A Problem Based Learning System for Linear Algebra with Simple Input Method

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Abstract—**Writing a matrix as an answer to a short-answer question is not easy thing to do. Therefore, there are not many linear algebra learning systems which give short-answer questions. Studying two most frequently used input methods to write a matrix, we are able to develop a simple input method to express a matrix. Furthermore, using this method, we develop a problem based learning system for linear algebra which is capable of assessing students' knowledge structure and skills.**

Index Terms—**Problem based learning, Adaptive system, knowledge structure, relative distance.**

I. INTRODUCTION

To design a problem based learning system for linear algebra which accurately diagnosing students' knowledge structure and skills, generated questions must be short-answer questions. For short-answer questions can derive more information about learners' knowledge than multiple choice questions can. Another reason for using a short-answer question can be seen in [1]. It is said that problem-solving and higher-order reasoning skills are better assessed through short-answer than multiple choice.

To generate a short-answer question for linear algebra, a problem based learning system for linear algebra must be equipped with a component which can express vectors, matrices, characteristic equations, and vector spaces. Currently, there are essentially two ways to express matrices. One is by using LaTeX. The other is by MathML.

Once a short-answer question is generated, it has to be displayed on screen. To display a matrix written in LaTeX format on screen, a matrix has to be converted into a graphic image. Therefore, the displayed matrix loses structure of the matrix. To display a matrix written in MathML on screen, a browser must be capable of rendering MathML. But not every browser can handle MathML, for example internet explorer 8 cannot handle MathML [2].

Now, writing an answer in matrix form in response to a short answer question, many methods are available. These methods are divided into two categories: writing a matrix using tab input and writing a matrix using character strings with pre-assigned rule. For example, Equation Editor [3] is one of the most widely used programs for writing a matrix by tab input. On the other hand, LaTeX [4] is one of the most widely used programs for writing a matrix by character strings with pre-assigned rule. But using either LaTeX or MathML, many steps are required to express a matrix.

After receiving a learner's input, the system has to be able to tell the learner for which the inputted answer is right or not. In a problem based learning system for linear algebra, a learner may input values, vectors, matrices, characteristic equations, and sets of vector spaces as answers. Thus, a problem based learning system for linear algebra must be equipped with a component capable of checking the correctness of inputted answers.

Any problem based learning system has to guide learners to understand the presented material by showing sequence of adaptive hints. Thus to develop a problem based learning system for linear algebra, the following obstacles must be overcome: the simplicity of expressing a matrix, the simple way of generating short-answer questions, the simple way of displaying short-answer questions, simple way of telling right answers from wrong ones, simple way of assessing learners' knowledge structure, and simple way of presenting adaptive hints.

In this paper, we show how we manage to express a matrix in a simple way and develop a problem based learning system for linear algebra which is capable of accurately assessing students' knowledge structure and skills.

II. METHODS OF INPUTTING MATRICES

A. Expressing a matrix using tab input

Using Equation Editor [3], to express a matrix such as $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$, the following steps must be performed:

1) Click a text where the equation should be inserted.
2) Click the insert tab.
3) Click the equation tab.
4) Select the matrix tab and choose the matrix tab with parenthesis.
5) Select the size of matrix.
6) Select the empty box and type 2, 3, 4, 3, 4, 5.

Obviously, the number of steps to express the matrix is so many that tab input method cannot be accepted by learners of a problem based learning system for linear algebra.

B. Expressing a matrix using pre-assigned rule

Using LaTeX [4], to express a matrix such as $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$, the following steps must be performed.

1. Click a text where the equation should be inserted.
2. Type "$\left(\begin{array}{ccc}2&3&4\\ 3&4&5\end{array}\right)$".

Here too, the number of steps is too many for a learner to continue using the system. Note that using AMS-LaTeX [5], the above matrix can be expressed by typing $\begin{pmatrix}2&3&4\\3&4&5\end{pmatrix}$. This is much shorter than the one by LaTeX. But it is still cumbersome for learners. Thus, we need new method of

inputting a matrix with much less steps.

C. Simple input method for a matrix

In introduction, we discussed which formats should be used to develop a problem based learning system for linear algebra. In essence, it is not important, since a text written in LaTeX can be translated into MathML and vice versa [6]. Thus, we can focus on the simplicity of expressing a matrix using LaTeX.

