Proceedings of the World Congress on Engineering and Computer Science 2012 Vol II
WCECS 2012, October 24-26, 2012, San Francisco, USA

Estimating Upper and Lower Bound of Domain of Attraction via Markov Models

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***Abstract*—In this paper a new approach for determining an upper and a lower bound for estimating Domain of attraction of dynamical systems is proposed. We analyze the stability of dynamical systems by Markov models of them and using invariant measure as a stability indicator. Markov modeling focuses on asymptotic behaviors of systems and ignores the transient ones. Since an important limitation of estimating of *DA* via Markov modeling is that the estimated *DA* is actually an upper bound of the real one, for overcoming this limitation and estimating *DA* more accurately, we propose a novel method for determining a low bound of *DA* which is obtained by detecting and eliminating the boundary partitions which are not completely placed in the actual *DA*. The results of simulations show the efficiency of proposed method.**

Index Terms— **Discrete dynamical systems, Domain of attraction, Invariant measure, Markov chain, Lower bound domain of attraction.**

I. Introduction

Using Markov models for extracting dynamical behaviors has some advantages. The statistical properties of this model often have closed forms and are easily computed numerically and in addition the transient effects of system can be removed and only the asymptotic behaviors of system are computed. So it takes less time than direct analysis of system orbits.

To perform the problem of estimating Domain of attraction (*DA*), in the form of a finite dimensional optimization one, the state space is divided into some subspaces and the average quantities of these subspaces is considered. Any analysis of dynamical systems involving average quantities requires a reference measure to average contributions from different regions of the phase space. The most popular measure used in these cases is the probability invariant measure, which is described by the distribution of the typical long trajectories of the system. During recent years, invariant measure has played an important role in the characterization of dynamical systems. It is an approximate tool to determine behaviors of dynamical systems which is effectively used for detecting invariant sets or cyclic behaviors of nonlinear systems [1].

Some papers propose methods for producing better Markov models via smarter partition selection.

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In all cases, one selects an initial coarse partition, computes the invariant measure of the Markov model, and on the basis of information contained in the invariant measure of the current model, a choice is made on which partition sets to refine and which not to refine. In [2] any partition sets which are assigned a measure greater than $1/n$, where n is the current number of partition sets, are refined. In [3] a high derivative approach is proposed and it is assumed that the physical measure is smooth; therefore, if the current estimate of the invariant measure has adjacent sets given very different measure, there must be an error in this region, and so one refines these sets to obtain better estimates. In [4] all partition sets are refined and a temporary transition matrix P_{temp} and invariant measure p_{temp} for the refined partition are constructed. The new invariant measure is compared with the previous one which is related to p_{old} and only sets in the old partition for which the measure according to p_{temp} and p_{old} is very different are split up. The transition matrix p_{temp} is then discarded.

An important limitation of estimating *DA* by Markov modeling is that the estimated *DA* is actually an upper bound of *DA* (*UDA*) that contains the actual one. Since boundary partitions of *UDA* are not completely placed in actual *DA*. This fact generates an estimation error. To overcome this limitation and have more accuracy in estimation, we find a low bound for *DA* (*LDA*) which is obtained by eliminating the boundary partitions of *UDA* on the basis of changes of invariant measures.

This work contains 4 sections. In the second section some definitions are summarized. Introducing the main idea of this work which is describing stability analysis according to Markov model of a system and estimating *UDA* and estimating *LDA* is the subject of the third section. And finally, in the fourth section the results are simulated.

II. Preliminaries

Let Ω be an n-dimensional open rectangular set in R^n, equipped with Lebesgue measure μ on σ-algebra of Borel sets $B(\Omega)$ and T be a measurable nonsingular transition operator [1] on the measurable space (Ω, B, μ) such that

$$\begin{aligned} &X(k+1) = T(X(k)) \qquad T:\Omega \to \Omega,\ \Omega \subset R^n \\ &X(k) = [x_1(k),\ldots,x_n(k)]^T, x_i(k+1) = T_i(X(k)) \end{aligned} \tag{1}$$

Definition 1 (State space partitioning): $\mathcal{A}$ is a state space partitioning for Ω if it divides Ω into cells A_i where

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$i = 1, \dots, N$ such that they satisfy the following two conditions:

$$\bigcup_{i=1}^{N} A_i = \Omega \quad , \qquad \overset{\circ}{A_i} \bigcap \overset{\circ}{A_j} = \phi \quad , \quad \forall i \neq j$$

Where $\overset{\circ}{A_i}$ is the interior of A_i set and ϕ is the empty set.

