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Experience with Teaching Mathematics for Engineers with the Aid of Wolfram Alpha

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Abstract—**Recent development of answer engines such as Wolfram Alpha brought a huge computational power to the end devices such as mobile phones and as a consequence students now can perform many calculus tasks very easily. Teachers could either forbid using such tools and insist on often quite long time consuming manual calculations or they can incorporate their usage in the education process that obviously requires update of syllabuses and teaching materials. In this paper we present two years experience with teaching mathematics for engineers with the aid of Wolfram Alpha that students can use not only in classes, but also during the written part of examination.**

Index Terms—**mathematics, calculus, Wolfram Alpha, multiple choice tests, generator for written exams, content management system.**

I. Introduction

STOP teaching calculating, start teaching math - this is a marketing phrase used by Wolfram Research to support their products Mathematica and Wolfram Alpha. Though being rather tentative it surprisingly opened a vivid discussion among academics. From a black and white perspectives there exist two groups of mathematics teachers, ones insisting on manual calculations being the only correct way how to train the brain and learn calculus properly, and the other group claiming that it is a waste of time doing time consuming calculations manually when a computer can give the answer almost immediately. Providing the list of all arguments of both groups goes beyond the scope of this paper. Choosing a gray position somewhere in between we rather present a practical information based on two years experience of using Wolfram Alpha as a practical aid that engineering students are allowed to use not only in classes, but also during the written part of examination (examinations of almost all courses have written and oral part).

We introduce Wolfram Alpha and its usefulness in the mathematics courses for engineers. Together with the system `TRIAL` that during a decade evolved from an online database of mathematics problems to a living web portal playing a role of learning management system for mathematics education. In the last section we list our experiences with practical usage of Wolfram Alpha by students and teachers, we show the most common types of errors students do, we list some problems that cannot be solved by Wolfram Alpha (at least in its current version) and provide some hints and advices for educators.

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II. Wolfram Alpha

Wolfram Alpha is an answer engine product created and developed by Wolfram Research. It is based on Mathematica and works as an online service that answers queries directly by computing the answers rather than providing a list of web pages where the answer can be found. It was released in May 2009 and currently it includes hundreds of datasets from many disciplines (historical, financial, economic, geographical, meteorological, lexical, and many more). From the mathematics point of view it encompasses computer algebra, symbolic and numerical computation, visualization, and statistics capabilities. Public part of Wolfram Alpha is available at the web address www.wolframalpha.com.

Wolfram Alpha brings the biggest added value to Mathematica, namely the free form input[1]. It is capable of responding to particularly phrased natural-language questions such as `plot x^2-y^2` or more complex questions such as `find saddle points of x^2-y^2`. It displays its "Input interpretation" of such a question using standardized phrases, e.g. `saddle points | x^2 - y^2`, or one can provide the standard Mathematica language input, e.g. `Plot[x^2-y^2, {x; -3; 3}, {y; -3; 3 }]`.

More details about Wolfram Alpha can be found directly at the web server including many examples. Here we mention some advantages of using it at the mathematics courses for engineers. At first it is **capable of almost all calculus tasks** students need, it finds the limits, it can differentiate and integrate, solve algebraic systems of equations, find properties of matrices, solve some simple differential equations, find the series expansions, and almost everything can be solved both symbolically and numerically.

Secondly, **it is free** and when online then it is available not only on computers with a web browser, but also on all modern end devices including mobile phones and tablets. Well, Wolfram Research recently released a Wolfram Alpha Pro paid subscription with more features that also led to an increase in advertisement on the free site, but the free version still covers all the tasks that students might require in the mathematics courses.

One of the disadvantage might be the **online access requirement**. During its three years history, Wolfram Alpha proved to be a reliable system allowing more than 500 students in the examination rooms to use it at the same time often without the need of an extra computational time. Requirement of the Internet access now therefore belongs to the category of blackout threats.

Other advantages and disadvantages with respect to mathematics teaching will be discussed below.

[1]The newest Mathematica now also offers the Wolfram Alpha input.