Using LaTeX to express the above matrix, we had to type so many characters. The main reason is that commands in LaTeX are more like sentences with backslash. To shorten the number of steps to express a matrix, the parser is created to translate the parenthesis "(" into "$\begin{pmatrix}" and ")" into "\end{pmatrix}". This way, it is only necessary for learners to input "(2&3&4\\3&4&5)" to express $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$. But then the parser cannot distinguish the parenthesis for matrices from the parenthesis for vectors. Since the vector (2,3,4) can be expressed by typing "(2,3,4)", to distinguish a matrix from a vector, we carefully studied the property of vectors and matrices expressions in LaTeX. The main difference between the matrix notation and the vector notation in LaTeX is that the matrix notation requires double backslashes but not the vector notation. Therefore, replacing double backslashes with some new symbol, the parser can distinguish a matrix from a vector. For double backslashes is only used for carriage return in LaTeX.

The new symbol for double backslashes should be easier to write and not used for other purpose in LaTeX. We searched all characters on key board and found that at-mark @ is not used in LaTeX. So, using at-mark for double backslashes, the above matrix is now written as "(2&3&4@3&4&5)" which is much simpler than the methods mentioned above. Therefore, we use this method for our problem based learning system for linear algebra.

III. PROBLEM BASED LEARING SYSTEM

The field we are in is college level linear algebra. The major component of educational process here is solving qualitative problems. According to Linear algebra by MIT open course [7] for first 18 sessions, it requires the students to be able to manipulate the followings:

1) Evaluating the addition and scalar multiplication of vectors.
2) Evaluating the dot product and the cross product of vectors.
3) Finding the angle between two vectors.
4) Evaluating the addition and scalar multiplication of matrices.
5) Solving a system of linear equations by Gaussian elimination.
6) Finding the rank of a matrix by using row echelon form.
7) Finding the inverse of a matrix.
8) Evaluating the determinant of a matrix.
9) Determining the given set of vectors is linearly independent or not.
10) Finding the basis for a vector space.
11) Finding the characteristic value of a matrix.
12) Finding the corresponding characteristic vector for the characteristic value of a matrix.

A. Generating a short-answer question

Steps 1), 2), 4), 6), 7), 10), and 12), the solutions must be expressed by either vectors or matrices. Suppose the question 8) is the one to be generated. Suppose also that the size of matrix to be generated is 3 by 3. Then the system randomly generates numbers between -10 and 10 and assign these numbers to the elements a_{ij} of the matrix A. This way we have matrices of the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ in 21^9 different ways, which is more than ten billion ways. Then the matrix for display can be created by the simple input method mentioned above.

For a question of finding determinant, there are in general two ways to answer. One way to answer this question is by using cofactor expansion method. The other way to answer this question is by the Leibnitz formula which states that the determinant of A is

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod A_{i,\sigma(i)}$$

Here the sum is computed over all permutations σ of the numbers $\{1, 2, \ldots, n\}$.

Suppose that the question is "Find the determinant of the given matrix by cofactor expansion". Then according to the key concepts, which are derived from experienced mathematics educators' concept maps, saved in the system, we split the question into the following finer questions:

1) Calculate the Minor of a_{12}.
2) Calculate the cofactor of a_{12}.
3) Using 1) and 2), evaluate the determinant of A by cofactor expansion along the 1st row.

This way, a learner can master how to find the determinant by using cofactor expansion.

For if the question is "Find the determinant of the given matrix by Leibnitz formula". Then for example, we split the question into the following finer questions:

1) Find the sign of a permutation denoted by $\text{sgn}(2,3,1)$.
2) Find the product of $a_{12}a_{23}a_{21}$.
3) Evaluate the determinant by Leibnitz formula.

This way, we can test to see how much a learner knows about Leibnitz formula.

B. Producing a correct answer

Suppose that the question to be generated is "Solve the following system of linear equation by Gaussian elimination":

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 2 \\ 3x_1 - 2x_2 - x_3 = 1 \end{cases}$$

Then the system also reads a command such as "GaussianElimination(coMat)". Here coMat is the array of the elements of the coefficient matrix of the above system of linear equation given by "2&3&4&2@3&-2&-1&1".

Now assume that our system is capable of executing "GaussianElimination(coMat). Then the system can produce a row reduced matrix. Thus, the question can be split into finer questions such as "Find the rank of the coefficient matrix" and "Find the row-reduced echelon matrix". Then the system also generates the sentence "Find Rank(coMat)" and "Find GaussianElimination(coMat)".

