Proceedings of the World Congress on Engineering and Computer Science 2010 Vol II
WCECS 2010, October 20-22, 2010, San Francisco, USA

Weak Convergence of Ishikawa Iterates for Nonexpansive Maps

Abdul Rahim Khan[1], Hafiz Fukhar-ud-din[2] *

Abstract—**We establish weak convergence of the Ishikawa iterates of nonexpansive maps under a variety of new control conditions and without employing any of the properties: (i) Opial's property (ii) Fréchet differentiable norm (iii) Kadec-Klee property.**

Keywords: uniformly convex Banach space, nonexpansive map, weak convergence, Ishikawa iterates, Kadec-Klee property

1 Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A map $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

Numerous problems in mathematics and physical sciences can be formulated in a fixed point problem for noexpansive maps.In view of practical importance of these problems, methods of finding fixed points of nonexpansive maps continue to be a flourishing topic in fixed point theory. Iterative construction of fixed points of these maps is a fascinating field of research (see, [1, 4, 7, 9, 10]). In 1967, Browder [1] studied the iterative construction of fixed points of nonexpansive maps on closed and convex subsets of a Hilbert space (see also [3]).

For a map T of C into itself, we consider the Ishikawa iteration scheme: $x_1 \in C$, and

$$\begin{cases} x_{n+1} = \alpha_n T y_n + (1 - \alpha_n) x_n, \\ y_n = \beta_n T x_n + (1 - \beta_n) x_n, \quad n \geq 1. \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Set

$$\delta(r) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq r \right\}.$$

A Banach space E is uniformly convex if for each $r \in (0, 2]$, the number $\delta(r) > 0$.

*The author A. R. Khan is grateful to King Fahd University of Petroleum & Minerals and SABIC for supporting RESEARCH PROJECT SB10012. Address[1]: Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dahran, 31261, Saudia Arabia Email: arahim@kfupm.edu.sa. Address[2]: Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan. Email: hfdin@yahoo.com

For a sequence, the symbol $\to$ (resp.$\rightharpoonup$) denotes norm (resp. weak) convergence. The space E is said to satisfy : (i) *Opial's property* [8] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$; (ii) *Kadec-Klee property [6]* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The norm of E is : (iii) *Gâteaux differentiable [10]* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S and (iv) *Fréchet differentiable [10]* if for each x in S, the above limit is attained uniformly for $y \in S$.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

One of the fundamental and celebrated results in the theory of nonexpansive maps is Browder's demiclosed principle[1] which states that if C is a nonempty closed convex subset of a uniformly convex Banach space E, then for every nonexpansive map $T : C \to E$, $I - T$ is demiclosed at zero, i.e., for any $\{x_n\} \subset C, x_n \rightharpoonup x$ and $(I - T)x_n \to 0$ imply that $Tx = x$.

The above stated demiclosed principle has played an important role in the study of approximation of fixed points of nonexpansive maps through weak(strong) convergence of certain iterates.

A suitable varient of Lemma 3.1 due to Górnicki [5] for nonexpansive maps in uniformly convex Banach space is as follows:

Lemma 1.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive map of C into itself. If $x_n \rightharpoonup x(\{x_n\} \subset C, x \in C)$, then there exists strictly increasing convex map $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$g(\|x - Tx\|) \leq \liminf_{n\to\infty} \|x_n - Tx_n\|.$$

Note that Browder's demiclosed principle is a simple consequence of Lemma 1.1.

Tan and Xu[10] and Takahashi and Tamura[9], respectively, proved the following interesting results.
Theorem A. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let T be a nonexpansive map of C into itself. Then the sequence $\{x_n\}$ given by (1.1) converges weakly to a fixed point of T* provided the following condition is satisfied:
$(C1)$ $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty, \sum_{n=1}^{\infty} \beta_n(1-\alpha_n) < \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$.
Theorem B. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let T be a nonexpansive map of C into itself. Suppose that $\{x_n\}$ in (1.1) satisfies the condition:*
$(C2)$ $\alpha_n \in [a,1]$ *and* $\beta_n \in [a,b]$ *or* $\alpha_n \in [a,b]$ *and* $\beta_n \in [0,b]$ *for some* $a, b \in [0,1]$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Note that Tan and Xu's result is applicable to the case: $\alpha_n = 1-1/n$ and $\beta_n = 1/n$ for all $n \geq 1$, while Takahashi and Tamura's result is applicable to the case: $\alpha_n = \beta_n = 1/2$ for all $n \geq 1$. Moreover, in both the results, the demiclosed principle based on strong convegence of the approximate sequence $\{x_n - Tx_n\}$ to 0 has been utilized.