III. Web-based system TRIAL

A system called **TRIAL** was created and is continuously being developed at the Department of Mathematics (www.kma.zcu.cz) at the Faculty of Applied Sciences at the University of West Bohemia in Plzeň, Czech Republic. The history of **TRIAL** goes back to year 2002 and its original purpose was to create an online database of mathematics problems for the first year students at the technical faculties. Problems were then used in written examination tests and students could practice the trial versions (here the name **TRIAL** comes from) of the examination. Since then the database has grown up to several hundreds types of problems (each problem is available in many random mutations) and **TRIAL** is used in classes and examinations of ten mayor courses at five faculties for roughly 3500 students every year. The public part of **TRIAL** is available at the web address trial.kma.zcu.cz. Thanks to its size and content the server became one of the most visited web server at the university with the overall access rate greater than 1.2 millions individual visits per year. Nowadays **TRIAL** offers the following services:

1) generating of written tests (problem, solution, results), usually number of students = number of different written tests,
2) displaying results of written tests,
3) generating lecture notes for students (definitions, theorems, proofs, etc.),
4) generating lecture notes for lecturers and assistants,
5) discussion forum.

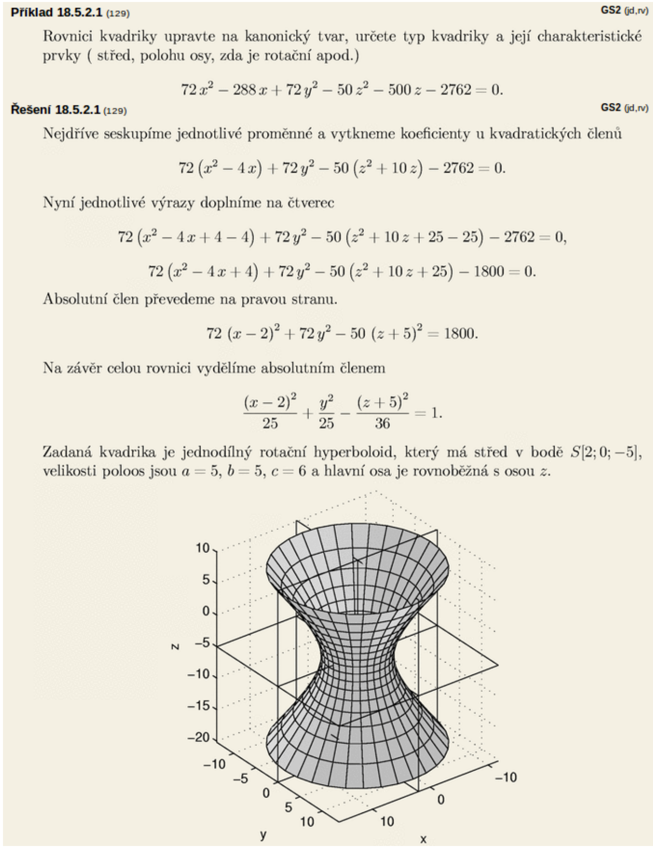
More details about **TRIAL** can be found in [1]. Here we mention two major advantages of using **TRIAL** together with Wolfram Alpha. The first is the **uniqueness of problems** in each student's test. None of the tests is the same, only the difficulty is consistent. This fact forces students to really work alone, individually formulate the questions for the answer engine and correctly get the results. A second advantage comes into account especially before the tests - a **discussion forum**. It is a place where students discuss with teachers and among themselves problems and their solution, they share the knowledge of questioning the answer engine and surprisingly they ask and discuss problems engineering students never asked before going for example deep into mathematical analysis. Having more than a hundred forum threads every term, mathematics became one of the most discussed topics for first year students.

Exercises in **TRIAL** are of the following types:

1) standard exercises, students has to solve given problems and provide step-by-step solution, see for example Exercises 1 and 2 at the end of this section.
2) multiple choice tests, students only mark correct answers, see for example Exercises 3 and 4,
3) fill-in tests, students have to calculate given problems (usually very easy) and provide only the result, they have to fill-in a missing information in the text or sketch a graph, see Exercises 5, 6 and 7.

Let us close this section by some exercises, which were automatically generated by the system **TRIAL** and that are regularly included in the written tests in the first term.

Exercise 1. *Let $\{a_n\}$ be a real sequence and let $a \in \mathbb{N}$ be*



Příklad 18.5.2.1 (129) **GS2** (jd,rv)

Rovnici kvadriky upravte na kanonický tvar, určete typ kvadriky a její charakteristické prvky (střed, polohu osy, zda je rotační apod.)

$$72\,x^2 - 288\,x + 72\,y^2 - 50\,z^2 - 500\,z - 2762 = 0.$$

Řešení 18.5.2.1 (129) **GS2** (jd,rv)