Definition 2 (Center of a partition): Let $\mathcal{A}$ be a state space partitioning for $\Omega \subset R^n$. For simplicity we suppose rectangular partitions as $A_i = [l_{1i}, h_{1i}] \times \ldots \times [l_{ni}, h_{ni}]$ $i = 1, \ldots, N$. The center of each partition A_i is a point like $C_i = [c_{1i}, \ldots, c_{ni}]^T$ where $c_{ji} = \dfrac{h_{ji} - l_{ji}}{2}$.

The state space partitioning for $n = 2$ is illustrated in figure 1.

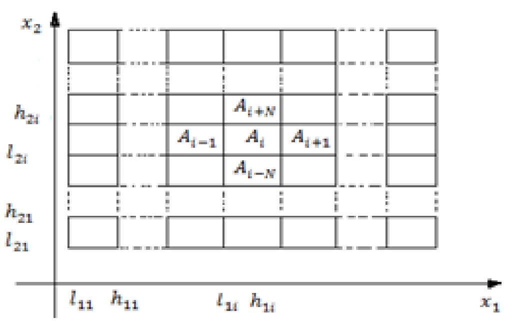


Fig. 1. State space partitioning for $n = 2$

Definition 3 (Domain of attraction): Consider nonlinear system (1). The domain of attraction of an asymptotic stable equilibrium point X_e [5] is $DA = \{X(k) \in \Omega | \lim_{k \to \infty} T^k(X) = X_e\}$.

Definition 4 (Upper bound of *DA*): Upper bound of *DA* (*UDA*) is an estimated *DA* which contains the actual *DA.*

Definition 5 (Lower bound of *DA*): Lower bound of *DA* (*LDA*) is an estimated *DA* which is surrounded by actual *DA*.

III. ESTIMATING DA BY MARKOV MODELING

Stability concepts in dynamical systems have both discrete-time and continuous-time versions. As results of discrete-time dynamical systems are easier to formulate, we focus on the discrete dynamical system of the equation (1). Results for the continuous-time systems can be deduced from the discrete ones. As we are concerned with estimating domain of attraction of certain (or uncertain) systems, we should analyze the long term orbits of the system, but it is not practically possible in many systems because it takes a long time and may lead to computer round off error. Therefore, in this paper we use the method of Markov modeling of dynamical systems to remove the transient effects and calculate the asymptotic behaviors.

In the next part, we review some properties and theorems that are necessary for stability analysis of nonlinear systems via Markov models.

A. *Stability Theorems*

For discrete system (1) a Markov chain can be constructed as [6]:

$$\phi = \left\{\phi_k \middle| \ \phi_k = X(k) = T^k(X), \ 0 \leq k < n+1\right\}. \qquad (2)$$

Definition 6: Let $X \in \Omega$ and $A \subset \Omega$. The n-step transition function, denoted by $p^n(X, A)$, shows the probability that a Markov chain ϕ starting from an arbitrary point like $\phi_0 = X$ remaining in the set A after n steps [5].

Proposition 1: For Markov chain (2) the Markov transition function is proposed as $P(X, A) = \lim_{n \to \infty} p^n(X, A)$

Proof: see [5, chapter 1, page 3].

In the sequel, it is considered that according to proposition 2, the uniformly distributed $P(X, A)$ depends only on A. For stability analysis of proposed Markov chain, we consider theorem1.

Theorem 1: The existence of a fixed point like X_e which is asymptotically stable in the set $A \subset \Omega$ is exactly equal to the existence of a nonzero unique solution for the following invariant equation:

$$m(A) = \int_{\Omega} P(X, A) \ dm(A)$$

Proof: see [5, chapter 1, page 20, asymptotic stability definition].