Using Lemma 1.1, we establish weak convergence of the Ishikawa iterates of nonexpansive maps under a variety of new parametric control conditions and without using any of the properties: (i) Opial's property (ii) Fréchet differentiable norm (iii) Kadec-Klee property.
In the sequel, we need the following lemmas.
Lemma 1.2 [11, Theorem 2]. *Let $r > 0$ be a fixed real number. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex map $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$,*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $\lambda \in [0,1]$.
Lemma 1.3 [12, Lemma 2.2]. *Let $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ be a strictly increasing map. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies $\lim_{n\to\infty} g(x_n) = 0$, then $\lim_{n\to\infty} x_n = 0$.*

2 Weak Convergence

We establish a pair of lemmas for the development of our convergence result.
Lemma 2.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be nonexpansive map with at least one fixed point. Suppose $\{x_n\}$ is given by (1.1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$.

Proof. For any $p \in F(T)$, utilizing (1.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|\alpha_n(Ty_n - p) + (1-\alpha_n)(x_n - p)\| \\
&\leq \alpha_n \|Ty_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n \|\beta_n(Tx_n - p) + (1-\beta_n)(x_n - p)\| \\
&\quad + (1-\alpha_n)\|x_n - p\| \\
&\leq (\alpha_n\beta_n + \alpha_n(1-\beta_n) + 1 - \alpha_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

This proves that $\{\|x_n - p\|\}$ is a non-increasing and bounded sequence and hence $\lim_{n\to\infty} \|x_n - p\|$ exists.
Lemma 2.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive map of C into itself with at least one fixed point. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ and satisfy one of the following three sets of conditions:
$(C3)$: $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\limsup_{n\to\infty} \beta_n < 1$;
$(C4)$: $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, $\liminf_{n\to\infty} \alpha_n > 0$;
$(C5)$: $0 \leq \alpha_n \leq b < 1, \sum_{n=1}^{\infty} \alpha_n = \infty, \beta_n \to 0$ as $n \to \infty$.

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.
Proof. Let $p \in F(T)$. With the help of Lemma 1.2 and the scheme (1.1), we have:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n(Ty_n - p) + (1-\alpha_n)(x_n - p)\|^2 \\
&\leq \alpha_n \|Ty_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \\
&\leq \alpha_n \|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \\
&\leq \alpha_n[\beta_n \|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)] \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \\
&\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \\
&\quad - \alpha_n\beta_n(1-\beta_n)g(\|x_n - Tx_n\|)
\end{aligned}$$

From the above estimate, we have the following two important inequalities:

$$\alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{2.1}$$

and

$$\alpha_n\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{2.2}$$

Case I: α_n and β_n satisfy $(C3)$.

Let $m \geq 1$. Then from the inequality (2.1), we have

$$\sum_{n=1}^{m} \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) \leq$$

$$\|x_1 - p\|^2 - \|x_{m+1} - p\|^2 < \infty$$

.

When $m \to \infty$ in the above inequality, we have

$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g(\|x_n - Ty_n\|) < \infty$. Since $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, therefore we have $\liminf_{n\to\infty} g(\|x_n - Ty_n\|) = 0$.

From Lemma 1.3, we get that $\liminf_{n\to\infty} \|x_n - Ty_n\| = 0$.

Since

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - Ty_n\| + \|Tx_n - Ty_n\| \\ &\leq \|x_n - Ty_n\| + \|x_n - y_n\| \\ &= \|x_n - Ty_n\| + \|x_n - y_n\| \\ &= \|x_n - Ty_n\| + \beta_n \|x_n - Tx_n\|, \end{aligned}$$

so we have

$$(1-\beta_n)\|x_n - Tx_n\| \leq \|x_n - Ty_n\|.$$

Therefore, from $\liminf_{n\to\infty} \|x_n - Ty_n\| = 0$ and $\limsup_{n\to\infty} \beta_n < 1$, we deduce that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Case II: α_n and β_n satisfy $(C4)$.

From the inequality (2.2), we have

$$\begin{aligned} &\sum\nolimits_{n=1}^{m} \alpha_n\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \\ &\leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 < \infty. \end{aligned}$$

Letting $m \to \infty$, we get that $\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) < \infty$.

Since $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, therefore $\liminf_{n\to\infty} \alpha_n g(\|x_n - Tx_n\|) = 0$.

That is, $(\liminf_{n\to\infty} \alpha_n)(\liminf_{n\to\infty} g(\|x_n - Tx_n\|)) = 0$.

As $\liminf_{n\to\infty} \alpha_n > 0$, therefore $\liminf_{n\to\infty} g(\|x_n - Tx_n\|) = 0$.

By Lemma 1.3, we get that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Case III: α_n and β_n satisfy $(C5)$.

Using the condition "$0 \leq \alpha_n \leq b < 1$" in the inequality(2.1), we have

$$\alpha_n(1-b)g(\|x_n - Ty_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Summing the first m terms of the above inequality, we have that

$$\begin{aligned} &(1-b)\sum\nolimits_{n=1}^{m} \alpha_n g(\|x_n - Ty_n\|) \\ &\leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 < \infty. \end{aligned}$$

When $m \to \infty$, we get $(1-b)\sum_{n=1}^{\infty} \alpha_n g(\|x_n - Ty_n\|) < \infty$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, therefore $\liminf_{n\to\infty} g(\|x_n - Ty_n\|) = 0$. Again from Lemma 1.3, we get that $\liminf_{n\to\infty} \|x_n - Ty_n\| = 0$.