Nejdříve seskupíme jednotlivé proměnné a vytkneme koeficienty u kvadratických členů

$$72\,(x^2 - 4\,x) + 72\,y^2 - 50\,(z^2 + 10\,z) - 2762 = 0.$$

Nyní jednotlivé výrazy doplníme na čtverec

$$72\,(x^2 - 4\,x + 4 - 4) + 72\,y^2 - 50\,(z^2 + 10\,z + 25 - 25) - 2762 = 0,$$

$$72\,(x^2 - 4\,x + 4) + 72\,y^2 - 50\,(z^2 + 10\,z + 25) - 1800 = 0.$$

Absolutní člen převedeme na pravou stranu.

$$72\,(x-2)^2 + 72\,y^2 - 50\,(z+5)^2 = 1800.$$

Na závěr celou rovnici vydělíme absolutním členem

$$\frac{(x-2)^2}{25} + \frac{y^2}{25} - \frac{(z+5)^2}{36} = 1.$$

Zadaná kvadrika je jednodílný rotační hyperboloid, který má střed v bodě $S[2;0;-5]$, velikosti poloos jsou $a = 5$, $b = 5$, $c = 6$ a hlavní osa je rovnoběžná s osou z.

Fig. 1. One sample problem (in Czech language) generated from the system **TRIAL** together with its step-by-step solution.

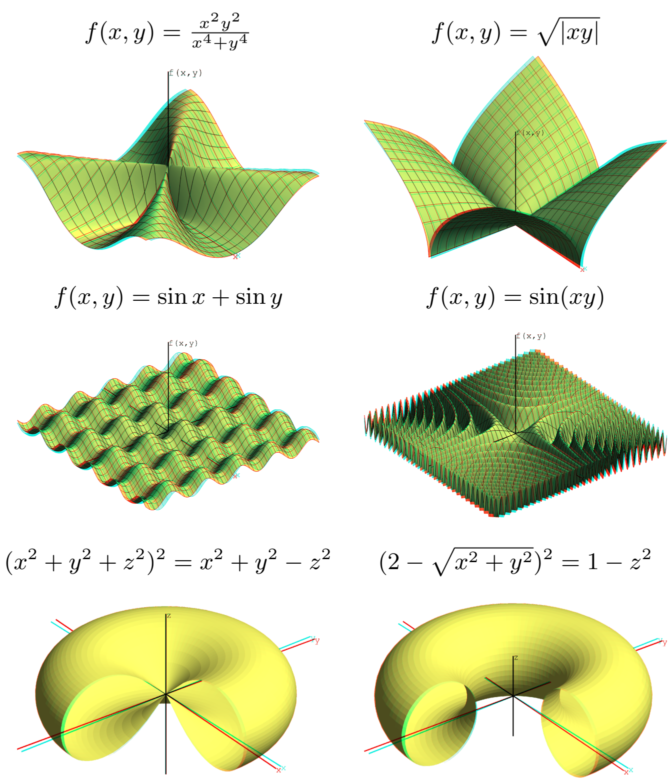


Fig. 2. System **TRIAL** contains also a lot of addons, for example graphs of functions of two variables and implicitly given surfaces for anaglyph 3D glasses.

such that

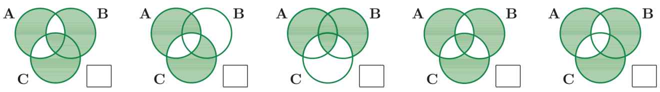
$$\exists\, \varepsilon > 0\ \forall\, n_0 \in \mathbb{N}\ \forall\, n \in \mathbb{N}:\ n > n_0\ \Rightarrow\ |a_n - a| \leq \varepsilon.$$

Firstly, give an example of such a sequence $\{a_n\}$ and sketch its graph. Secondly, describe the set of all such sequences.

Exercise 2. *Given a real function $h: y = \sqrt{t+2} + 3$, $t \in [2, +\infty)$, find the inverse function and sketch graphs of both functions.*

Exercise 3. *Which picture depicts the set*

$$M = (A \cap B \cap C) \cup (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B))?$$



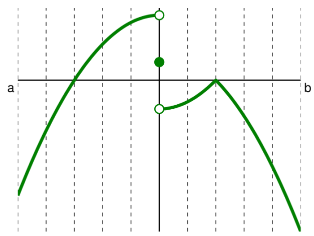
Exercise 4. *Check all functions, which have the following properties on their whole domain of definition:*

	e^x	$\sin x$	$\arcsin x$	$\sinh x$	$4x$
bounded from above	☐	☐	☐	☐	☐
convex	☐	☐	☐	☐	☐
periodic	☐	☐	☐	☐	☐

