How to Prove the Riemann Hypothesis

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Abstract- **To prove the Riemann Hypothesis is to show that the nontrivial zeros of the Riemann zeta function, which are complex, have real part equal to 0.5 . The proof given herein is divided into two parts. Integral calculus is used in the first part, while variational calculus is employed in the second part. Given that (a) is the real part of any nontrivial zero of the Riemann zeta function, it is assumed that (a) is a fixed exponent in the equations 50-59 and hence it is verified that a=0.5. In the remaining equations beginning in equation (60) (a) is treated as a parameter (a<0.5) and a contradiction is obtained.**

At the end of the proof (from equation (73) onward), it is verified again that a = 0.5

***Index Terms*-Functional Equation, L'Hospital's Rule, Variational Caculus.**

***Subj-class*: Functional analysis, complex variables, general mathematics.**

I. Introduction

The Riemann zeta function is the function of the complex variable s = a + bi (i = $\sqrt{-1}$), defined in the half plane a >1 by the absolute convergent series

$$\zeta(s)=\sum_{1}^{\infty}\frac{1}{n^{s}} \qquad (1)$$

and in the whole complex plane by analytic continuation.

The function $\zeta(s)$ has zeros at the negative even integers -2, -4, … and one refers to them as the trivial zeros. The Riemann Hypothesis states that the nontrivial zeros of $\zeta(s)$ have real part equal to 0.5.

II, Proof of The Riemann Hypothesis

We begin with the equation

$$\zeta(s)=0 \qquad (2)$$

where

$$s= a+bi \qquad (3)$$

i.e.

$$\zeta(a+bi)=0 \qquad (4)$$

It is known that the nontrivial zeros of $\zeta(s)$ are all complex. Their real parts lie between zero and one.

If 0 < a < 1 then

$$\zeta(s) = s\int_{0}^{\infty}\frac{[x]-x}{x^{s+1}}dx \qquad (0 < a < 1) \qquad (5)$$

[x] is the integer function

Hence

$$\int_{0}^{\infty}\frac{[x]-x}{x^{s+1}}dx = 0 \qquad (6)$$

Therefore

$$\int_{0}^{\infty}([x]-x)x^{-1-a-bi}dx=0 \qquad (7)$$

$$\int_{0}^{\infty}([x]-x)x^{-1-a}x^{-bi}dx=0 \qquad (8)$$

$$\int_{0}^{\infty}x^{-1-a}([x]-x)(\cos(b\log x)-i\sin(b\log x))dx=0 \qquad (9)$$

Separating the real and imaginary parts we get

$$\int_{0}^{\infty}x^{-1-a}([x]-x)\cos(b\log x)dx=0 \qquad (10)$$

$$\int_{0}^{\infty}x^{-1-a}([x]-x)\sin(b\log x)dx=0 \qquad (11)$$

According to the functional equation, if $\zeta(s)$=0 then $\zeta(1-s)$=0.

Hence we get besides (11)

$$\int_{0}^{\infty}x^{-2+a}([x]-x)\sin(b\log x)dx=0 \qquad (12)$$

In (11) replace the dummy variable x by the dummy variable y

$$\int_{0}^{\infty}y^{-1-a}([y]-y)\sin(b\log y)dy=0 \qquad (13)$$

We form the product of the integrals (12)and (13).This is justified by the fact that both integrals (12) and (13) are absolutely convergent .As to integral (12) we notice that

$$\int_{0}^{\infty}x^{-2+a}([x]-x)\sin(b\log x)dx \le \int_{0}^{\infty}\left| x^{-2+a}([x]-x)\sin(b\log x)\right| dx$$

$$\le \int_{0}^{\infty}x^{-2+a}((x))dx$$

(where ((z)) is the fractional part of z , 0$\le$ ((z))<1)

$$= \lim(t\to 0)\int_{0}^{1-t}x^{-1+a}dx + \lim(t\to 0)\int_{1+t}^{\infty}x^{-2+a}((x))dx$$

(t is avery small positive number) (since ((x)) =x whenever 0$\le$x<1)

$$= \frac{1}{a} + \lim(t\to 0)\int_{1+t}^{\infty}x^{-2+a}((x))dx$$

$$< \frac{1}{a} + \lim(t\to 0)\int_{1+t}^{\infty}x^{-2+a}dx = \frac{1}{a}+\frac{1}{a-1}$$

