# Variable Satellite Usage in GPS Receivers

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*Abstract*—**Consumer product global positioning system (GPS) receivers use inexpensive clocks that introduce a significant error in the location process. While only three satellites are adequate for high quality, synchronized clocks, four or more satellites are typically necessary for reasonable accuracy when inexpensive, low quality, unsynchronized clocks are in the receiver. We present an inexpensive, fast algorithm to improve the location procedure for consumer level products with low quality, unsynchronized clocks.**

*Index Terms*—**Global positioning system, GPS, multiple satellite resolution, nonlinear algorithms.**

## I. INTRODUCTION

Global positioning systems (GPS) consist of three components (called GPS segments), namely, (1) satellites with highly accurate atomic clocks in planar zones (there are currently 6 zones), (2) an almost unlimited number of receivers on or near the surface of Earth, and (3) satellite monitoring base stations on Earth. The primary goal is to accurately trilocate receivers using the center of the Earth as the origin in a three dimensional axis. Assuming a direct radio frequency connection between the set of GPS satellites and the receiver, at least 6 satellites can provide data for trilocation [1].

To solve the trilocation problem, we must

- define a model, and
- choose a nonlinear or linearized algorithm.to solve the discrete problem arising from the model.

When high quality, synchronized (with the satellites) clocks are in receivers, only 3 satellites are needed to provide accurate trilocation.

The pseudo ranging four-point problem (P4P) [2] has been used extensively and produces a system of unknowns consisting of three spatial locations and a clock basis [1]. Inexpensive clocks on the receivers are assumed in the P4P model.

The differential GPS (DGPS) [3] adds either clock dependent errors or satellite errors. Four satellites must provide positioning data for this method to be accurate.

Direct and iterative methods exist for solving the

trilocation problem. Direct methods provide closed form solutions that are fast, but are intolerant to errors in location caused by clock inconsistencies between the satellites and the receiver. Most direct methods assume that the P4P model is deterministic, which is a completely false assumption in practice [4].

Iterative methods are usually based on solving the nonlinear problem using a Newton-Raphson (NR) algorithm [5]. Errors in the initial guess do not cause the same harm as in a direct method and lead to a better trilocation when an inexpensive clock is in the receiver.

In Section II we define a model and the solution algorithm. In Section III, we demonstrate the effectiveness of the model and algorithm. In Section IV, we draw some conclusions.

## II. THE MODEL AND SOLUTION ALGORITHM

Assume there are $1 \le i \le m$ satellites $S_i$ whose exact location is defined as $(x_i, y_i, z_i)$ with exact distances to the receiver (squared) of $\rho_i^2$, respectively. Assume that the estimated location of the receiver is given by

$$\rho^e = (x^e, y^e, z^e).$$

Then for $1 \le i \le m$,

$$\rho_i = \sqrt{(x^e - x_i)^2 + (y^e - y_i)^2 + (z^e - z_i)^2} \qquad (1)$$

Assuming that we know $\{\rho_i, (x_i, y_i, z_i)\}_{i=1}^{m=3}$, then we can solve for $\rho^e$ exactly. We can approximate each $\rho_i$ using the speed of light $c$ and the exact time $s_i$ the data was sent by satellite $S_i$ and the exact time $t_i$ when it is received:

$$\rho_i = (t_i - s_i)c. \qquad (2)$$

In reality, (2) is more precisely written using the average speed of transmission $c_i^a$,

$$\rho_i = (t_i - s_i)c_i^a. \qquad (3)$$

Unlike $s_i$, both $c_i^a$ and $t_i$ are difficult to measure accurately enough.

A much more realistic model for estimating the distance between the receiver and each $S_i$ is given by

$$\rho_i^e = \rho_i + \varepsilon^R + \varepsilon_i^S, \qquad (4)$$

where $\varepsilon^R$ and $\varepsilon_i^S$ are the errors dependent on the receiver and on each $S_i$, respectively. Let the errors in the receiver's clock and average transmission speed be defined by

$$t_i^e = t_i + \Delta t_i \text{ and } c_i^a = c + \Delta c_i,\tag{5}$$

respectively. Then

$$\varepsilon^R = c\Delta t_i \text{ and } \varepsilon_i^S = (t_i - s_i)\Delta c_i.\tag{6}$$

If we substitute (4) directly into (1), then we have a system of equations with 3 equations and 7 unknowns. We could require the use of 7 satellites, but we cannot guarantee receiving transmissions from more than 6 satellites at a time. Clearly this is unsatisfactory.

