Optimal Control Approaches for Some Geometric Optimization Problems

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Abstract—This work is a survey on optimal control methods applied to shape optimization problems. The unknown character of the domain where the state system is defined creates major difficulties both for the theoretical analysis and for the implementation of the numerical approximation methods.

The optimal control and the controllability methods for elliptic equations allow the development of approximation procedures defined in a given domain that contains all the admissible subdomains for the original optimal design problem. This class of methods enters the so-called fixed domain admissible subdomains for the original optimal design problem. See the monographs of Pironneau [15], Sokolowski and Zolesio [16], Allaire [1], Neitaanmäki and Tiba [13], adapted to the problem (1) - (3).

One may ask supplementary assumptions on $f,u$ such that the unique solution $y$ of (5), (6) is continuous in $D$ and the constraints (7), (8) make sense.

Proposition 1.1 The problem (4) - (8) is a subproblem of (1) - (3).

To each admissible control $u$, we associate the open set $\Omega_u = \text{int} \{ x \in D; y_u(x) \geq 0 \}$ and the subdomain $\Omega_u$ given by the connected component of $\Omega_u$ that contains $E$, $\Omega_u \supset E$. It is clear that the cost (4), corresponding to $u$, is equal with the cost (1) corresponding to $\Omega_u$. The mapping $u \rightarrow \Omega_u$ is one-to-one due to the analyticity of harmonic functions in $D$. The set of admissible controls for the problem (4) - (8) is nonvoid iff $u \equiv 0$ is admissible (due to the comparison property for (5), (6)).

Remark The domain variation or boundary variation techniques, employed in the literature for the solution of (1) - (3), require in each iteration of the algorithm the definition of a new finite element mesh, the computation of the new mass matrix and are very time consuming. The advantage of the control problem (4) - (8) is that it is defined in the fixed domain $D$ (which avoids remeshing) and it is convex. In [13], it is shown that under a certain controllability hypothesis one...
can prove a partial converse statement for P 1.1. In this sense, we may say that the solution \( u^* \) of (4) - (8) generates a suboptimal domain \( \Omega_{u^*} \) for the shape optimization problem (1) - (3).

In the next section, we discuss various controllability properties for elliptic systems that are useful for shape optimization problems, including applications in free boundary problems. Section 3 is devoted to further applications of optimal control methods.

We end the paper with some short conclusions and hints on future work.

II. CONTROLLABILITY

We indicate several boundary controllability and distributed controllability properties for elliptic equations, with applications to problems involving unknown domains. We recall first the following approximate controllability result from the classical monograph of Lions [10]:

**Theorem 2.1** Let \( \Omega \in \mathbb{R}^d \) be a bounded domain with smooth boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset \). For every \( u \in L^2(\Gamma_1) \), let \( y_u \in L^2(\Omega) \) be the unique solution (in the transposition sense) of the elliptic problem

\[
\Delta y_u = 0 \quad \text{in} \; \Omega, \quad \tag{9}
\]

\[
y_u = u \quad \text{on} \; \Gamma_1; \quad y_u = 0 \quad \text{on} \; \Gamma_2. \quad \tag{10}
\]

Then \( \left\{ \frac{\partial y_u}{\partial n}; u \in L^2(\Gamma_1) \right\} \) forms a dense subspace in \( H^{-1}(\Gamma_2) \).

**Remark** A constructive proof of this density result may be found in [12], §5.2.3. The interpretation as an approximate controllability property is a consequence of the fact that the range of the mapping \( u \to \frac{\partial y_u}{\partial n} \) is dense in \( H^{-1}(\Gamma_2) \). An important application is in the identification of the region occupied by the plasma in the void chamber of a tokamak, from outside measurements [5], [12], via an optimization method.

We consider now the case of distributed controls both for the Dirichlet problem (2), (3) and for the case of Neumann boundary conditions. In order to avoid working in unknown domains \( \Omega \), the idea is to extend (at least in an approximating sense) the boundary value problem (2), (3) to the given domain \( D \). We denote by \( \chi \) the characteristic function of \( \Omega \) in \( D \) and we introduce the control system:

\[
-\Delta y = f + (1-\chi)u \quad \text{in} \; D, \quad \tag{11}
\]

\[
y = 0 \quad \text{on} \; \partial D, \quad \tag{12}
\]

with the cost functional

\[
\frac{1}{2} \int_{D \setminus \Omega} y^2 \, dx. \quad \tag{13}
\]