As to integral (13) $\int_{0}^{\infty}y^{-1-a}([y]-y)\sin(b\log y)dy$

$$\le \int_{0}^{\infty}\left| y^{-1-a}([y]-y)\sin(b\log y)\right| dy$$

$$\le \int_{0}^{\infty}y^{-1-a}((y))dy$$

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$$= \lim (t \to 0) \int_0^{1-t} y^{-a}\, dy + \lim(t \to 0) \int_{1+t}^{\infty} y^{-1-a}((y))dy$$

(t is avery small positive number) (since ((y)) =y whenever $0 \le y<1$)

$$= \frac{1}{1-a} + \lim(t \to 0) \int_{1+t}^{\infty} y^{-1-a}((y))dy$$

$$< \frac{1}{1-a} + \int_{1+t}^{\infty} y^{-1-a} dy = \frac{1}{1-a} + \frac{1}{a}$$

Since the limits of integration do not involve x or y , the product can be expressed as the double integral

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y) \sin(b\log y)\sin(b\log x)dxdy = 0 \quad (14)$$

Thus

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y) (\cos(b\log y + b\log x) - \cos(b\log y - b\log x))dxdy = 0 \quad (15)$$

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y) (\cos(b\log xy) - \cos(b\log \frac{y}{x}))dxdy = 0 \quad (16)$$

That is

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy =$$

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log \frac{y}{x})dxdy \quad (17)$$

Consider the integral on the right-hand side of (17)

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log \frac{y}{x})dxdy \quad (18)$$

In this integral make the substitution $x = \frac{1}{z}$ $dx = \frac{-dz}{z^2}$

The integral becomes

$$\int_{\infty}^{0}\int_0^{\infty} z^{2-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)\frac{-dz}{z^2}dy \quad (19)$$

That is

$$-\int_{\infty}^{0}\int_0^{\infty} z^{-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy \quad (20)$$

This is equivalent to

$$\int_0^{\infty}\int_0^{\infty} z^{-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy \quad (21)$$

If we replace the dummy variable z by the dummy variable x, the integral takes the form

$$\int_0^{\infty}\int_0^{\infty} x^{-a} y^{-1-a}([\frac{1}{x}]-\frac{1}{x})([y]-y)\cos(b\log xy)dxdy \quad (22)$$

Rewrite this integral in the equivalent form

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy \quad (23)$$

Thus (17) becomes

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy =$$

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy \quad (24)$$

Write (24) in the form

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([y]-y)\cos(b\log xy) \{(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})-([x]-x)\}dx\,dy=0 \quad (25)$$

Let p >0 be an arbitrary small positive number.we consider the following regions in the x –y plane.

The region of integration I = $[0,\infty) \times [0,\infty)$ (26)

The large region I1 =$[p,\infty) \times [p,\infty)$ (27)

The narrow strip I 2 =$[p,\infty) \times [0,p]$ (28)

The narrow strip I 3 = $[0,p] \times [0,\infty)$ (29)

Note that

I = I1 $\bigcup$ I 2 $\bigcup$ I 3 (30)

Denote the integrand in the left hand side of (25) by F (x,y) =

$$x^{-2+a} y^{-1-a}([y]-y)\cos(b\log xy) \{(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})-([x]-x)\} \quad (31)$$

Let us find the limit of F (x,y) as $x \to \infty$ and $y \to \infty$. This limit is given by

$$\text{Lim } x^{-a} y^{-1-a} [-((y))] \cos(b\log xy) [-((\frac{1}{x})) + ((x))\, x^{2a-2}] \quad (32)$$

The above limit vanishes ,since all the functions [- ((y))] , cos (blog xy), $-((\frac{1}{x}))$, and ((x)) remain bounded as $x \to \infty$ and $y \to \infty$