We can reduce the number of required satellites to just 4 if we can compensate for either clock or satellite dependent errors (but not both at the same time). Using the DGPS methodology [3] gives us a problem similar to (1) for $1 \le i \le m = 4$:

$$\rho_i^e = \sqrt{(x^e - x_i)^2 + (y^e - y_i)^2 + (z^e - z_i)^2} + \varepsilon.\tag{7}$$

We solve (7) to get $(x^e, y^e, z^e)$ and $\varepsilon$, where $\varepsilon$ is either $\varepsilon^R$ or $\varepsilon_i^S$.

Per the DGPS methodology, we consider only $\varepsilon$ for either clock or satellite dependent errors introducing a new variable $t_u$ of which $\varepsilon = t_u c$. In practice, $t_u$ is computed by (a) acquiring accurate standard time from external time-keeping providers or (b) using the clock bias calculated by the NR method. In this paper, we obtain the value $t_u$ by (b). We define a new variable $\rho_i^E := \rho_i^e - t_u c$, then

$$\sqrt{(x^e - x_i)^2 + (y^e - y_i)^2 + (z^e - z_i)^2} = \rho_i^E.\tag{8}$$

By squaring and expanding (8), we obtain the following equations:

$$\left( \begin{array}{c} \left(x^e\right)^2 + \left(y^e\right)^2 + \left(z^e\right)^2 + x_i^2 + y_i^2 + z_i^2 \\ -2x_i x^e - 2y_i y^e - 2z_i z^e \end{array} \right) = \left(\rho_i^E\right)^2\tag{9}$$

for $i = 1, \cdots, m$. Since $\rho^e = (x^e, y^e, z^e)$ is the common variable of the equations, the quadratic terms can be eliminated by subtraction. We subtract the first equation from the rest of equations and we have a system of $(m-1)$ linear equations:

$$\left((x_j - x_1)x^e + (y_j - y_1)y^e + (z_j - z_1)z^e\right)$$
$$= \frac{1}{2}\left((x_j^2 - x_1^2) + (y_j^2 - y_1^2) + (z_j^2 - z_1^2) - \left(\left(\rho_j^E\right)^2 - \left(\rho_1^E\right)^2\right)\right),$$

for $j = 2, \cdots, m.$ The equations can be re-written using a matrix representation of the form,

$$\mathbf{Ax} = \mathbf{d}\tag{10}$$

where

$$\mathbf{A} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ \vdots & \vdots & \vdots \\ x_m - x_1 & y_m - y_1 & z_m - z_1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x^e \\ y^e \\ z^e \end{bmatrix},$$

and

$$\mathbf{d} = \frac{1}{2}\begin{bmatrix} (x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2) - \left(\left(\rho_2^E\right)^2 - \left(\rho_1^E\right)^2\right) \\ (x_3^2 - x_1^2) + (y_3^2 - y_1^2) + (z_3^2 - z_1^2) - \left(\left(\rho_3^E\right)^2 - \left(\rho_1^E\right)^2\right) \\ \vdots \\ (x_m^2 - x_1^2) + (y_m^2 - y_1^2) + (z_m^2 - z_1^2) - \left(\left(\rho_m^E\right)^2 - \left(\rho_1^E\right)^2\right) \end{bmatrix}$$

If $m = 4$, the equation is directly solvable when $\mathbf{A}^{-1}$ exists, and if $m > 4$, the problem is reduced to a linear optimization problem minimizing the norm of the residual. In the following subsections, we describe two different ways to solve the given optimization problem by changing the definition of the norm.

*A. Ordinary least squares method.*

If $m > 4$, the problem (10) leads an over-determined system and we need methods like least squares method. The ordinary least squares (OLS) method is the minimization of the residual on the $l^2$ norm, i.e.,

$$\min_{\mathbf{x}} \|\mathbf{d} - \mathbf{Ax}\|^2, \text{ where } \|\mathbf{x}\|^2 = \mathbf{x}^t \mathbf{x}.$$

This minimization problem has a unique solution, provided that the 3 columns of the matrix $\mathbf{A}$ are linearly independent. The solution is given by the solution of the normal equation,

$$\mathbf{A}^t \mathbf{Ax} = \mathbf{A}^t \mathbf{d}.$$

The optimality of the solution, in the sense of the best linear unbiased estimator, is provided under the conditions of the Gauss-Markov theorem:

$$\mathrm{E}\left[\varepsilon_i\right] = 0, \mathrm{V}\left[\varepsilon_i\right] = \sigma^2 < \infty, \text{ and } \mathrm{cov}\left[\varepsilon_i, \varepsilon_j\right] = 0,\tag{11}$$

where $\varepsilon_i = (\mathbf{d} - \mathbf{Ax})_i$.