In the above theorem $m \in M$ and M is the set of all probability Lebesgue measures on the topological space Ω.

According to the *DA* definition, we propose Lemma 1 which is the direct conclusion of theorem 1.

Lemma1: Closure of the Domain of attraction of nonlinear system (1), $\overline{DA} \subset \Omega$, is the union of the members of support of probably measure m and obtained from following equation:

$$\overline{DA} = SUPP\{m\}$$

where

$$SUPP\{m\} = \cup\{A | \ m(A) = \int_{\Omega} P(X, A) \ dm(A) \neq 0\}$$

B. *Estimating UDA*

It is not practically possible to estimate domain of attraction of system (1) using Lemma 1, because it leads to an infinite dimensional problem in space M. In other words since $DA \subset \Omega$, we should calculate $P(X, A) = \lim_{n \to \infty} p^n(X, A)$ for every $X \in \Omega$ which leads

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

to an infinite dimensional problem. So we use the rough idea of [7] and partition the state space Ω (according to definition 1). Assuming that $P(X,A)$ has a uniform distribution, we can calculate probability of partitions transitions instead of calculating every point $X\in\Omega$ transition. So in the sequel we convert the infinite dimensional problem of estimating *DA*, proposed in Lemma1, to a finite dimensional one. To investigate the stability of state partitions we use the discrete-time Markov chain which is a Markov process ϕ_n having a countable number of states A_n [6].

Definition 7 (Markov transition matrix): Consider nonlinear system (1). For $\mathcal{A}$ partitioning of Ω, the $N\times N$ Markov transition matrix P is defined as:

$$P^{(n_1,n_2)}=[p_{ij}^{(n_1,n_2)}]=[prob\,[X(n_2)\in A_j \mid X(n_1)\in A_i]];\quad \sum_j p_{ij}^{(n_1,n_2)}=1 \qquad (3)$$

Definition 8 (state probability vector): The state probability vector for state space partitioning $\mathcal{A}$ in n[th] transition is defined as $\vartheta(n)=(\vartheta_1(n),....,\vartheta_N(n))$, where $\vartheta_i(n)$ is the probability of existing Markov chain in A_i state in n[th] transition [5] in other words $\vartheta(n)=prob[X(n)\in A_i]$ and $\sum_{i=1}^{N}\vartheta_i(n)=1\ \ \forall n$ [6].

Definition 9 (n-step Markov transition matrix): n-step Markov transition matrix for a homogenous Markov process is defined as:

$$P^{(n)}=[p_{ij}^{(n)}]=prob(X(k+n)\in A_j / X(k)\in A_i) \qquad (4)$$

Proposition 2: For uniformly distributed $P(X,A)$, $p_{ij}^{(1)}$ can also be presented as:

$$p_{ij}^{(1)}=\frac{m(T^{-1}(A_j)\cap A_i)}{m(A_i)}\quad i,j=1,...,N \qquad (5)$$

Proof: see [7].

Proposition 3: For a homogeneous process we have:

i- $P^{(n)}=P^n$ (6)

ii- $\vartheta(n)=\vartheta(0)P^{(n)}=\vartheta(0)P^n=\vartheta(n-1)P$ (7)

Proof: see equation (16-110) of [6] .

Proposition 4: For a stationary Markov process, the state probability vector ϑ does not depend on n and is called a **stationary distribution** (or **invariant measure**) vector.

Theorem2 (Perron-Frobenius theorem): For irreducible and aperiodic Markov chains there exists a unique invariant measure vector ϑ. In addition $P^{(n)}$ converges to a rank-one matrix in which each row is the stationary distribution ϑ that is:

$$\lim_{n\to\infty}P^{(n)}=\mathbf{1}\vartheta$$

Where **1** is the column vector with all entries equal to 1.
Proof: See [6].