Note that the function F (x,y) is defined and bounded in the region I 1. We can prove that the integral

$$\iint_{I1} F(x,y)\,dx\,dy \text{ is bounded as follows} \quad (33)$$

$$\iint_{I1} F(x,y)\,dx\,dy = \iint_{I1} x^{-a} y^{-1-a}[-((y))]\cos(b\log xy)[-((\frac{1}{x})) + ((x))\,x^{2a-2}]\,dx\,dy \quad (34)$$

$$\le \left| \iint_{I1} x^{-a} y^{-1-a}[-((y))]\cos(b\log xy)[-((\frac{1}{x})) + ((x))\,x^{2a-2}]\,dx\,dy \right|$$

$$= \left| \int_p^{\infty} (\int_p^{\infty} x^{-a} \cos(b\log xy)[-((\frac{1}{x})) + ((x))\,x^{2a-2}]\,dx)\, y^{-1-a}[-((y))]dy \right|$$

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$$\leq \int_p^\infty \left| \left(\int_p^\infty x^{-a} \cos(b\log xy) \left[-\left(\left(\frac{1}{x}\right)\right) + ((x)) \right.\right.\right.$$

$$\left. x^{2a-2} \right] dx) \Big| \; \Big| \; y^{-1-a} [-((y))] \Big| \; dy$$

$$\leq \int_p^\infty \left(\int_p^\infty x^{-a} \Big| \cos(b\log xy) \Big| \; \Big| \left[-\left(\left(\frac{1}{x}\right)\right) + \right.\right.$$

$$\left. ((x))\, x^{2a-2} \right] \Big| \; dx) \Big| \; y^{-1-a} [-((y))] \Big| \; dy$$

$$< \int_P^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$$

$$\int_P^\infty y^{-1-a}$$

$$= \frac{1}{ap^a} \int_P^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$$

$$= \frac{1}{ap^a} \{ \lim(t \to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \right.$$

$$\left. x^{2a-2} \right] dx + \lim(t \to 0)$$

$$\int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \}$$

where t is a very small arbitrary positive. number.Since the integral

$$\lim(t \to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$$

is bounded, it remains to show that $\lim(t \to 0)$

$$\int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \text{ is bounded.}$$

Since x >1 ,then $\left(\left(\frac{1}{x}\right)\right) = \frac{1}{x}$ and we have

$$\lim(t \to 0) \int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$$

$$= \lim(t \to 0) \int_{1+t}^\infty x^{-a} \left[\frac{1}{x} + ((x))\, x^{2a-2} \right] dx$$

$$= \lim(t \to 0) \int_{1+t}^\infty \left[x^{-a-1} + ((x))\, x^{a-2} \right] dx$$

$$< \lim(t \to 0) \int_{1+t}^\infty \left[x^{-a-1} + x^{a-2} \right] dx$$

$$= \frac{1}{a(1-a)}$$

Hence the boundedness of the integral $\iint_{I1}$ F(x,y) dx dy is proved.

Consider the region

$$I4 = I2 \cup I3 \tag{35}$$

We know that

$$0 = \iint_I F(x,y)\,dx\,dy = \iint_{I1} F(x,y)\,dx\,dy + \iint_{I4} F(x,y)\,dx\,dy \tag{36}$$

and that

$$\iint_{I1} F(x,y)\,dx\,dy \text{ is bounded} \tag{37}$$

From which we deduce that the integral

$$\iint_{I4} F(x,y)\,dx\,dy \text{ is bounded} \tag{38}$$

Remember that

$$\iint_{I4} F(x,y)\,dx\,dy = \iint_{I2} F(x,y)\,dx\,dy + \iint_{I3} F(x,y)\,dx\,dy \tag{39}$$

Consider the integral

$$\iint_{I2} F(x,y)\,dx\,dy \leq \Big| \iint_{I2} F(x,y)\,dx\,dy \Big| \tag{40}$$

$$= \Big| \int_0^p \left(\int_p^\infty x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x))\, x^{2a-2} \right\} \right.$$

$$\left. \cos(b\log xy)\, dx \right) \frac{1}{y^a}\, dy \Big|$$

$$\leq \int_0^p \Big| \int_p^\infty \left(x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x))\, x^{2a-2} \right\} \right.$$

$$\left. \cos(b\log xy)\, dx \right) \Big| \; \frac{1}{y^a}\, dy$$

$$\leq \int_0^p \left(\int_p^\infty \Big| x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x))\, x^{2a-2} \right\} \Big| \right.$$