To answer the original question, the question of "Write the answer in vector form" is generated and the same time "VectorForm(gMat)" is executed. Here gMat is the result of the Gaussian elimination. Thus, producing a correct answer is the same as executing the commands. Therefore, a correct

answer can be generated from the list of methods and the list of arguments. In other words, when a student chooses "Solving system by Gaussian elimination" from the menu, our system makes a one to one correspondence between the items chosen and the methods GaussianElimination( ), Rank( ), and VectorForm( ). This way, we can generate not only short-answer questions but also corresponding correct answers.

C. Testing learners' answers

Suppose the question to be generated is "Find the characteristic value of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Then as we explained in III.A, the question is divided into finer questions. For example, there might be a question asking "Find the characteristic equation of the matrix A". Then the answer to this question is given by polynomial. Now note that to test a learner's answer is correct or not, we cannot simply compare the learner's answer with the right answer generated. For the inputted polynomial might be factored, but the answer generated by the system might not be factored. To overcome this problem, we use the method called one-point criterion [8] which is developed for testing two functions are the same or not. This means that our system does not require neither computer algebra system such as Mathematica, Maple, or Maxima nor numerical analysis software such as Matlab. Therefore, our system is lightweight.

D. Assessing learner's knowledge structure

How experienced mathematics educators assess students' knowledge structure from the solution written on the paper is studied by asking the following questions to my colleagues:

1) If a student writes $\det \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ as the answer to the question of "Find the (1,2) minor of $\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$". How do you asses this student's knowledge structure.

2) If a student writes $\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ as the answer to the question of "Find the (1,2) cofactor of $\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$". How do you asses this student's knowledge structure.

3) If a student writes $\det \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ as the answer to the question of "Find the cofactor expansion of $\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ along the 1st row". How do you asses this student's knowledge structure.

For the question 1), most of my colleagues responded by saying that this student knows that the minor of a matrix is given by determinant. But the person does not know how to find it. For the question 2), most of my colleagues responded by saying that this student may know a little bit about cofactor. But forgetting a sign means that his/her knowledge about cofactor is not enough. For the question 3), most of my colleague responded by saying that this student has no idea about cofactor expansion.

From these responses, we are inclined to say that the knowledge structure of experienced mathematics educators is very similar. Even though the knowledge of an individual expert consists of both a cognitive element— the individual's viewpoints and beliefs, and a technical element—the individual's context specific skills and abilities [9], we can use experienced mathematics educators' knowledge structure as the basic knowledge about how to solve questions in problem based linear algebra system.

To assess a learner's knowledge structure, one well known method is concept mapping. According to [10], to construct a good concept map, it is important to begin with a domain of knowledge structure that is very familiar to the person constructing the map. Since a short-answer question generated by the system is split into finer short-answer questions which are more familiar to the learner, by checking the learner's answer to the finer short-answer questions, we can construct a concept map for each person. Now using the knowledge structure of experienced mathematics educators, it is possible to identify the key concepts that apply to this domain. Thus, the concept map of experienced educators' knowledge structures is used to generate a short-answer question.

Now by defining the ratio or the difference of the evaluated values of a learner's input and a generated correct answer, it is possible to construct a learner's concept map. To do so, first define the distance d: Let vin and vca be defined as follows:

vin = the value of the input evaluated at certain point.
vca = the value of the correct answer evaluated at certain point.

Then define

$$d = \begin{cases} \text{the difference of } vin \text{ and } vca \\ \text{the ratio of } vin \text{ and } vca \end{cases} .$$

With this distance d, it is possible to construct a learner's concept map with some weight.