Since the optimal control problem (11) - (13) is ill-posed in general (it may have no solution), a Tikhonov regularization of the cost is considered:

\[
\frac{1}{2} \int_{D \setminus \Omega} y^2 \, dx + \frac{\varepsilon}{2} |u|_{L^2(D)}^2, \; \varepsilon > 0. \tag{14}
\]

We denote by \([y_e, u_e] \in [H^2(D) \cap H^1_0(D)] \times L^2(D)\) the unique optimal pair of (11), (12), (14) and by \( p_e \in H^2(D) \cap H^1_0(D) \) the corresponding adjoint state:

\[
-\Delta p_e = (1-\chi)y_e \quad \text{in} \; D, \quad p_e = 0 \quad \text{on} \; \partial D. \tag{15}
\]

The first order optimality conditions are given by (11), (12), (15) and

\[
\varepsilon u_e + (1-\chi)p_e = 0 \quad \text{in} \; D. \quad \tag{16}
\]

One can eliminate \( u_e \) from the above relations and obtain the system:

\[
-\Delta y_e = f - \frac{1}{\varepsilon} (1-\chi)^2 p_e \quad \text{in} \; D, \quad \tag{17}
\]

\[
-\Delta p_e = (1-\chi)y_e \quad \text{in} \; D, \quad \tag{18}
\]

\[
y_e = p_e = 0 \quad \text{on} \; \partial D \quad \tag{19}
\]

which is equivalent with the optimal control problem (11), (12), (14).

**Theorem 2.2** We have:

\[
p_e \to 0 \quad \text{strongly in} \; H^1(D \setminus \Omega), \quad \tag{20}
\]

\[
y_e \to 0 \quad \text{weakly in} \; L^2(D \setminus \Omega). \tag{21}
\]

**Remark** Since \( \chi \) is a characteristic function, then \((1-\chi)^2 = (1-\chi)\) and the system (17) - (19) can be written in a more symmetric form. The result of Thm. 2.2 is an approximate extension result for the equation (2), (3). By the Mazur theorem, strong convergence may be obtained for sequences of convex combinations of the optimal controls. Notice that the extension system has a different form than the initial equation. This may be interpreted as a generalization of usual controllability properties.

In the case of Neumann boundary value problems, the situation is more difficult:

\[
\int_{\Omega} \left[ \sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} + a_{0} y v \right] \, dx = \int_{\Omega} f v \, dx, \quad \forall \; v \in H^1(\Omega), \tag{22}
\]
where \( a_{ij}, a_{ij} \in L^\infty(D) \) and \( (a_{ij})_{i,j=1}^{17} \) is an elliptic matrix. Denote \( \Omega_0 = D \setminus \bar{\Omega} \) and assume that \( \partial D \subset \Omega_0 \), that is \( \Omega_0 \) is a relative neighbourhood of \( \partial D \). Let \( \Gamma = \partial \Omega_0 \setminus \partial D \).

We introduce again an optimal control problem:

\[
\begin{align*}
\text{Min} \quad & \frac{1}{2} \int_{D} \| u - w \|^2_{H^1/2(\Gamma)} + \frac{\varepsilon}{2} \frac{|u|^2}{L^2(\Omega_0)} \bigg\}, \quad \varepsilon > 0, \quad (23) \\
\text{s.t.} \quad & \int_{\Omega_0} \sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} + a_{0} y z \bigg\} \, dx = \int_{\Omega_0} u z \, dx, \quad (24) \\
\forall z \in H^1(\Omega_0),
\end{align*}
\]

where \( w \in H^{1/2}(\Gamma) \) is some given element. We denote by \( \{y_\varepsilon, u_\varepsilon\} \) the unique optimal pair of (23), (24). Notice that (24) is the weak formulation of the homogeneous Neumann problem in \( \Omega_0 \).

**Remark** This approximate controllability result is a distributed variant of Theorem 2.1 since \( y_\varepsilon \) satisfies in a generalized sense the condition \( \frac{\partial y_\varepsilon}{\partial n_A} = 0 \) in \( \Gamma \) (the null conormal derivative) as well.

Under smoothness assumptions on \( \Omega_0, w, u \), one can apply directly the trace theorem in Sobolev spaces. The basic idea of Thm. 2.3 is that \( y_\varepsilon \) can provide an approximate extension of the solution of (22) from \( \Omega \) to \( D \), if \( u \) is chosen appropriately, that is the trace on \( \Gamma \) of the solution of (22).