$$\left. \Big| \cos(b\log xy) \Big| \; dx \right) \frac{1}{y^a}\, dy$$

$$\leq \int_P^\infty \Big| x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x))\, x^{2a-2} \right\} \Big| \; dx \times$$

$$\int_0^P \frac{1}{y^a}\, dy$$

(This is because in this region ((y)) = y). It is evident that the integral $\int_P^\infty \Big| x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x))\, x^{2a-2} \right\} \Big| \, dx$ is bounded,this was proved in the course of proving that the integral $\iint$ F(x,y) dx dy is bounded .Also it is evident that the integral I1

$$\int_0^P \frac{1}{y^a}\, dy$$

is bounded. Thus we deduce that the integral (40) $\iint_{I2}$ F(x,y) dx dy is bounded

Hence ,according to (39),the integral $\iint_{I3}$ F(x,y) dx dy is bounded.

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Now we consider the integral

$$\iint_{I3} F(x,y)\,dx\,dy \qquad (41)$$

and write it in the form

$$\iint_{I3} F(x,y)\,dx\,dy = \int_0^p \left(\int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx \qquad (42)$$

(This is because in this region ((x)) = x)

$$\leq \left| \int_0^p \left(\int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx \right|$$

$$\leq \int_0^p \left| \left(\int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy\right) \right| \left| \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a} \right| dx$$

$$\leq \int_0^p \left(\int_0^\infty y^{-1-a}\,((y))\,dy\right) \left| \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a} \right| dx$$

Now we consider the integral with respect to y

$$\int_0^\infty y^{-1-a}\,((y))\,dy \qquad (43)$$

$$= (\lim t \to 0) \int_0^{1-t} y^{-1-a} \times y\,dy + (\lim t \to 0) \int_{1+t}^\infty y^{-1-a}\,((y))\,dy$$

(where t is a very small arbitrary positive number) .(Note that ((y))=y whenever $0 \leq y<1$).

Thus we have $(\lim t \to 0) \int_{1+t}^\infty y^{-1-a}\,((y))\,dy <$

$$(\lim t \to 0) \int_{1+t}^\infty y^{-1-a}\,dy = \frac{1}{a}$$

and $(\lim t \to 0) \int_0^{1-t} y^{-1-a} \times y\,dy = \frac{1}{1-a}$

Hence the integral (43) $\int_0^\infty y^{-1-a}\,((y))\,dy$ is bounded.

Since $\left| \int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy \right| \leq \int_0^\infty y^{-1-a}\,((y))\,dy$, we conclude that the integral

$\left| \int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy \right|$ is a bounded function of x

. Let this function be H(x) . Thus we have

$$\left| \int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy \right| = H(x) \leq K \qquad (44)$$

(K is a positive number)

Now (44) gives us

$$-K \leq \int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy \leq K \qquad (45)$$

According to (42) we have

$$\iint_{I3} F(x,y)\,dx\,dy = \int_0^p \left(\int_0^\infty y^{-1-a}\,((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx$$

$$\geq \int_0^p (-K) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx$$

$$= K \int_p^0 \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx \qquad (46)$$

Since $\iint F(x,y)\,dx\,dy$ is bounded, then

$\int_p^0 \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx$ is also bounded. Therefore the integral

$$G = \int_0^p \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}\,dx \text{ is bounded} \qquad (47)$$

We denote the integrand of (47) by

$$F = \frac{1}{x^a}\{((\frac{1}{x})) - x^{2a-1}\} \qquad (48)$$

Let δ G [F] be the variation of the integral G due to the variation of the integrand δ F.

Since

$$G[F] = \int F\,dx \text{ (the integral (49) is indefinite)} \qquad (49)$$

(here we do not consider a as a parameter, rather we treat it as a given exponent)

We deduce that $\frac{\delta G[F]}{\delta F(x)} = 1$

that is

$$\delta G[F] = \delta F(x) \qquad (50)$$

But we have

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δ G[F] = $\int$ dx $\frac{\delta G[F]}{\delta F(x)}$ $\delta F(x)$ (51)

(the integral (51) is indefinite)
Using (50) we deduce that

δ G[F] = $\int$ dx $\delta F(x)$ (52)

(the integral (52) is indefinite)
Since G[F] is bounded across the elementary interval [0,p], δ G[F] is bounded across this interval (53)
From (52) we conclude that