*B. General least squares method.*

As described in [6], the condition (11) is not satisfied for the problem (10), and thus OLS method does not guarantee the optimality of the solution. Instead, the general least squares (GLS) method extends the Gauss-Markov theorem to the case where the error vector has a non-zero covariance matrix as the optimal. The GLS method minimizes the residual on the weighted norm based on the covariance matrix of $\mathbf{d}$, $\mathbf{W}^{-1}$. So the minimization problem is defined

as follows:

$$\min_{\mathbf{x}} \|\mathbf{d} - \mathbf{Ax}\|_{\mathbf{W}^{-1}}^2 , \text{ where } \|\mathbf{v}\|_{\mathbf{W}^{-1}}^2 = \mathbf{v}'\mathbf{W}^{-1}\mathbf{v}.$$

The solution of the minimization problem is given by the solution of the normal equation,

$$\mathbf{A}'\mathbf{W}^{-1}\mathbf{Ax} = \mathbf{A}'\mathbf{W}^{-1}\mathbf{d}.$$

The covariance matrix $\mathbf{W}^{-1}$ can be estimated from the set of previous observation data, namely,

$$\mathbf{W}^{-1} = \frac{1}{n-1}\sum_{k=1}^{n}(\mathbf{d}^k - \mu)(\mathbf{d}^k - \mu)^t ,$$

where $\mu = \frac{1}{n}\sum_{k=1}^{n}\mathbf{d}^k$ and $k = 1, \cdots, n$ denote different observation data at different times. To eliminate the computational cost for computing $t_u$ for each data set $k$, we ignore the term $t_u$ and use the approximation

$$\widetilde{\mathbf{W}}^{-1} = \frac{1}{n-1}\sum_{k=1}^{n}(\tilde{\mathbf{d}}^k - \tilde{\mu})(\tilde{\mathbf{d}}^k - \tilde{\mu})^t , \quad (12)$$

where

$$\tilde{\mathbf{d}}^k = \frac{1}{2}\begin{bmatrix} \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right) + \left(z_2^2 - z_1^2\right) - \left(\left(\rho_2\right)^2 - \left(\rho_1\right)^2\right) \\ \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right) + \left(z_3^2 - z_1^2\right) - \left(\left(\rho_3\right)^2 - \left(\rho_1\right)^2\right) \\ \vdots \\ \left(x_m^2 - x_1^2\right) + \left(y_m^2 - y_1^2\right) + \left(z_m^2 - z_1^2\right) - \left(\left(\rho_m\right)^2 - \left(\rho_1\right)^2\right) \end{bmatrix}$$

and $\tilde{\mu} = \frac{1}{n}\sum_{k=1}^{n}\tilde{\mathbf{d}}^k$. Since the magnitude of $t_u c$ is negligible compared to $\rho_i^E$, the use of $\rho_i$ is effective for the approximation. Numerical results confirm the validity of the approximation regardless of the number of samples for the estimation.

### C. Algorithms

We summarize the discussion in the following algorithms:

**Algorithm OLS**

Step 1.　Calculate $t_u$ using the NR method.

Step 2.　Calculate $\mathbf{A}$ and $\mathbf{d}$ in (10).

Step 3.　Solve the least squares method using the equation $\mathbf{A}'\mathbf{Ax} = \mathbf{A}'\mathbf{d}$.

**Algorithm OLS**

Step 1.　Calculate $t_u$ using the NR method.

Step 2.　Calculate the approximation of the covariance matrix, $\widetilde{\mathbf{W}}^{-1}$ in (12).

Step 3.　Calculate $\mathbf{A}$ and $\mathbf{d}$ in (10).

Step 4.　Solve the general least squares method using the equation,
$\mathbf{A}'\widetilde{\mathbf{W}}^{-1}\mathbf{Ax} = \mathbf{A}'\widetilde{\mathbf{W}}^{-1}\mathbf{d}$.

