# Preconditioned SSOR Iterative Method For Linear System With $M$-Matrices

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*Abstract*— **A preconditioned SSOR iterative method with a multi-parameters preconditioner $I+\widetilde{S}_\alpha$ is proposed. Some convergence and comparison results for $\alpha_i \in [0,1]$ are given when the coefficient matrix of linear system $A$ is a nonsingular $M-$matrix. Numerical example shows that our methods are superior to the basic SSOR iterative method.**

***Keywords: Preconditioner, SSOR iterative method, M-matrix***

## 1 Introduction

$$Ax = b \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown.

For simplicity, we let $A = I - L - U$, where $I$ is the identity matrix, $L$ and $U$ are strictly lower and strictly upper triangular matrices, respectively.

Assume that $A = M - N$, where

$$M = \frac{(I-\omega L)(I-\omega U)}{\omega(2-\omega)} \tag{2}$$

$$N = \frac{[(1-\omega)I+\omega L][(1-\omega)I+\omega U]}{\omega(2-\omega)} \tag{3}$$

$\omega$ is real parameter with $\omega \neq 0$ and $\omega \neq 2$.
Then the iterative matrix of the SSOR (Symmetric SOR) iterative method [1] for solving the linear system (1) is

$$\begin{aligned} S_\omega &= (I-\omega U)^{-1}[(1-\omega)I+\omega L](I-\omega L)^{-1} \\ &\quad [(1-\omega)I+\omega U] \\ &= (I-\omega U)^{-1}(I-\omega L)^{-1}[(1-\omega)I+\omega L] \\ &\quad [(1-\omega)I+\omega U] \\ &= [(I-\omega L)(I-\omega U)]^{-1}[(1-\omega)I+\omega L] \\ &\quad [(1-\omega)I+\omega U] \end{aligned} \tag{4}$$

When $\omega = 1$, SSOR iterative method becomes Symmetric Gauss-seidel method.

Now, we consider a preconditioned system of (1)

$$PAx = Pb$$

where $P$ is a nonsingular matrix.
In [2]-[7], some different preconditioners have been proposed by several authors.
In this paper, we propose a multi-parameters preconditioned SSOR iterative method with a preconditioner as following:

$$\widetilde{P} = I + \widetilde{S}_\alpha$$

where

$$\widetilde{S}_\alpha = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 0 & 0 & \ddots & \vdots \\ 0 & -\alpha_3 a_{32} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha_n a_{nn-1} & 0 \end{pmatrix}$$

Now, we consider the preconditioned linear system

$$\widetilde{A}x = \widetilde{b} \tag{5}$$

where $\widetilde{A} = (I+\widetilde{S}_\alpha)A$ and $\widetilde{b} = (I+\widetilde{S}_\alpha)b$
We express the coefficient matrix $\widetilde{A}$ of (5) as

$$\widetilde{A} = \widetilde{D} - \widetilde{L} - \widetilde{U}$$

where $\widetilde{D}$ is the diagonal matrix, $\widetilde{L}$ and $\widetilde{U}$ are strictly lower and strictly upper triangular matrices, respectively.

Then the corresponding iterative matrix of the above preconditioned SSOR method is

$$\begin{aligned} \widetilde{S}_\omega &= (\widetilde{D}-\omega\widetilde{U})^{-1}[(1-\omega)\widetilde{D}+\omega\widetilde{L}](\widetilde{D}-\omega\widetilde{L})^{-1} \\ &\quad [(1-\omega)\widetilde{D}+\omega\widetilde{U}] \\ &= (I-\omega\widetilde{D}^{-1}\widetilde{U})^{-1}[(1-\omega)I+\omega\widetilde{D}^{-1}\widetilde{L}](I \\ &\quad -\omega\widetilde{D}^{-1}\widetilde{L})^{-1}[(1-\omega)I+\omega\widetilde{D}^{-1}\widetilde{U}] \\ &= (I-\omega\widetilde{D}^{-1}\widetilde{U})^{-1}(I-\omega\widetilde{D}^{-1}\widetilde{L})^{-1}[(1-\omega)I \\ &\quad +\omega\widetilde{D}^{-1}\widetilde{L}][(1-\omega)I+\omega\widetilde{D}^{-1}\widetilde{U}] \end{aligned} \tag{6}$$

Assume that

$$\widetilde{M} = \frac{(\widetilde{D}-\omega\widetilde{L})\widetilde{D}^{-1}(\widetilde{D}-\omega\widetilde{U})}{\omega(2-\omega)} \tag{7}$$

$$\widetilde{N} = \frac{[(1-\omega)\widetilde{D}+\omega\widetilde{L}]\widetilde{D}^{-1}[(1-\omega)\widetilde{D}+\omega\widetilde{U}]}{\omega(2-\omega)} \tag{8}$$

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$\omega$ is real parameter with $\omega \neq 0$ and $\omega \neq 2$.