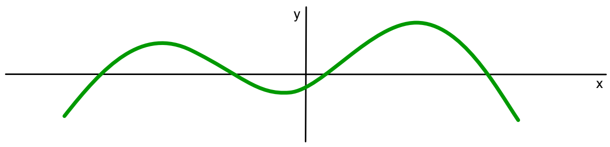
Exercise 5. *Calculate the limit:*

$$\lim_{x \to 0+} \operatorname{sgn}(4x) = \underline{\hspace{3cm}}.$$

Exercise 6. *Let f be a given function on $[a, b]$. A partition of the interval $[a, b]$ is given in the graph of f. Draw the lower integral sum of f on $[a, b]$.*



Exercise 7. *Given a graph of function f sketch the graph of its derivative and one of its antiderivatives:*



IV. Two years experience

In the first year at the University of West Bohemia, engineering students take the two term course Mathematics and supporting Seminar of Differential Calculus, Seminar of Integral Calculus and Seminar of Differential Equations. In winter term students learn

1) real sequences and series,
2) real functions of one real variable,
3) differential and integral calculus in $\mathbb{R}$,

and in the summer term then

1) function sequences and series,
2) vector functions,
3) real functions of more real variables,
4) differential and integral calculus in $\mathbb{R}^n$.

More detailed description of the courses can be found in the university course catalog (ects.zcu.cz), teaching mathematics courses is in particular discussed in [2] and [3].

Even if Wolfram Alpha is allowed to use during examination, students still make mistakes. What has changed is the character of these errors. Usage of Wolfram Alpha reduced the numbers of numerical errors students usually did in the past. Unfortunately we have to say that the overall number of errors students do because they wrongly or at all understand the problem has not changed much. With respect to Wolfram Alpha, students do the following type of errors.

Syntax errors: As already mentioned, Wolfram Alpha uses free form syntax, so one could assume that it is almost "impossible" to make such an error. Unfortunately wrong query yields wrong results. Surprisingly, problems with correct query formulation are more frequent for students that have some programming capabilities and are influenced by the strict syntax form of programming languages. A natural language formulation of queries is hence the most preferred way of input by the vast majority of students.

Wrong interpretation of results: Using Wolfram Alpha, one has to take care to interpret the answers correctly. The vast majority of the mathematics courses in the first term is taught only on the set of real numbers. However, Wolfram Alpha often returns results in complex numbers that students interpret incorrectly. For example, they believe that the real logarithm function $y = \log x$ is an even function since Wolfram Alpha generates the graph of the real part as a symmetric graph with respect to y-axis (see Figure 3).

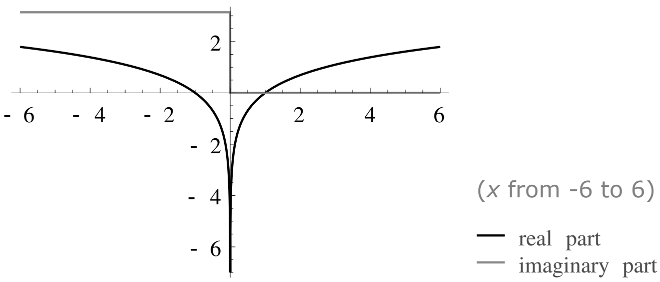


Fig. 3. The graph of the logarithm function $y = \log x$ by Wolfram Alpha.

Moreover, some students think that the equation $\sin x = 2$ has some (real) solutions since the output generated by Wolfram Alpha can tempt to make wrong conclusions (see Figure 4).

$$x = 2\,\pi\, n + \pi - \sin^{-1}(2), \quad n \in \mathbb{Z}$$

$$x = 2\,\pi\, n + \sin^{-1}(2), \quad n \in \mathbb{Z}$$

Fig. 4. All solutions of the equation $\sin x = 2$ found by Wolfram Alpha.

Even if the query for Wolfram Alpha is clearly formulated, the answer engine does not have to understand it correctly. As claimed by Stephen Wolfram in his blog [4], Wolfram Alpha is now on average giving complete, successful responses to more than 90% of the queries entered on its website. In his blog, Stephen Wolfram shows several absurd examples of wrong input interpretations. He claims that there are two main issues that seem to combine to produce most of the artificial stupidity – Wolfram Alpha either tries to give an

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answer even if it does not really know what it is talking about, or Wolfram Alpha simply does not know enough. Many calculus problems do not have general solutions that could be easily transfered into algorithm and if so, these algorithms might be really hard to implement. To mention just a few: finding the domain and range of functions, determining the convergence (and especially uniform convergence) of sequences and series, etc. And there are of course many open problems in mathematical analysis itself, so Wolfram Alpha for example cannot solve many differential equations.