### III. NUMERICAL EXPERIMENTS

We downloaded four different data sets from land observation stations [7]. Table 3.1 provides the specifications of the data. The measurement of the pseudo ranges is based on the C/A-Code on L1. Since the positions of satellites are provided every 15 minutes, we need to interpolate the positions at every second. The method suggested in [8] is used for this purpose.

The result of the GLS method depends on the number of samples for $\widetilde{\mathbf{W}}^{-1}$, which we denote as GLS(#), where GLS uses # number of samples. The samples are chosen from historical observation data sets (see Table 3.1) and from specified observation times to calculate the covariance.

#### A. Observations on accuracy

The accuracy of each proposed method is compared with the standard NR method. Figure 3.1 depicts the Euclidean distance error versus the number of satellites for each method. From Figure 3.1, the errors of the NR and OLS methods are almost indistinguishable, but GLS shows superb results compared to the others. Although the accuracy of the OLS method is similar to the NR method, the linear property of the OLS method eliminates the iterations and makes the method favorable in the sense of computational cost.

The GLS method shows very stable results with 6 or more satellites: there are small changes in errors from 6 satellites. Note that when using at least 6 satellites the errors of the GLS method are approximately half of the other methods.

#### B. Observations on the number of samples for $\widetilde{\mathbf{W}}^{-1}$

Figure 3.2 shows the accuracy for different numbers of samples for the estimation of the covariance matrix. As shown in the (c) BJCO case of Figure 3.2, a too small number of samples (10 samples in this case) can cause an irregular result. The error with 30 samples is larger than with 15 samples is caused by the accumulation of the errors inherited from the estimation of satellites positions. The numerical results show that 15 samples is a useful sample size for the estimation of the covariance.

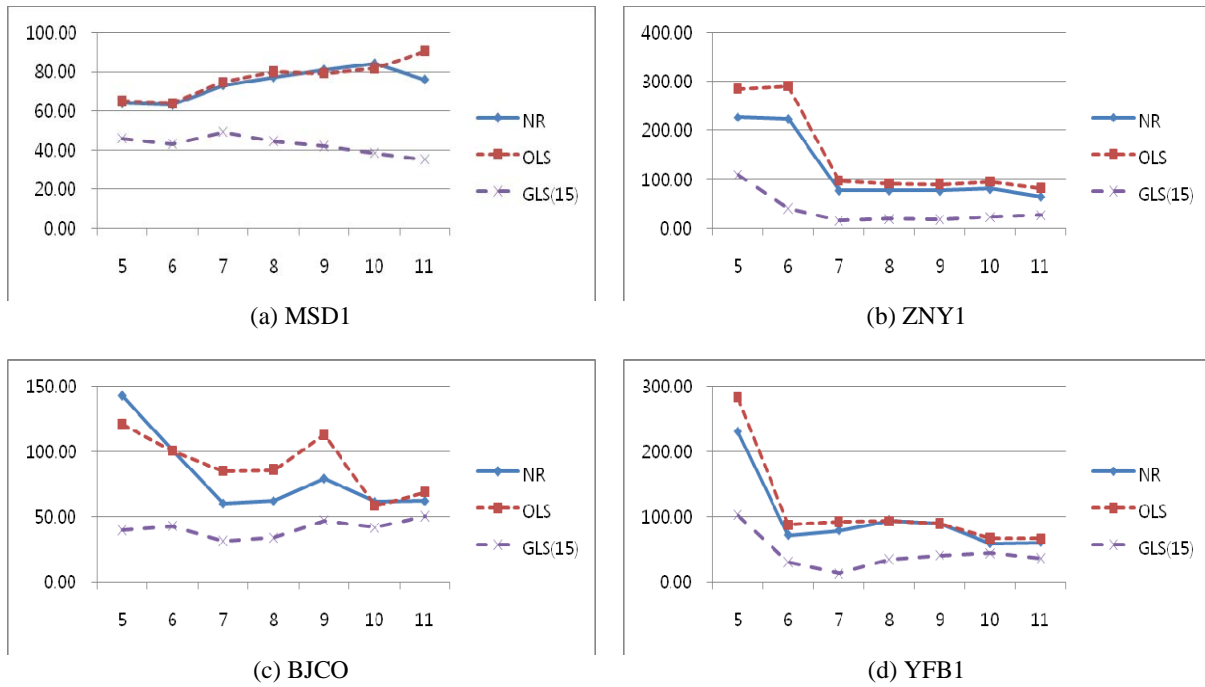| No. | Site ID | Coordinate (X,Y,Z) (m) | Time of observation |
|-----|---------|------------------------|---------------------|
| 1 | MSD1 | (-1979518.886, -5523223.079, 2493106.300) | 7/6/2010 09:00 30.00 |
| 2 | ZNY1 | (1406144.797, -4627343.637, 4144321.697) | 7/6/2010 09:00 30.00 |
| 3 | BJCO | (6333076.505, 270973.252, 704551.808) | 7/6/2010 09:00 30.00 |
| 4 | YFB1 | (1035381.682, -2634289.442, 5696539.054) | 7/6/2010 09:00 30.00 |

Table 3.1 The specification of the data.