As in Thm. 2.2, we shall find a system of equations, defined in \( D \), that approximately extends (22) to \( D \). We consider the constrained control problem:

\[
\begin{align*}
\text{Min} \quad & \frac{1}{2} \int_{D} u^2 \, dx \\
\text{s.t.} \quad & \int_{D} \sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} + a_{0} y z \bigg\} \, dx = \int_{D} u z \, dx, \quad (25) \\
\frac{\partial y}{\partial n_A} = 0 \quad \text{on } D, \quad (26) \\
\text{with the state constraint} \quad & F(y_u) = \sum_{i,j=1}^{d} a_{ij} \frac{\partial y_u}{\partial x_j} (\nabla g \cdot e_i) = 0 \quad \text{on } \partial \Omega, \quad (27)
\end{align*}
\]

Above, \( \chi \) is the characteristic function of \( \Omega \) in \( D \) and it is assumed that

\[
\Omega = \{ x \in D; \, g(x) > 0 \} \quad (29)
\]

where \( g \in C^1(D) \). Then, restriction (28) signifies that \( \frac{\partial y_u}{\partial n_A} = 0 \) on \( \partial \Omega \). In the case of the Laplace operator, (28) has the simple form \( \nabla g \cdot \nabla y = 0 \) on \( \partial \Omega \).

We penalize the state constraint (28) in the cost functional and we define the approximating unconstrained control problem

\[
\begin{align*}
\text{Min} \quad & \frac{1}{2} \int_{D} u^2 \, dx + \frac{1}{2 \varepsilon} \int_{\partial \Omega} \frac{F(y_u)^2}{d \sigma}, \quad \varepsilon > 0 \quad (30)
\end{align*}
\]

subject to (26), (27). Here \( [g \equiv 0] \) is a notation for \( \partial \Omega \) defined in (29). This underlines the fact that the mapping \( g \) may be used as the main unknown instead of the unknown domain \( \Omega \).

The existence and uniqueness of the optimal pair \( \{y_\varepsilon, u_\varepsilon\} \in H^2(D) \times L^2(D) \), if \( \partial D \) and \( a_{ij} \) are smooth enough, is standard. One can prove that \( u_\varepsilon \to \hat{u} \) strongly in \( L^2(D) \), \( y_\varepsilon \to \hat{y} \) strongly in \( H^1(D) \), where \( \{\hat{g}, \hat{u}\} \) is the unique optimal pair for the control problem (25) - (28).

The adjoint equation and its solution \( p_\varepsilon \in L^2(D) \) may be defined by the transposition method, Lions [10]:

\[
\int_{D} p_\varepsilon \bigg\} \, dx = \frac{1}{\varepsilon} \int_{\partial \Omega} \frac{F(y_u)}{d \sigma}, \quad \varepsilon > 0 \quad (31)
\]

The maximum principle has again the form (16) and one can eliminate the control \( u_\varepsilon \) from the state equation (26):

\[
- \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial y_u}{\partial x_i} \right) + a_{0} y_u = f - \frac{1}{\varepsilon} (1-\chi) p_\varepsilon \quad \text{in } D. \quad (32)
\]

**Remark** Relations (31), (32) and the boundary condition (27) give the approximate extension of the Neumann problem (22) from \( \Omega \) to \( D \), given by (29), to \( D \). It is possible, via a similar procedure, to obtain an approximating extension of (22) involving Dirichlet conditions on \( \partial D \).

**Remark** Coming back to the shape optimization problem (1) - (3) (or with (3) replaced by the Neumann condition), we consider that the admissible class of domains \( \Omega \) is defined by (29) with \( g \in X \subset C^1(D) \) some given subset. The state system in \( \Omega \), given by (2) or, alternative, by (22) can be approximated by (17) - (19), respectively (31), (32). We note that \( \chi = H(g) \), where \( H \) is the Heaviside function. It is possible to regularize \( H(\cdot) \) by the Yosida approximation \( [3] \) of its maximal monotone extension in \( R \times R \).

In this way the initial geometric optimization problem is transformed into an optimal control problem (by the coefficients) with respect to \( g \in X \).