$\delta G = \int_0^P$ dx $\delta F(x) = \int_0^P$ dx $\frac{dF}{dx}$ δx = [F δx] (at x = p)- [F δx] (at x = 0)
(54)

Since the value of [F δx] (at x = p)is bounded, we deduce from (54) that
lim (x $\rightarrow 0$) F δ x must remain bounded. (55)
Thus we must have that

(lim x $\rightarrow 0$) [δ x $\frac{1}{x^a}$ { (($\frac{1}{x}$))- x^{2a-1} }] (56)

is bounded .
First we compute

(lim x$\rightarrow 0$) $\frac{\delta x}{x^{\ a}}$ (57)

Applying L 'Hospital ' rule we get

(lim x$\rightarrow 0$) $\frac{\delta x}{x^{\ a}}$ = (lim x$\rightarrow 0$)$\frac{1}{a}\times$ x^{1-a} $\times \frac{d(\delta x)}{dx} = 0$ (58)

We conclude from (56) that the product

0 $\times$(lim x$\rightarrow 0$) { (($\frac{1}{x}$))- x^{2a-1} } (59)

must remain bounded.
Assume that a =0.5 .(remember that we are treating a as a given exponent)This value a =0.5 will guarantee that the quantity

{ (($\frac{1}{x}$))- x^{2a-1} }

will remain bounded in the limit as (x $\rightarrow 0$) .Therefore , in this case (a=0.5) (56) will approach zero as (x $\rightarrow 0$) and hence remains bounded .
Now suppose that a< 0.5 .In this case we consider a as a parameter.Hence we have

(60) G $_a$ [x]= $\int$ dx $\frac{F(x,a)}{x}x$ (60)

(the integral (60) is indefinite)
Thus

$\frac{\delta G_{\ a}[\,x\,]}{\delta x} = \frac{F(x,a)}{x}$ (61)

But we have that

$\delta G_{\ a}[\,x\,] = \int dx\ \frac{\delta G_a[x]}{\delta x}\ \delta\, x$ (62)

(the integral (62) is indefinite)
Substituting from (61) we get

$\delta G_{\ a}[\,x\,] = \int dx\ \frac{F(x,a)}{x}\delta x$ (63)

(the integral (63) is indefinite)
We return to (49) and write

G = lim (t $\rightarrow 0$) $\int_t^p Fdx$ (64)

(t is a very small positive number 0<t<p)

= { F x(at p) – lim (t $\rightarrow 0$) Fx (at t) } - lim (t $\rightarrow 0$) $\int_t^p$ x dF

Let us compute

lim (t $\rightarrow$ 0) Fx (at t) = lim (t $\rightarrow$ 0) $t^{1-a}\,((\frac{1}{t})) - t^{a} = 0$ (65)

Thus (64) reduces to

G – Fx (at p) = - lim (t $\rightarrow 0$) $\int_t^p$ x dF (66)

Note that the left – hand side of (66) is bounded. Equation (63) gives us

δ G $_a$ =lim (t$\rightarrow 0$) $\int_t^p$ dx $\frac{F}{x}\delta x$ (67)

(t is the same small positive number 0<t<p)

We can easily prove that the two integrals $\int_t^p$ x dF and $\int_t^p$ dx $\frac{F}{x}\delta x$ are absolutely convergent .Since the limits of integration do not involve any variable , we form the product of (66) and (67)

K = lim(t $\rightarrow 0$) $\int_t^p \int_t^p$ xdF$\times$ dx $\frac{F}{x}\delta x$

= lim(t $\rightarrow 0$) $\int_t^p$ FdF $\times$ $\int_t^p$ δxdx (68)

(K is a bounded quantity)
That is

K = lim(t $\rightarrow 0$) [$\frac{F^2}{2}$ (at p) - $\frac{F^2}{2}$ (at t)] $\times$ [δ x (at p) - δ x (at t)] (69)

We conclude from (69) that

{ [$\frac{F^2}{2}$ (at p) - lim(t $\rightarrow 0$) $\frac{F^2}{2}$ (at t)] $\times$[δ x (at p)] }(70)

is bounded .
(since lim(x $\rightarrow 0$) δ x = 0 , which is the same thing as lim(t $\rightarrow 0$) δ x = 0)