Theorem 3: The (closure of) domain of attraction of nonlinear system (1) with N state partitioning $\mathcal{A}$ can be estimated from the support of invariant measure vector ϑ. Where ϑ is calculated from following equations:

$$\vartheta=P\vartheta\quad;\vartheta=(\vartheta_1,....,\vartheta_N)\quad,\quad\sum_{i=1}^{N}\vartheta_i=1 \qquad (8)$$

Proof: Propositions 3 and 4 implies that $\vartheta=P\vartheta$, where $\sum_{i=1}^{N}\vartheta_i=1$ and $\vartheta=(\vartheta_1,....,\vartheta_N)$ is unique (see Perron-Frobenius theorem). In addition as ϑ_i is the probability of existing Markov chain in A_i state we can conclude that ϑ_i is an stability weight or in other words $\vartheta_i=0$ shows that orbits do not exist in A_i and leave this state so domain of attraction includes states with nonzero invariant measure or equally *DA* is the support of ϑ. □

Proposed analytic form of Markov Matrix

Considering theorem 3, to estimate *DA*, we should calculate Markov matrix. There are different numerical algorithms to calculate P matrix from equation (5) [see chapter 6 of reference 7]. In the sequel, we provide a new analytic formula to determine P which is more accurate.

Proposition 5: Some useful properties of the (probability) Lebesgue measure m and characteristic function χ are:

a- $m(A\cap B)=\int_A \chi_B(X)\,dX=\int_B \chi_A(X)\,dX$

Proof: From [8] we have $\chi_{(A\cap B)}=\chi_{(A)}.\chi_{(B)}$, which yields;

$$m(A\cap B)=\int_\Omega \chi_A(X).\,\chi_B(X)\,dX=\int_A \chi_B(X)\,dX=\int_B \chi_A(X)\,dX$$

b- $\chi_{T^{-1}(A)}(X)=\chi_A(T(X))$

Proof: Science T is nonsingular we have

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\chi_{T^{-1}(A)}(X)=1 \Leftrightarrow X \in T^{-1}(A) \Leftrightarrow T(X) \in A$$
$$\Leftrightarrow [T_1(X),...,T_n(X)]^T \in A \Leftrightarrow \chi_{(A)}(T(X))=1$$

C. Estimating LDA

In this section a novel algorithm for eliminating the Boundary partitions and determining *LDA* base on the changes of invariant measures is proposed.

For simplicity the algorithm is applied for two dimensional systems but without loss of generality this can be used for n dimensional systems too.

Definition 10 (neighbor partition): Let A_i and A_j be rectangular sets in Ω and consider A_i, $A_j \in \mathcal{A}$. A_i is denoted by a neighbor of A_j if $A_i \cap A_j \neq \Phi$. If we define $A_i \cap A_j = \{S\}$, it is clear that S includes a line (vertical or horizontal neighbor) or a node (diagonal neighbor).

Definition 11 (boundary partitions): The partitions which have nonzero invariant measure but they are not completely placed in the actual *DA*.

Definition 12 (Horizontal gradient): Horizontal gradients at each partition can be defined as the central difference between invariant measures of two horizontal neighbors. Horizontal gradient is calculated from the following equation:

$$G_x = \frac{\partial \vartheta}{\partial x} \approx \frac{\vartheta_{i+1} - \vartheta_i}{2} \qquad (9)$$

Where ϑ_i is invariant measure of current partition A_i.

Definition 13 (Vertical gradient): Difference between invariant measures of two vertical neighbor leads to vertical gradient that can define as follows:

$$G_y = \frac{\partial \vartheta}{\partial y} \approx \frac{\vartheta_{i+N} - \vartheta_i}{2} \qquad (10)$$

Proposition 6 (absolute magnitude of gradient): The absolute magnitude of gradient (G) is illustrated by the mean square root of the horizontal (G_x) and vertical (G_y) gradients. That is $|G| = \sqrt{{G_x}^2 + {G_y}^2}$. To reduce the computational cost of magnitude, it is often approximated with absolute sum of the horizontal and vertical gradients ($|G| = |G_x| + |G_y|$).

According to definition 8, we introduce invariant measure of a partition A_i by a constant value (ϑ_j); so variations of invariant measure from a partition like A_j to one of its neighbor like $A_{i+1,j}$ can be modeled by a step signal.

