Preconditioned AOR Iterative Method And Comparison Theorems For Irreducible L-matrices

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Abstract—**A preconditioned AOR iterative method is proposed with the preconditioner $I + S^*_{\alpha\beta}$. Some comparison theorems are given when the coefficient matrix of linear system A is an irreducible $L-$matrix. The convergence rate of AOR iterative method with the preconditioner $I + S^*_{\alpha\beta}$ is faster than the convergence rate with the preconditioner $I + S_\alpha$ by Li et al. Numerical example verifies comparison theorems.**
Keywords: Preconditioner, AOR iterative method, irreducible L-matrix

1 Introduction

$$Ax = b \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown.

For simplicity, we let $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the AOR iterative method [1] for solving the linear system (1) is

$$L_{\gamma,\omega} = (I - \gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U] \tag{2}$$

where ω and γ are real parameters with $\omega \neq 0$.

we consider a preconditioned system of (1)

$$PAx = Pb \tag{3}$$

where P is a nonsingular matrix.
In [2], the author proposed the preconditioner $P = I + \widehat{S}_\alpha$, where

$$\widehat{S}_\alpha = \begin{pmatrix} 0 & 0 & \cdots & -\frac{a_{1n}}{\alpha} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where α is a real parameter.

Now, we consider the preconditioned linear system

$$\widehat{A}x = \widehat{b} \tag{4}$$

where $\widehat{A} = (I + \widehat{S}_\alpha)A$ and $\widehat{b} = (I + \widehat{S}_\alpha)b$
We express the coefficient matrix $\widehat{A}$ of (4) as

$$\widehat{A} = \widehat{D} - \widehat{L} - \widehat{U}$$

where $\widehat{D}$, $-\widehat{L}$ and $-\widehat{U}$ are diagonal, strictly lower and strictly upper triangular matrices of $\widehat{A}$, respectively.

$$\widehat{D} = \begin{pmatrix} 1 - \frac{a_{1n}a_{n1}}{\alpha} & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\widehat{L} = L = \begin{pmatrix} 0 & & & \\ -a_{21} & 0 & & \\ \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{pmatrix}$$

$$\widehat{U} = \begin{pmatrix} 0 & -\frac{a_{1n}a_{n2}}{\alpha} - a_{12} & \cdots & (\frac{1}{\alpha} - 1)a_{1n} \\ & 0 & \cdots & -a_{2n} \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}$$

Therefore, the preconditioned AOR iterative matrix is

$$\widehat{L}_{\gamma,\omega} = (\widehat{D} - \gamma\widehat{L})^{-1}[(1-\omega)\widehat{D} + (\omega-\gamma)\widehat{L} + \omega\widehat{U}] \tag{5}$$

In [3], H.J.Wang et al.proposed the preconditioned AOR iterative method with the preconditioner $I + S'_{\alpha\beta}$. In this paper, we propose the preconditioned AOR iterative method with $I + S^*_{\alpha\beta}$ and give some comparison theorems.

2 Preliminaries

In this paper, $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 2.1([4]). For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times n}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ holds for all $i, j = 1, 2, \cdots, n$. Calling A nonnegative matrix if $A \geq 0 (a_{ij} \geq 0,\ i, j = 1, 2, \cdots, n)$.

Definition 2.2([5]). A matrix A is a L-matrix if $a_{ii} \geq 0$, $i = 1, 2, \cdots, n$ and $a_{ij} \leq 0$ for all $i, j = 1, 2, \cdots n$, $i \neq j$.

Definition 2.3([4]). A matrix A is irreducible if the directed graph associated to A is strongly connected.

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Lemma 2.1([4]). Let A be a nonnegative $n \times n$ nonzero matrix. Then
(a) $\rho(A)$, the spectral radius of A, is an eigenvalue;
(b) A has a nonnegative eigenvector corresponding to $\rho(A)$;
(c) $\rho(A)$is a simple eigenvalue of A;
(d) $\rho(A)$ increases when any entry of A increases.

Lemma 2.2([2]). Let A be an irreducible L-matrix with $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$, If $0 \leq \gamma \leq \omega \leq 1 (\omega \neq 0, \gamma \neq 1)$, then $L_{\gamma,\omega}$ by (2) is nonnegative and irreducible.

Lemma 2.3([6]). Let A be a nonnegative matrix. Then
(1) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.
(2) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$.