Since $\frac{F^2}{2}$ (at p) is bounded , we deduce at once that $\frac{F^2}{2}$ must remain bounded in the limit as (t $\rightarrow 0$), which is the same thing as saying that F must remain bounded in the limit as (x$\rightarrow 0$) .
Therefore .

lim (x $\rightarrow 0$) $\frac{((\frac{1}{x})) - x^{2a-1}}{x^{a}}$ (71)

must remain bounded

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But

$$\lim(x\to 0)\ \frac{((\frac{1}{x}))-x^{2a-1}}{x^{a}} = \lim(x\to 0)$$

$$\frac{x^{1-2a}}{x^{1-2a}}\times\frac{((\frac{1}{x}))-x^{2a-1}}{x^{a}}$$

$$= \lim(x\to 0)\frac{x^{1-2a}((\frac{1}{x}))-1}{x^{1-a}}$$

$$= \lim(x\to 0)\ \frac{-1}{x^{1-a}} \qquad (72)$$

It is evident that this last limit is unbounded. This contradicts our conclusion (71) that

$$\lim(x\to 0)\ \frac{((\frac{1}{x}))-x^{2a-1}}{x^{a}}$$ must remain bounded (for a< 0.5)

Therefore the case a<0.5 is rejected .We verify here that ,for a = 0.5 (71) remains bounded as $(x\to 0)$.

We have that

$$((\frac{1}{x}))-x^{2a-1} < 1- x^{2a-1} \qquad (73)$$

Therefore

$$\lim(a\to 0.5)\ (x\to 0)\ \frac{((\frac{1}{x}))-x^{2a-1}}{x^{a}} < \lim(a\to 0.5)$$

$$(x\to 0)\ \frac{1-x^{2a-1}}{x^{a}} \qquad (74)$$

We consider the limit

$$\lim(a\to 0.5)\ (x\to 0)\ \frac{1-x^{2a-1}}{x^{a}} \qquad (75)$$

We write

$$a = (\lim x\to 0)\ (0.5 + x) \qquad (76)$$

Hence we get

$$\lim(a\to 0.5)\ (x\to 0)\ x^{2a-1} = \lim(x\to 0)\ x^{2(0.5+x)-1}$$

$$=\lim(x\to 0)\, x^{2x} = 1 \qquad (77)$$

(Since $\lim(x\to 0)\ x^{x} = 1$)

Therefore we must apply L 'Hospital ' rule with respect to x in the limiting process (75)

$$\lim(a\to 0.5)\ (x\to 0)\ \frac{1-x^{2a-1}}{x^{a}} = \lim(a\to 0.5)$$

$$(x\to 0)\ \frac{-(2a-1)x^{2a-2}}{ax^{a-1}} \qquad (78)$$

$$= \lim(a\to 0.5)\ (x\to 0)\ \frac{(\frac{1}{a}-2)}{x^{1-a}}$$

Now we write again

$$a = (\lim x\to 0)\ (0.5 + x) \qquad (79)$$

Thus the limit (78) becomes

$$\lim(a\to 0.5)\ (x\to 0)\ \frac{(\frac{1}{a}-2)}{x^{1-a}}$$

$$= \lim\ (x\to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5-x}}$$

$$= \lim\ (x\to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}\times x^{-x}}$$

$$= \lim\ (x\to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}}$$

(Since $\lim\ (x\to 0)\ x^{-x} = 1$) (80)

We must apply L 'Hospital ' rule

$$\lim\ (x\to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}} \qquad (81)$$

$$= \lim\ (x\to 0)\ \frac{-(0.5+x)^{-2}}{0.5x^{-0.5}}$$

$$= \lim\ (x\to 0)\ \frac{-2\times x^{0.5}}{(0.5+x\ )^{2}} = 0$$

Thus we have verified here that ,for a = 0.5 (71) approaches zero as $(x\to 0)$ and hence remains bounded.

We consider the case a >0.5 . This case is also rejected, since according to the functional equation, if ($\zeta(s)$ =0) (s = a+ bi) has a root with a>0.5 ,then it must have another root with another value of a < 0.5 . But we have already rejected the case with a<0.5

Thus we are left with the only possible value of a which is a = 0.5

Therefore a = 0.5

This proves the Riemann Hypothesis .

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