The remainder of the present paper is organized as follows. Next section is the preliminaries. The convergence of the proposed methods and comparison theorems are studied in section 3. In section 4, we present numerical example to confirm our theoretical analysis.

## 2 Preliminaries

In this paper, $\rho(\cdot)$ denotes the spectral radius of a matrix.

**Definition 2.1**([1]). A matrix $A$ is a $L$-matrix if $a_{ii} \geq 0$, $i = 1, 2, \cdots, n$ and $a_{ij} \leq 0$ for all $i, j = 1, 2, \cdots n$, $i \neq j$. A nonsingular $L$-matrix $A$ is a nonsingular $M$-matrix if $A^{-1} \geq 0$.

**Lemma 2.1**([8]). Let $A$ be a nonnegative $n \times n$ nonzero matrix. Then
(a) $\rho(A)$, the spectral radius of $A$, is an eigenvalue;
(b) $A$ has a nonnegative eigenvector corresponding to $\rho(A)$;
(c) $\rho(A)$is a simple eigenvalue of $A$;
(d) $\rho(A)$ increases when any entry of $A$ increases.

**Definition 2.2**([8]). For $n \times n$ real matrices$A$, $M$ and $N$, $A = M - N$ is a regular splitting of the matrix $A$ if $M$ is nonsingular with $M^{-1} \geq 0$and $N \geq 0$. Similarly, $A = M - N$ is a weak regular splitting of the matrix $A$ if $M$ is nonsingular with $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

**Theorem 2.1**([9]). Let $A^{-1} \geq 0$ and $A = M_1 - N_1 = M_2 - N_2$ be weak regular splittings, In either of the cases
(1) $N_1 \leq N_2$
(2) $M_1^{-1} \geq M_2^{-1}$, $N_1 \geq 0$
(3) $M_1^{-1} \geq M_2^{-1}$, $N_2 \geq 0$
the inequality $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$ holds.

**Lemma 2.2**([2]). Let $A$ be a nonnegative matrix. Then
(1) If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$.
(2) If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector $x$, then

$$\alpha < \rho(A) < \beta$$

and $x$ is a positive vector.

**Theorem 2.2**([8]). Let $A = M - N$ be a weak regular splitting of the matrix $A$. Then, $A$ is nonsingular with $A^{-1} \geq 0$ if and only if $\rho(M^{-1}N) < 1$.

## 3 Comparison theorems

**Lemma 3.1** Let $A$ and $\widetilde{A}$ be the coefficient matrices of the linear system (1) and (5), respectively. Let $A$ is a nonsingular $M$-matrix, Assume that $A = M - N$ and $\widetilde{A} = \widetilde{M} - \widetilde{N}$, where $M$, $N$, $\widetilde{M}$ and $\widetilde{N}$ are defined by (2), (3), (7) and (8), respectively. If $0 < \omega \leq 1$ and $0 \leq \alpha_i \leq 1 (i = 2, 3, \cdots, n)$, then $A = M - N$ and $\widetilde{A} = \widetilde{M} - \widetilde{N}$ are regular splittings of $A$ and $\widetilde{A}$, respectively.

**Proof.** Since $A$ is a nonsingular $M-$ matrix and $0 < \omega \leq 1$,

$$\begin{aligned} M^{-1} &= \omega(2-\omega)[(I-\omega L)(I-\omega U)]^{-1} \\ &= \omega(2-\omega)(I-\omega U)^{-1}(I-\omega L)^{-1} \\ &= \omega(2-\omega)[I+\omega U+(\omega U)^2+\cdots][I+ \\ &\omega L+(\omega L)^2+\cdots] \\ &\geq 0 \end{aligned}$$

$$N = \frac{[(1-\omega)I+\omega L][(1-\omega)I+\omega U]}{\omega(2-\omega)} \geq 0$$

We know that $A = M - N$ is regular splitting of $A$ by definition 2.2.

$$\begin{aligned} \widetilde{M}^{-1} &= \omega(2-\omega)[(\widetilde{D}-\omega\widetilde{L})\widetilde{D}^{-1}(\widetilde{D}-\omega\widetilde{U})]^{-1} \\ &= \omega(2-\omega)(\widetilde{D}-\omega\widetilde{U})^{-1}\widetilde{D}(\widetilde{D}-\omega\widetilde{L})^{-1} \\ &= \omega(2-\omega)(I-\omega\widetilde{D}^{-1}\widetilde{U})^{-1}(I-\omega\widetilde{D}^{-1}\widetilde{L})^{-1}\widetilde{D}^{-1} \end{aligned}$$

