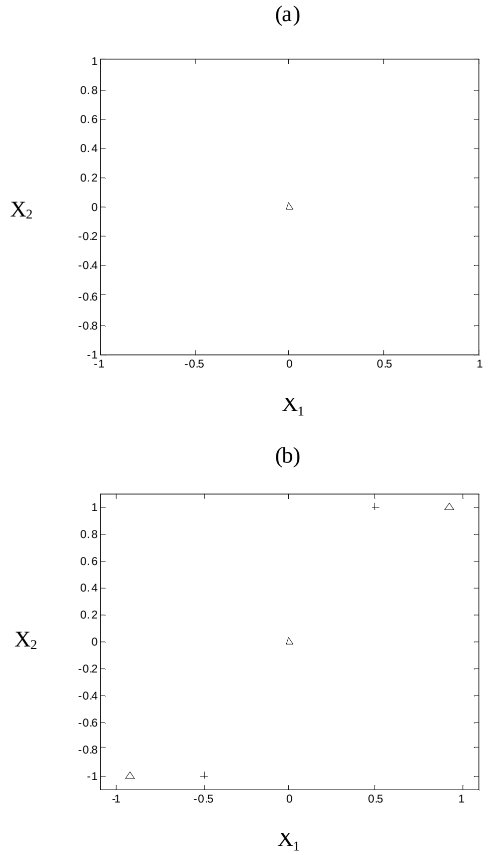

Fig. 4. The dynamical behavior when the Fold bifurcation is produced. + and $\triangle$ is the saddle and source fixed point, respectively. (a): w_{11}=2.5, w_{12}=-1, w_{21}=4 and w_{22}=3 ; (b): w_{11}=2.5, w_{12}=-0.7, w_{21}=4 and w_{22}=3 .

VI. Conclusion

In this paper we have considered a simple discrete recurrent two-neuron network model. We have analyzed the dynamical configurations and explored the very rich dynamics of this network. We have discussed the number of fixed points and the kind of stability. We have shown the bifurcation Fold direction and the dynamical behavior associated.

The two-neuron networks discussed above are quite simple, but they are potentially useful since the complexity found in these simple cases might be carried over to larger discrete recurrent neural network. There exists the possibility of generalizing some of these results to higher dimensions and use them to design training algorithms that avoid the problems associated with the learning process.

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