Partitions of Ω space can be divided in to two major groups, the partitions that are inside *DA* (*DA* mode) and the partitions that are outside *DA* (Ω-*DA* mode), it is clear that the most considerable invariant measure variations occurs in boundary partitions because in these partitions the stability mode changes. This fact leads to the idea that we can effectively detect the boundary partitions by determining the partitions with considerable variations in their invariant measure. To obtain this aim, we compare the invariant measure gradient with a proper threshold.

For determining the lower boundary by deleting boundary partitions, we propose the following algorithm:

1. Calculate horizontal and vertical gradient of state partitions which are contained in *UDA*.
2. Determine absolute magnitude of gradient of each partition.
3. Select the partitions that their absolute magnitude of gradient are larger than their neighbors in either the horizontal or vertical directions and are larger than the threshold which obtained by try and error as boundary partitions.
4. Eliminate the remaining partitions (since *DA* is a connected set).

IV. SIMULATION RESULTS

Consider the following Vander Pol oscillator:

$$x_1[k+1] = (-x_2[k])T_s + x_1[k]$$
$$x_2[k+1] = \{x_1[k] - (1-(x_1[k])^2)x_2[k]\}T_s + x_2[k]$$

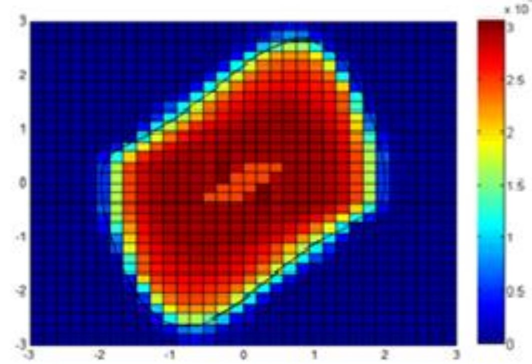
Considering $\Omega = [-3,3]\times[-3,3]$, $N_1 = N_2 = 32$ that yields to $\Delta_1 =, \Delta_2 = 0.187$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ $i = 1,..,32^2$,

$$l_{1i} = -3 + 0.187 remaider\left(\frac{i}{32}\right),$$

$$l_{2i} = -3 + 0.187 quotient\left(\frac{i}{32}\right)$$

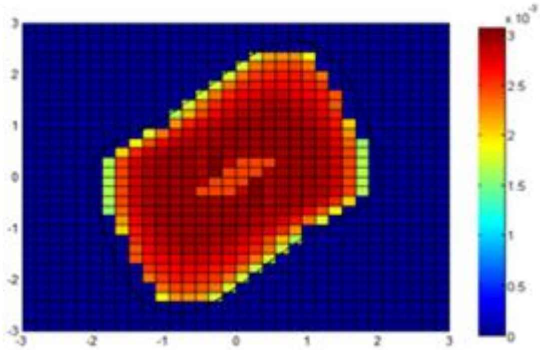
and $h_{1i} = l_{1i} + 0187$, $h_{2i} = l_{2i} + 0.187$,

The estimated *UDA* and *LDA* are obtained according to algorithm 1 and 2. Figure 2 shows the *UDA* and *LDA* of Vander Pol system.



a)estimated *UDA*

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)



b) estimating *LDA* by eliminating boundary partitions of *UDA*

Fig. 2. Comparison of *UDA* (2.a), *LDA* (2.b) and actual *DA* (black curve) of Vander Pol oscillator.

V. Conclusion

In this work we propose a novel method for estimating upper and lower bounds of domain of attraction. This method is based on Markov modeling of nonlinear systems which considers the average quantities of state space and is able to effectively find estimated *DA*. As the estimated *DA* which is obtain trough this method doesn't completely surrounded by *DA*, in the other words it is a upper bound for *DA*, we propose an algorithm to omit the boundary partitions of estimated *DA* and find a more accurate estimate which is denoted by lower boundary of *DA*.

In this approach according to the changes of invariant measures, the boundary partitions which are contained in estimated *DA* but are not completely placed in the actual *DA* detected and deleted. The efficiency of proposed methods is shown via simulations.

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ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)