3 Preconditioned AOR iterative method

We consider the preconditioned linear system

$$A^* x = b^* \tag{6}$$

where $A^* = (I + S^*_{\alpha\beta})A$ and $b^* = (I + S^*_{\alpha\beta})b$

$$S^*_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & \cdots & -\frac{a_{1n}}{\alpha} - \beta \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where α and β are real parameters.
If β is equal to zero, then $S^*_{\alpha\beta} = \widehat{S}_\alpha$
We express the coefficient matrix A^* of (6) as

$$A^* = D^* - L^* - U^*$$

where D^*, $-L^*$ and $-U^*$ are diagonal, strictly lower and strictly upper triangular matrices of A^* , respectively. where

$$D^* = \begin{pmatrix} (-\frac{a_{1n}}{\alpha} - \beta)a_{n1} + 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$L^* = L = \begin{pmatrix} 0 & & & & \\ -a_{21} & 0 & & & \\ -a_{31} & -a_{32} & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & 0 \end{pmatrix}$$

$$U^* = \begin{pmatrix} 0 & (\frac{a_{1n}}{\alpha} + \beta)a_{n2} - a_{12} & \cdots & (\frac{a_{1n}}{\alpha} + \beta) - a_{1n} \\ & 0 & \cdots & -a_{2n} \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}$$

Then the preconditioned AOR iteration matrix is

$$L^*_{\gamma,\omega} = (D^* - \gamma L^*)^{-1}[(1-\omega)D^* + (\omega - \gamma)L^* + \omega U^*] \tag{7}$$

where γ and ω are real parameters.

4 Comparison theorems

Lemma 4.1 Let A, $\widehat{A}$ and A^* be the coefficient matrices of the linear system (1), (4) and (6), respectively. Let A be an irreducible $L-$matrix with $0 < a_{1n}a_{n1} < \alpha(\alpha > 1)$, Assume that $L_{\gamma,\omega}$, $\widehat{L}_{\gamma,\omega}$ and $L^*_{\gamma,\omega}$ are defined by (2), (5) and (7), respectively. If $0 \leq \gamma \leq \omega \leq 1$, $(\gamma \neq 1, \omega \neq 0)$ and $\beta \in (-\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha}) \cap ((1 - \frac{1}{\alpha})a_{1n}, \frac{a_{1n}}{\alpha})$, then $L_{\gamma,\omega}$, $\widehat{L}_{\gamma,\omega}$ and $L^*_{\gamma,\omega}$ are nonnegative and irreducible.

Proof. The proof of $L_{\gamma,\omega}$ and $\widehat{L}_{\gamma,\omega}$ have been given in [2].
Now, we prove that $L^*_{\gamma,\omega}$ is nonnegative and irreducible.

$$\begin{aligned} L^*_{\gamma,\omega} &= (D^* - \gamma L^*)^{-1}[(1-\omega)D^* + (\omega-\gamma)L^* + \omega U^*] \\ &= (I - \gamma D^{*-1}L^*)^{-1}[(1-\omega)I + (\omega-\gamma)D^{*-1}L^* \\ &+\omega D^{*-1}U^*] \\ &= [I + \gamma D^{*-1}L^* + (\gamma D^{*-1}L^*)^2 + (\gamma D^{*-1}L^*)^3 \\ &+ \cdots][(1-\omega)I + (\omega-\gamma)D^{*-1}L^* + \omega D^{*-1}U^*] \\ &= (1-\omega)I + (\omega-\gamma)D^{*-1}L^* + \omega D^{*-1}U^* \\ &+[\gamma D^{*-1}L^* + (\gamma D^{*-1}L^*)^2 + (\gamma D^{*-1}L^*)^3 + \cdots] \\ &[(1-\omega)I + (\omega-\gamma)D^{*-1}L^* + \omega D^{*-1}U^*] \end{aligned}$$

If $\beta \in (-\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha}) \cap ((1 - \frac{1}{\alpha})a_{1n}, \frac{a_{1n}}{\alpha})$ and $0 < a_{1n}a_{n1} < \alpha$, then $L^*_{\gamma,\omega} \geq 0$. If A is irreducible and $0 \leq \gamma \leq \omega \leq 1$, then $(1-\omega)I + (\omega-\gamma)D^{*-1}L^* + \omega D^{*-1}U^*$ is irreducible. Thus, $L^*_{\gamma,\omega}$ is irreducible. This completes the proof.