Since $\|x_n - Tx_n\| \leq \|x_n - Ty_n\| + \beta_n \|x_n - Tx_n\| \leq \|x_n - Ty_n\| + \beta_n M$ for some $M > 0$ and $\beta_n \to 0$, therefore we get

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.3}$$

Now we prove our convergence result.

Theorem 2.3: Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive map of C into itself with at least one fixed point. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ and satisfy one of the three sets of conditions of Lemma 2.2. Then the sequence $\{x_n\}$ defined by (1.1), converges weakly to a fixed point of T.

Proof. Let $\omega_w(x_n)$, the weak ω-limit set of $\{x_n\}$, be given by:

$$\omega_w(x_n) = \{y \in E : x_{n_k} \rightharpoonup y \text{ for } \{x_{n_k}\} \subseteq \{x_n\}\}.$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$, therefore the sequence $\{x_n\}$ is bounded.Without any loss of generality, we can suppose that C is bounded. This gives that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in \omega_w(x_n)$ as $i \to \infty$ and vice versa. This shows that $\omega_w(x_n) \neq \phi$ and so by Lemma 1.1, $g(\|p - Tp\|) \leq \liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\|$. But $\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$ by Lemma 2.2. That is, $g(\|p - Tp\|) = 0$. By the properties of g, we get that $\|p - Tp\| = 0$. That is $p \in F(T)$ and hence $\omega_w(x_n) \subset F(T)$. Next, we follow Chang et. al[2]to prove the weak convergence of the sequence. For any $p \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} \rightharpoonup p \text{ as } j \to \infty. \tag{2.4}$$

Hence from (2.4) and continuity of T, it follows that

$$Tx_{n_i} \rightharpoonup p. \tag{2.5}$$

Now from (1.1), (2.4) and (2.5), we get that

$$y_{n_i} = (1-\beta_{n_i})x_{n_i} + \beta_{n_i} Tx_{n_i} \rightharpoonup p. \tag{2.6}$$

From (2.4), (2.6) and the continuity of T, we have that

$$Ty_{n_i} = (Ty_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p. \tag{2.7}$$

Again from (1.1) and (2.7), we conclude that

$$x_{n_i+1} = (1-\alpha_{n_i})x_{n_i} + \alpha_{n_i} Ty_{n_i} \rightharpoonup p$$

Continuing in this way, by induction, we can prove that, for any $m \geqslant 0$,

$$x_{n_i+m} \rightharpoonup p.$$

ISBN: 978-988-18210-0-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

By induction, we get that $\bigcup_{m=0}^{\infty}\{x_{n_j+m}\}$ converges weakly to p as $j \to \infty$; in fact $\{x_n\}_{n=n_1}^{\infty} = \bigcup_{m=0}^{\infty}\{x_{n_j+m}\}_{j=1}^{\infty}$ gives that $x_n \rightharpoonup p$ as $n \to \infty$.

Remark 2.4. A comparison of Theorem 2.3 with Theorem A reveals that the assumption $\sum_{n=1}^{\infty}\beta_n(1-\alpha_n) < \infty$ in Theorem A is superflous. Also, it is obvious that the assumption $(C2)$ in Theorem B implies $(C3) - (C4)$. Also Theorem A and Theorem B are established under the Opial's property or Fréchet differentiable norm. Moreover, Kadec-Klee property is required in Theorem 4.1[4] to establish the weak convergence of the Ishikawa iterates. The weak convergence theorem presented in this paper is valid in any uniformly convex Banach space.

References

[1] F. E. Browder, *Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Rational Mech. Anal.24 (1967), 82-90.

[2] S.S. Chang, Y.J. Cho, H. Zhou, *Demi-closed principal and weak convergence problems for asymptotically nonexpansive mappings,* J. Korean Math. Soc. 38 (2001), No. 6, 1245–1260.

[3] J. B. Diaz, F. T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc.73(1967), 516-519.

[4] H. Fukhar-ud-din, A. R. Khan, *Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces,* Comput. Math. Appl. 53 (2007), 1349-1360.

[5] J. Gornicki, *Nonlinear ergodic theorems for asymptotically non-expansive mappings in Banach spaces satisfying opial's condition*, J. Math. Anal. Appl. 161(1991), 440–446.

[6] W. Kaczor, *Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups,* J. Math. Anal. Appl. 272 (2002), 565-574.

[7] A. R. Khan, N. Hussain, *Iterative approximation of fixed points of nonexpansive maps*, Sci. Math. Jpn. 54(2001), 503-511.

[8] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[9] W. Takahashi, T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal. 5 (1995), 45–58.

[10] K.K.Tan, H.K. Xu, *Approximatig fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993), 301–308.

[11] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. TMA 16(12) (1991), 1127–1138.

[12] H.Y. Zhou, G. T. Guo, H. J. Hwang, Y. J. Cho, *On the iterative methods for nonlinear operator equations in Banach spaces,* PanAmer Math. J., 14(4)(2004), 61-68.

ISBN: 978-988-18210-0-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)