**Remark** Proofs, examples and more details on this analytic approach in shape optimization problems may be found in [11], [14]. For another fixed domain approach in optimal design, using finite elements, we quote [6], [17].
We close this section with an approximate controllability result for the coincidence set in variational inequalities of obstacle type. The classical formulation of this problem is:

\[
\Delta y = 0 \text{ in } \{x \in D; y(x) > \varphi(x)\}, \\
\Delta y \leq 0 \text{ in } D, \\
y \geq \varphi \text{ in } D, \\
y = u \text{ on } \partial D.
\] (33) - (36)

Here \( D \) is a bounded domain in \( R^d, d \geq 2 \) with \( \partial D \) a smooth manifold of class \( C^m \), \( u \in H^{m-1/2}(\partial D) \) and \( \varphi \in C^m(\overline{D}) \) is the "obstacle" (with \( u \geq \varphi \) on \( D \) assumed in order that (33) - (36) make sense). The natural number \( m \) is "big" such that \( y, \varphi \) and \( u \) are continuous by the Sobolev lemma. We define the coincidence set associated to (33) - (36) by

\[ C_u = \{x \in D; y(x) = \varphi(x)\}. \]

If \( \Omega \subset D \) is some given subdomain of class \( C^m \), we study the boundary controllability problem

\[
\text{find } u \in H^{m-1/2}(\partial D) \text{ such that } \Omega \subset C_u. 
\] (37)

We denote by \( \Omega_0 = D \setminus \Omega \). An alternative formulation of (37) is:

\[
\text{find } u \in H^{m-1/2}(\partial D) \text{ such that } y_u = \varphi \text{ on } \partial \Omega, \quad y_u \geq \varphi \text{ in } \Omega_0, 
\] (38)

where \( y_u \in H^m(\Omega_0) \) is the solution to the elliptic problem

\[
\Delta y_u = 0 \text{ in } \Omega_0, \\
\frac{\partial y_u}{\partial n} = \frac{\partial \varphi}{\partial n} \text{ on } \partial \Omega, \\
y_u = u \text{ on } \partial D.
\] (39) - (40)

Notice that the problem (38) - (40) is a variant of the one studied in Thm. 2.1, with the notable difference that \( y_u \geq \varphi \) in \( \Omega_0 \) (a state constraint is imposed). One can compare the problem (38) - (40) with (5) - (8) as well and we shall give now an approximation result.

The idea of replacing (37) by (38) - (40) is that \( y_u \) may be extended in \( \Omega \) by \( \varphi \), preserving regularity in \( H^2(D) \). Consequently, \( \Omega \subset C_u \) and (33) - (36) are satisfied. See [4] for details, where the following approximate geometrical controllability results are established.

**Theorem 2.4** If \( \Delta \varphi \leq 0 \text{ in } \Omega \) and \( \frac{\partial \varphi}{\partial n} \geq 0 \text{ on } \partial \Omega \), there is a sequence \( \{(u_n, y_n)\} \subset H^{m-1/2}(\partial D) \times H^m(\Omega_0) \), satisfying (39), (40) and \( y_n \geq \varphi \text{ in } \Omega_0 \) and having the following property:

\[
\text{For every smooth part } \Gamma \text{ of } \partial D \text{ there is no domain } \Pi \subset \Omega_0 \text{ such that } \partial \Pi \cap \partial \Omega = \Gamma, \partial \Pi \cap \partial D \neq \emptyset \text{ is a smooth submanifold of } \partial D \text{ and }
\]

\[ y_n(x) > \varphi(x), \quad \forall x \in \Pi, \]

for some subsequence \( n_k \to \infty \).

**Theorem 2.5** For every \( \varepsilon > 0 \), there is a connected open subset \( Q_\varepsilon \subset \Omega_0 \) such that \( \text{meas}(\Omega_0 \setminus Q_\varepsilon) \leq \varepsilon \) and a sequence \( \{(u_n, y_n)\} \subset H^{m-1/2}(\partial D) \times H^m(\Omega_0) \) satisfying (39), (40) and

\[
y_n \geq \varphi \text{ a.e. in } Q_\varepsilon, \\
y_n \rightharpoonup \varphi \text{ weakly in } L^2(\partial \Omega).
\]

**Remark** Such geometrical approximate boundary controllability properties may be obtained in the case of Neumann boundary conditions as well.