Theorem 4.1 Let A is an irreducible L-matrix with $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$, Assume that $L_{\gamma,\omega}$, $\widehat{L}_{\gamma,\omega}$ and $L^*_{\gamma,\omega}$ are defined by (2), (5) and (7), respectively. If $\beta \in (-\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, 0) \cap ((1 - \frac{1}{\alpha})a_{1n}, 0)$ and $0 \leq \gamma \leq \omega \leq 1$, $(\omega \neq 0, \gamma \neq 1)$, then
$\rho(L^*_{\gamma,\omega}) \leq \rho(\widehat{L}_{\gamma,\omega}) < \rho(L_{\gamma,\omega})$(If $\rho(L_{\gamma,\omega}) < 1$))

Proof. When $\rho(L_{\gamma,\omega}) < 1$, the proof of $\rho(\widehat{L}_{\gamma,\omega}) < \rho(L_{\gamma,\omega})$ has been given in [2].
Now we prove that $\rho(L^*_{\gamma,\omega}) \leq \rho(\widehat{L}_{\gamma,\omega})$
$\widehat{L}_{\gamma,\omega}$ and $L^*_{\gamma,\omega}$ are nonnegative and irreducible matrices from Lemma 4.1. We know that there exists a nonnegative eigenvector x, such that $\widehat{L}_{\gamma,\omega}x = \lambda x$ from Lemma 2.1.
Assume $\lambda = \rho(\widehat{L}_{\gamma,\omega})$
From (5), we have

$$[(1-\omega)\widehat{D} + (\omega-\gamma)\widehat{L} + \omega\widehat{U}]x = \lambda(\widehat{D} - \gamma\widehat{L})x \tag{8}$$

From (8), we have
$[(1-\omega)\widehat{D} + (\omega - \gamma + \lambda\gamma)\widehat{L} + \omega\widehat{U} - \lambda\widehat{D}]x = 0$
It is easy to see that

$$L^* = L$$

$$D^* - U^* = \widehat{D} - \widehat{U} - S\widehat{L} + S \tag{9}$$

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where

$$S = \begin{pmatrix} 0 & 0 & \cdots & -\beta \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

If $\beta \in (-\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, 0) \cap ((1 - \frac{1}{\alpha})a_{1n}, 0)$
From (8) and (9), we have

$$\begin{aligned} L^*_{\gamma,\omega}x - \lambda x &= (D^* - \gamma L^*)^{-1}[(1-\omega)D^* + (\omega-\gamma)L^* + \\ &\omega U^* - \lambda(D^* - \gamma L^*)]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\omega)D^* + (\omega-\gamma)\widehat{L} + \\ &\omega(D^* - \widehat{D} + \widehat{U} + S\widehat{L} - S) - \lambda(D^* - \\ &\lambda\widehat{L})]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\lambda)D^* - \omega\widehat{D} + (\omega- \\ &\gamma)\widehat{L} + \omega S\widehat{L} + \omega\widehat{U} - \omega S + \lambda\gamma\widehat{L}]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\lambda)D^* - \omega\widehat{D} + (\omega- \\ &\gamma + \lambda\gamma)\widehat{L} + \omega S\widehat{L} + \omega\widehat{U} - \omega S]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\lambda)D^* - \omega\widehat{D} - (1- \\ &\omega - \lambda)\widehat{D} + \omega S\widehat{L} - \omega S]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\lambda)D^* - (1-\lambda)\widehat{D} + \\ &\omega(S\widehat{L} - S)]x \\ &= (D^* - \gamma L^*)^{-1}[(1-\lambda)(D^* - \widehat{D}) + \\ &\omega(S\widehat{L} - S)]x \end{aligned}$$

From (8), we have

$$[(1-\omega)\widehat{D} + (\omega-\gamma)\widehat{L} + \omega\widehat{U}]x = \lambda(\widehat{D} - \gamma\widehat{L})x$$

Thus,

$$(\omega - \gamma + \lambda\gamma)\widehat{L}x = [(\lambda - 1 + \omega)\widehat{D} - \omega\widehat{U}]x$$