We know that the elements of $\widetilde{D} = (\widetilde{d}_{ij})$ are $\widetilde{d}_{ii} = 1 - \alpha_i a_{ii-1} a_{i-1i}$ when $2 \leq i \leq n$ and $\widetilde{d}_{11} = 1$.
Since $A$ is a nonsingular $M-$ matrix, from [10], we know $1 - a_{ii-1}a_{i-1i} > 0$. If $0 \leq \alpha_i \leq 1$, Then $\widetilde{d}_{ii} = 1 - \alpha_i a_{ii-1} a_{i-1i} > 0$. Thus, $\widetilde{D}^{-1} \geq 0$,
We have

$$\widetilde{U} = (\widetilde{u}_{ij}) = \begin{cases} 0, & if\ i \geq j, \\ -(a_{ij} - \alpha_i a_{ii-1} a_{i-1i}), & if\ i < j, i \neq 1. \\ -a_{1j}, & if\ i = 1 \end{cases}$$

Since $A$ is a nonsingular $M-$ matrix, $\widetilde{U} \geq 0$.
Similarly,

$$\widetilde{L} = (\widetilde{l}_{ij}) = \begin{cases} 0, & if\ i \leq j, \\ -(a_{ij} - \alpha_i a_{ii-1} a_{i-1i}), & if\ i > j. \end{cases}$$

Thus, $\widetilde{L} \geq 0$.
When $0 < \omega \leq 1$, we have

$$\begin{aligned} \widetilde{M}^{-1} &= \omega(2-\omega)(I-\omega\widetilde{D}^{-1}\widetilde{U})^{-1}(I-\omega\widetilde{D}^{-1}\widetilde{L})^{-1}\widetilde{D}^{-1} \\ &= \omega(2-\omega)[I+\omega\widetilde{D}^{-1}\widetilde{U}+(\omega\widetilde{D}^{-1}\widetilde{U})^2+\cdots] \\ &[I+\omega\widetilde{D}^{-1}\widetilde{L}+(\omega\widetilde{D}^{-1}\widetilde{L})^2+\cdots] \\ &\geq 0 \end{aligned}$$

$$\widetilde{N} = \frac{[(1-\omega)\widetilde{D}+\omega\widetilde{L}]\widetilde{D}^{-1}[(1-\omega)\widetilde{D}+\omega\widetilde{U}]}{\omega(2-\omega)} \geq 0$$

Therefore, $\widetilde{A} = \widetilde{M} - \widetilde{N}$ is regular splittings of $\widetilde{A}$.
This completes the proof of lemma 3.1.

**Theorem 3.1** Let $A$ be a nonsingular $M-$matrix, $S_\omega$ and $\widetilde{S}_\omega$ are be defined by (4) and (6), respectively. Assume that $0 < \omega \leq 1$ and $0 \leq \alpha_i \leq 1$, $i = 2, 3, \cdots, n$, then

$$\rho(\widetilde{S}_\omega) \leq \rho(S_\omega) < 1$$

**Proof.** For $\widetilde{A} = (I+\widetilde{S}_\alpha)A = \widetilde{M} - \widetilde{N}$, we get $A = (I+\widetilde{S}_\alpha)^{-1}\widetilde{M} - (I+\widetilde{S}_\alpha)^{-1}\widetilde{N}$.
Let $\widetilde{E} = (I+\widetilde{S}_\alpha)^{-1}\widetilde{M}$, $\widetilde{F} = (I+\widetilde{S}_\alpha)^{-1}\widetilde{N}$, we note that $\widetilde{E}^{-1}\widetilde{F} = \widetilde{M}^{-1}\widetilde{N}$.
Since $I+\widetilde{S}_\alpha \geq I$, $\widetilde{d}_{ii} = 1-\alpha_i a_{ii-1}a_{i-1i} \leq 1$ for $i = 2, 3, \cdots, n$ and $\widetilde{d}_{11} = 1$. We know $-(a_{ij} - \alpha_i a_{ii-1}a_{i-1i}) \geq -a_{ij}$, we obtain $\widetilde{D}^{-1} \geq I$, $\widetilde{U} \geq U$, $\widetilde{L} \geq L$
Therefore

$$\begin{aligned}\widetilde{E}^{-1} &= \widetilde{M}^{-1}(I+\widetilde{S}_\alpha)\\ &\geq \omega(2-\omega)[I+\omega\widetilde{D}^{-1}\widetilde{U}+(\omega\widetilde{D}^{-1}\widetilde{U})^2+\cdots]\\ &[I+\omega\widetilde{D}^{-1}\widetilde{L}+(\omega\widetilde{D}^{-1}\widetilde{L})^2+\cdots]\\ &\geq \omega(2-\omega)[I+\omega U+(\omega U)^2+\cdots]\\ &[I+\omega L+(\omega L)^2+\cdots]\\ &= M^{-1}\\ &\geq 0\end{aligned}$$