(a) MSD1

(b) ZNY1

(c) BJCO

(d) YFB1

Figure 3.1 Accuracy comparisons: Euclidean distance errors versus number of satellites



(a) MSD1

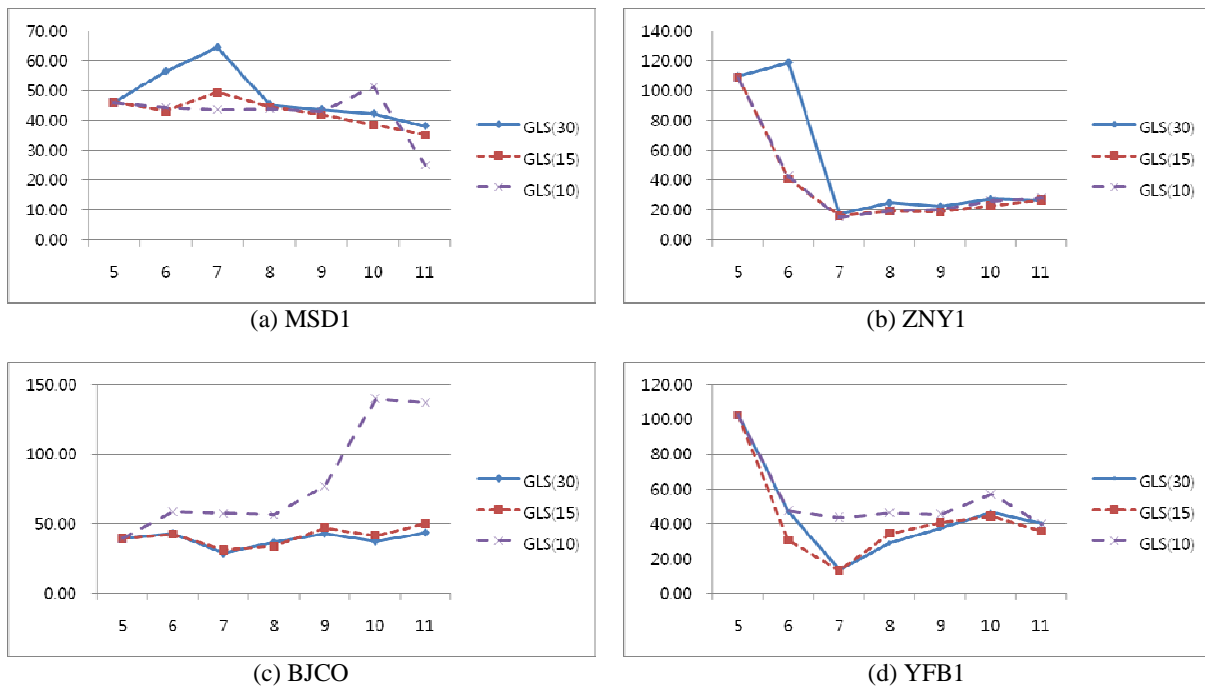(b) ZNY1

(c) BJCO

(d) YFB1

Figure 3.2 Number of samples for estimating the covariance matrix based on Euclidean distance errors versus number of satellites.

In particular, if we use $\mathbf{W}^{-1}$ instead of $\widetilde{\mathbf{W}}^{-1}$, the difference between results are less than $1m$. Thus the use of $\widetilde{\mathbf{W}}^{-1}$ is much more efficient than the exact covariance estimation.

## IV. CONCLUSIONS

We introduced new algorithms for the trilocation problem based on variable satellite usage. Due to the linearization technique used in the algorithm, we eliminate an iteration required in the standard Newton-Raphson algorithm and thus the computational cost is significantly reduced. The accuracy is improved by applying the general least squares method. The numerical results show approximately half the error compared to the standard Newton-Raphson algorithm for practical problems.

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