All these facts give teachers the opportunity to formulate many new problems, some of them of a little complex character where Wolfram Alpha can be used in some parts, but the overall solution lies on the students. Instead of performing sometimes long lasting manual calculations, Wolfram Alpha allows students to break into the more difficult parts of mathematical analysis, for example function sequences and series, in particular power and Fourier series. The following two exercises are parts of the regular written exams in the second term.

Exercise 8. *Give an example of a function sequence* $\{f_n(x)\}$ *such that* $f_n \xrightarrow{\mathbb{R}} f$, *all the functions* f_n *are differentiable on* $\mathbb{R}$ *and* $f(x) = |x|$.

Using Wolfram Alpha, a student can easily check his answer by writing `Limit[2/Pi x ArcTan[n x], n->Infinity]`. Moreover, the graph of the difference between the limit function f and the n-th member f_n can be easily visualized as `plot | Table[Abs[x] - 2/Pi x ArcTan[n x],{n,1,10}]` (see Figure 5).

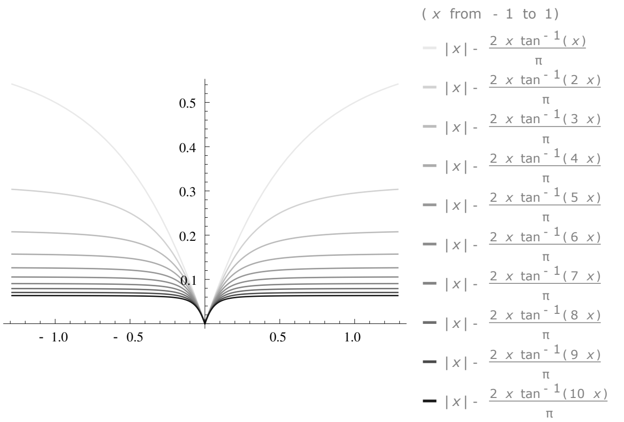

Fig. 5. Graphs of the differences $f - f_n$ for $n = 1, \ldots, 10$, where $f(x) = |x|$ and $f_n(x) = \frac{2}{\pi} x \arctan nx$, generated by Wolfram Alpha.

Exercise 9. *Give an example of a power series with the domain of convergence* $K = (-4, 0]$.

One of the infinitely many correct answers

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n 2^n} (x+2)^n$$

can be checked by writing

```
Sum[(-1)^n/(n 2^n)(x+2)^n,{n,1,Infinity}].
```

The following simple examples illustrate that probably no answer engine product can substantially help to solve them or even check the answer.

Exercise 10. *Give an example of a function sequence* $\{f_n(x)\}$ *such that* $f_n \xrightarrow{\mathbb{R}} f$, *all the functions* f_n *are not injective (one-to-one) on* $\mathbb{R}$ *and* f *is injective on* $\mathbb{R}$.

What one only need is to really understand the injectivity property and the pointwise convergence of a function sequence to give a possible answer

$$f_n(x) = \begin{cases} x & \text{for } x \neq 0, \\ \frac{1}{n} & \text{for } x = 0, \end{cases} \qquad f(x) = x.$$

Exercise 11. *Let* $\{a_n\}$ *be a sequence of real numbers. Which of the following implications holds?*

$$\sum_{n=1}^{+\infty} a_n^2 \quad \textit{converges} \qquad \begin{array}{c} \Longrightarrow \\ \Longleftarrow \end{array} \qquad \lim_{n \to +\infty} n a_n = 0$$

Justify and prove your conclusions. Find a suitable counterexample if the implication does not hold.

The first implication does not hold. The convergence of the series $\sum_{n=1}^{+\infty} a_n^2$ does not imply necessarily that $n a_n$ tends to zero. The choice of $a_n = \frac{1}{n}$ is a suitable counterexample. On the other hand, the opposite implication is valid. Indeed, using the assumption $n a_n \to 0$, we conclude that $a_n^2 < \frac{1}{n^2}$ for all sufficiently large $n \in \mathbb{N}$. Finally, using the comparison test (see [5]), we obtain that the series $\sum_{n=1}^{+\infty} a_n^2$ converges.

We conclude this section by several recommendations for mathematics teachers not only at the undergraduate level at the university. At first **learn and get used to Wolfram Alpha** , understand its capabilities and limitations, your students probably already know it, so be ready to answer their questions. **Update your teaching materials**, introducing tools like Wolfram Alpha can only enhance your classes and you can take your students deeper into the beauty of mathematics.

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