We know that $S\widehat{D} = S$, $D^* - \widehat{D} \le 0$
and $S\widehat{L}x = [\frac{\lambda-1+\omega}{\omega-\gamma+\lambda\gamma}S\widehat{D} - \frac{\omega}{\omega-\gamma+\lambda\gamma}S\widehat{U}]x$
$= [\frac{\lambda-1+\omega-(\omega-\gamma+\lambda\gamma)}{\omega-\gamma+\lambda\gamma}\overrightarrow{S} - \frac{\omega}{\omega-\gamma+\lambda\gamma}S\widehat{U}]x$
Since $S\widehat{U} = 0$, We obtain
$(S\widehat{L} - S)x = [\frac{(1-\gamma)(\lambda-1)}{\omega-\gamma+\lambda\gamma}S - \frac{\omega}{\omega-\gamma+\lambda\gamma}S\widehat{U}]x$
$= \frac{(1-\gamma)(\lambda-1)}{\omega-\gamma+\lambda\gamma}Sx$
Since $(D^* - \gamma L^*)^{-1} \ge 0$, if $\lambda < 1, 0 \le \gamma \le \omega \le 1$, then $L^*_{\gamma,\omega}x - \lambda x \le 0$. From Lemma 2.3, we have $\rho(L^*_{\gamma,\omega}) \le \lambda$. Therefore, $\rho(L^*_{\gamma,\omega}) \le \rho(\widehat{L}_{\gamma,\omega})$. This completes the proof.

Remark 4.1 If $\gamma = \omega$, AOR iterative method becomes SOR iterative method. Thus, we obtain the comparison theorem of the preconditioned SOR iterative method.

Corollary 4.1 Let L_ω, $\widehat{L}_\omega$ and L^*_ω be the iterative matrices of the SOR iterative method associated to (1), (4) and (6), respectively. If the matrix A of (1) is an irreducible $L-$ matrix with $0 < a_{1n}a_{n1} < \alpha(\alpha > 1)$ and $\beta \in (\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, 0) \cap ((1 - \frac{1}{\alpha})a_{1n}, 0)$, and $0 < \omega < 1$, then $\rho(L^*_\omega) \le \rho(\widehat{L}_\omega) < \rho(L_\omega)$ (If $\rho(L_\omega) < 1$)

Remark 4.2 Let $\gamma = 0$ and $\omega = 1$, AOR iterative method becomes Jacobi iterative method. Thus, we obtain the comparison theorem of the preconditioned Jacobi iterative method.

Corollary 4.2 Let $L_{0,1}$, $\widehat{L}_{0,1}$ and $L^*_{0,1}$ be the iterative matrices of the Jacobi iterative method associated to (1), (4) and (6), respectively. If the matrix A of (1) is an irreducible $L-$ matrix with $0 < a_{1n}a_{n1} < \alpha(\alpha > 1)$ and $\beta \in (\frac{a_{1n}}{\alpha} + \frac{1}{a_{n1}}, 0) \cap ((1 - \frac{1}{\alpha})a_{1n}, 0)$, then $\rho(L^*_{0,1}) \le \rho(\widehat{L}_{0,1}) < \rho(L_{0,1})$ (If $\rho(L_{0,1}) < 1$)

5 Numerical example

In this section, we give the following example to illustrate the results obtained in section 4.
Example The coefficient matrix A of (1) is given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

Table 1 The comparison of the spectral radius of AOR iterative matrix

γ	ω	α	β	$\rho(L_{\gamma,\omega})$	$\rho(\widehat{L}_{\gamma,\omega})$	$\rho(L^*_{\gamma,\omega})$
0.8	0.9	2	-0.05	0.5735	0.5699	0.5681
0.7	0.8	2	-0.05	0.6400	0.6363	0.6345
0.6	0.7	4	-0.05	0.6994	0.6976	0.6958
0.5	0.6	4	-0.05	0.7531	0.7515	0.7498
0.1	0.2	5	-0.1	0.9288	0.9283	0.9270
0.9	0.9	2	-0.05	0.5476	0.5447	0.5433
0.5	0.5	2	-0.1	0.7943	0.7932	0.7904
0	1	2	-0.05	0.6551	0.6484	0.6449
0	1	5	-0.1	0.6551	0.6525	0.6456

From Table 1, we know that when $\rho(L_{\gamma,\omega}) < 1$, $\rho(L^*_{\gamma,\omega}) < \rho(\widehat{L}_{\gamma,\omega}) < \rho(L_{\gamma,\omega})$

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