Since $\widetilde{E}^{-1}\widetilde{F} = \widetilde{M}^{-1}\widetilde{N} \geq 0$, it follows from definition 2.2 that $A = \widetilde{E} - \widetilde{F}$ is a weak regular splitting of $A$.
Thus $A = M - N = \widetilde{E} - \widetilde{F}$ is weak regular splitting of $A$ and $N \geq 0$. Since $A$ is a nonsingular $M-$matrix, $A^{-1} \geq 0$, it follows Theorem 2.1, we have $\rho(\widetilde{E}^{-1}\widetilde{F}) \leq \rho(M^{-1}N)$. From lemma 3.1, $A = M - N$ is regular splitting of $A$. Thus, $A = M - N$ is weak regular splitting of $A$. Since $A$ is a nonsingular $M-$matrix, we know $\rho(S_\omega) = \rho(M^{-1}N) < 1$ by Theorem 2.2.
That is

$$\rho(\widetilde{S}_\omega) \leq \rho(S_\omega) < 1$$

This completes the proof of Theorem 3.1.

In Theorem 3.1, If we take the parameter $\omega = 1$, then we obtain the comparison theorem of Symmetric Gauss-seidel iterative method .

## 4 Numerical example

**Example** The coefficient matrix $A$ of (1) is given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

We obtain the spectral radius of SSOR iterative matrix under the different preconditioners with real parameters $\omega$ and $\alpha_i (i = 2, 3, \cdots, n)$. If we denote the spectral radius of the preconditioned SSOR iterative matrix by $\rho(\widetilde{S}_{\omega 1})$ when $\alpha_2 = 0.1$, $\alpha_3 = 0.2$, $\alpha_4 = 0.3$, $\alpha_5 = 0.5$, we denote the spectral radius of the preconditioned SSOR iterative matrix by $\rho(\widetilde{S}_{\omega 2})$ when $\alpha_2 = 0.2$, $\alpha_3 = 0.3$, $\alpha_4 = 0.4$, $\alpha_5 = 0.6$, we denote the spectral radius of the preconditioned SSOR iterative matrix by $\rho(\widetilde{S}_{\omega 3})$ when $\alpha_2 = 0.5$, $\alpha_3 = 0.8$, $\alpha_4 = 0.5$, $\alpha_5 = 1$, we denote the spectral radius of the preconditioned SSOR iterative matrix by $\rho(\widetilde{S}_{\omega 4})$ when $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$, then we obtain the table 1.

**Table 1 The comparison of the spectral radius of SSOR iterative matrix**

| $\omega$ | $\rho(S_\omega)$ | $\rho(\widetilde{S}_{\omega 1})$ | $\rho(\widetilde{S}_{\omega 2})$ | $\rho(\widetilde{S}_{\omega 3})$ | $\rho(\widetilde{S}_{\omega 4})$ |
|---|---|---|---|---|---|
| $\omega = 0.10$ | 0.9300 | 0.9285 | 0.9279 | 0.9258 | 0.9240 |
| $\omega = 0.20$ | 0.8582 | 0.8553 | 0.8541 | 0.8501 | 0.8464 |
| $\omega = 0.30$ | 0.7850 | 0.7808 | 0.7791 | 0.7732 | 0.7680 |
| $\omega = 0.40$ | 0.7111 | 0.7056 | 0.7035 | 0.6960 | 0.6893 |
| $\omega = 0.50$ | 0.6373 | 0.6309 | 0.6283 | 0.6194 | 0.6115 |
| $\omega = 0.60$ | 0.5650 | 0.5577 | 0.5548 | 0.5448 | 0.5358 |
| $\omega = 0.70$ | 0.4959 | 0.4881 | 0.4849 | 0.4740 | 0.4641 |
| $\omega = 0.80$ | 0.4327 | 0.4246 | 0.4212 | 0.4097 | 0.3992 |
| $\omega = 0.90$ | 0.3791 | 0.3710 | 0.3676 | 0.3559 | 0.3449 |
| $\omega = 1$ | 0.3405 | 0.3328 | 0.3295 | 0.3178 | 0.3066 |

From Table 1, we can see that the preconditioned SSOR method proposed in Section 1 is superior to the basic SSOR iterative method. Especially, when the parameter $\alpha_i (i = 2, 3, \cdots, n)$ are equal to 1, the spectral radius of SSOR iterative matrix is the smallest when $0 < \omega \leq 1$.

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