For if the inputted value of a learner is far from the correct value, the distance d becomes large. In this case, even an experience educator cannot conclude whether a student knows a material a little or none. Thus, define the relative distance by

$$rd = \frac{d}{\text{evaluated value of a correct answer}}$$

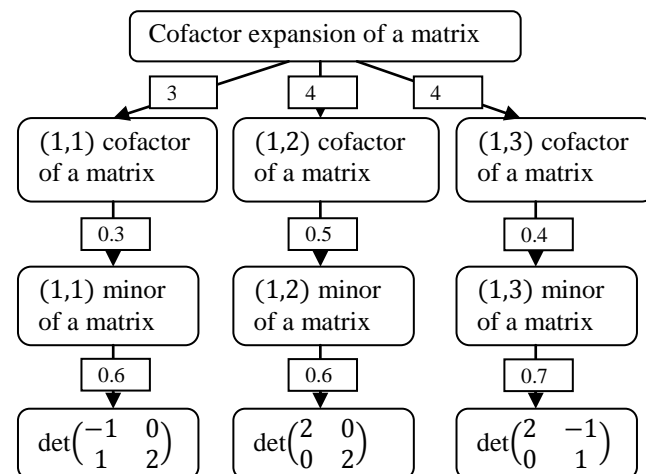Then the concept map of the cofactor expansion of the above matrix becomes like Fig1.

Fig1 Concept map with relative distances

Now using the concept map with weight, the relative distance can be used to assess a student's knowledge structure.

E. Feedback

It is noted in [11] that any educational software needs to give a quick feedback to encourage a learner to study more. After a learner's input is read, interpreted, and the relative distance is calculated, depending on the value of rd and the number of wrong input, two different types of hints will be displayed. Since the value of rd itself cannot assess a student's knowledge structure completely, the number of mistakes made adds more valuable information about how much the person knows.

To generate a hint, the system search the concept map of experienced educators' knowledge structure and uses the value of relative distance.

IV. IMPLEMENTING PROBLEM BASED LINEAR ALGEBRA SYSTEM

A. System design

1) Interface part is composed of the following forms and controls
   (a) A menu to choose a subject to study.
   (b) Timer to choose a number of questions and time to study.
   (c) Textbox for ID and Button for start.
   (d) Picture box for displaying a generated short-answer question.
   (e) Textbox for inputting an answer.
   (f) Picture box for showing inputted math expression.
   (g) Message box to tell the correctness of inputted answer. Message box to tell that hint is ready.

2) Problem and solution part is composed of the following components and methods:
   (a) A component to generate short-answer questions.
   (b) A component to generate a solution to a short-answer question.
   (c) A component to read a learner's input and test to see whether the inputted answer is correct or not.

3) Knowledge structure assessment part is composed of the following components and methods:
   (a) If the answer is correct, then the next question is generated. Otherwise, the relative distance between the correct answer and a submitted answer is measured.
   (b) If the relative distance is small, then return hint and ask a student to answer the question again. If the relative distance is large, return more detailed hint and ask a student to answer the question again.

B. Implementing the system

To implement the system, interface is created as shown in figure2.

Fig2 Menu

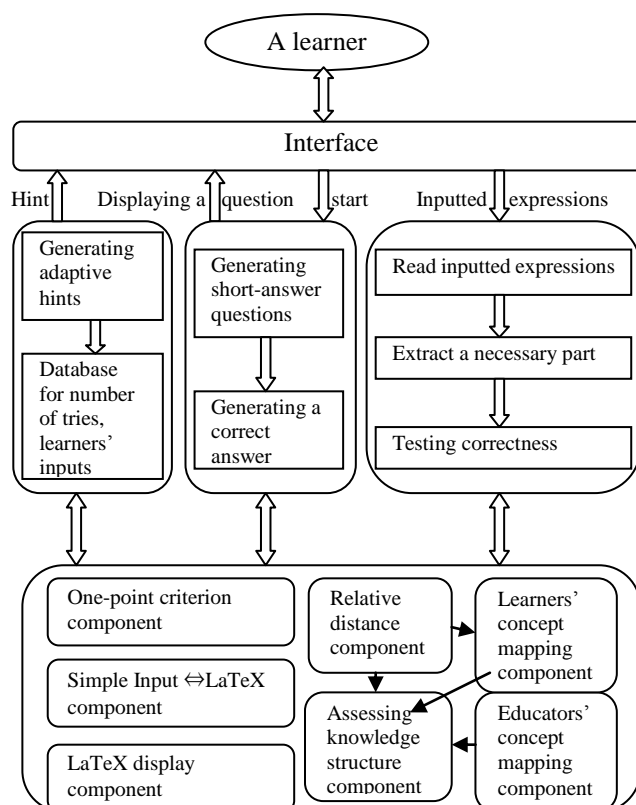To implement 2) and 3), the following scheme is developed.

Fig 3 Problem Based Learning Linear Algebra System

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