### III. Optimal Control Approach

We fix now the general optimal design problem

\[
\text{Min}_{\Omega} \int_E j(x, y_\Omega(x))dx, \\
\text{subject to } \frac{\partial y_\Omega}{\partial n} = f \text{ in } \Omega, \\
B_y = 0 \text{ on } \partial \Omega.
\] (41) - (43)

All the notations are as before and \( B \) is a general boundary operator: identity for Dirichlet conditions, \( B_y = \frac{\partial y}{\partial n} \) for the Neumann problem, \( B_y = \frac{\partial y}{\partial n} + \alpha y \) (\( \alpha \in R \)) for the Robin condition, etc. The admissible domains are Lipschitzian and contain \( E \) (given).

To the problem (41) - (43), we associate the optimal control problem

\[
\text{Min}_{u \in L^2(\Omega)} \left\{ \int_E j(x, y_u(x))dx + \eta \int_{E_{y_u}} (u - f)^2dx \right\}
\] (44)

subject to

\[
\text{subject to } \frac{\partial y_u}{\partial n} = f \text{ in } \Omega, \\
B_y = 0 \text{ on } \partial \Omega, \\
\text{where } \eta > 0 \text{ is some "big" constant and } E_{y_u} \subset D \text{ is the smallest (or some) Lipschitzian subdomain such that } E \subset E_{y_u} \subset D \text{ and } B_y = 0 \text{ on } \partial E_{y_u}. \text{ One may allow } E_{y_u} \text{ to be an open set, not necessarily connected (for instance, if } E \text{ is not connected).}
For any $\Omega$ Lipschitzian, admissible for the problem (41) - (43), by the trace theorem, we can find $\tilde{y} \in H^2(D)$ with $B\tilde{y} = 0$ on $\partial D$ and $\tilde{y}|_{\partial \Omega} = y_0$.

In particular, $B \tilde{y} = 0$ on $\partial \Omega$ too.

Define $v \in L^2(D \setminus \Omega)$ by

$$v = - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial \tilde{y}}{\partial x_j} \right) + a_0 \tilde{y}, \tag{47}$$

$$\tilde{u} = \begin{cases} f & \text{in } \Omega, \\ v & \text{in } D \setminus \Omega \end{cases} \tag{48}$$

It is clear that $\tilde{y}$ is a strong solution of (45), (46) corresponding to $\tilde{u}$ given by (39), (40). The open set $E_y^{-}$ is contained in $\Omega$ since $B \tilde{y} = 0$ on $\partial \Omega$.

Then, for every $\eta > 0$, the second term in the cost (44) is zero since $\tilde{u} = f$ in $E_y^{-} \subset \Omega$.

**Proposition 3.1** For any $\eta > 0$, the cost (36) corresponding to $\tilde{u}$ is equal to the cost (41) corresponding to $\Omega$. Moreover

$$\text{Inf}_2 \leq \text{Inf}_1 \tag{49}$$

where $\text{Inf}_1$, $\text{Inf}_2$ denote the infimum of the problem (41) - (43), respectively (44) - (46).

A partial converse of P 3.1 may be formulated for solutions with cost close to the infimal value.

**Theorem 3.2** Let $\eta \in R_+$. If $[y_1, u_1]$ is a $\delta_\eta$-optimal pair for (44) - (46), then $E_y^{-}$ is an $\varepsilon_\eta$-optimal domain for (41) - (43), where $\varepsilon_\eta > 0$ depends on $\delta_\eta > 0$ and decreases with $\delta_\eta$.

We recall that $\delta_\eta$-optimal pair means that the cost (44) associated to $[y_\eta, u_\eta]$ is less than $\text{Inf}_1 + \delta_\eta$. Similar definition for $\varepsilon_\eta$-optimal domain. Related results and proofs, numerical examples may be found in [12].

**Remark** In the case of the Dirichlet boundary conditions, $E_y$ is just a level set. For the Neumann boundary condition $\frac{\partial y}{\partial n} = 0$ (i.e. corresponding to the Laplace operator), the boundary of $E_y$ is orthogonal to the level surfaces.

In the case of the Robin boundary condition (in $R^2$) the angle between the boundary of $E_y$ and the level lines of $y$ is given by:

$$\cos \theta(x) = \pm \frac{\alpha y(x)}{\nabla y(x)}$$

where the sign depends on the choice of the tangent direction.

**IV. Conclusion**

In this work, we review various controllability and optimal control techniques for the solution of shape optimization or free boundary problems. The main idea is to replace problems involving geometric unknowns (domains or boundaries) by analytic formulations, approximating the original problems.

**REFERENCES**


