Bipolar-valued Fuzzy Finite Switchboard State Machines

J. Kavikumar, *Member, IAENG,* Azme Bin Khamis and Rozaini Bin Roslan

***Abstract*—This paper introduces and investigates related properties of bipolar fuzzy finite switchboard state machines. Thus, the notion of bipolar valued fuzzy finite state machine, the concept of bipolar submachine, bipolar connected, bipolar retrievable are utilized.**

***Index Terms*—bipolar fuzzy finite state machine, bipolar switching, bipolar submachine, bipolar connected, bipolar retrievable.**

I. Introduction

IN 1965, Zadeh [11] introduced the notion of fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, pattern recognition, robotics, computer networks, decision making and automata theory.

In 1994, Zhang [12], [13] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property (see [2], [15]), and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets [8]. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [3], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity tend to occur too. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions, but this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object , thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

Malik et al. [10] introduced the notions of submachine of a fuzzy finite state machine, retrievable, separated and connected fuzzy finite state machines and discussed their basic properties. They also initiated a decomposition theorem for fuzzy finite state machines in terms of primary submachines. On the other hand, Kumbhojkar and Chaudhari [7] provided several ways of constructing products of fuzzy finite state machines and their mutual relationship, through isomorphism and coverings. Li and Pedrycz [9] indicated that fuzzy finite state automata can be viewed as a mathematical model of computation in fuzzy systems. Recently, a higher order set with imprecision has been extended to automata. Based on Atanassov's intuitionistic fuzzy sets [1], Jun proposed intuitionistic fuzzy finite state machines in [4] and also intuitionistic fuzzy finite switchboard state machines in [5]. Zhang and Li [14] presented the properties of intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. Thus, using the notion of bipolar fuzzy valued sets, the present author [6] introduced the concepts of bipolar fuzzy finite state machines, bipolar successors, bipolar subsystems and studied related properties. He established a condition for bipolar fuzzy finite state machine to satisfy the bipolar exchange property.

In this paper, using the notion of bipolar-valued fuzzy sets, concept of bipolar submachines, bipolar connected, bipolar retrievable and bipolar fuzzy finite switchboard state machines (bffssm) is introduced and related properties are investigated.

II. Preliminaries

Let X be the universe of discourse. A *bipolar-valued fuzzy set* φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. The positive membership degree $\varphi^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x

Manuscript received March 30, 2012; revised July 10, 2012. This work was supported by the Ministry of Higher Education, Malaysia under FRGS Grant, Vot No: 0821.

J. Kavikumar, Azme Bin Khamis and Rozaini Bin Roslan are with the Center for Research in Computational Mathematics, Faculty of Science, Technology and Human Development, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor, Malaysia e-mail: (kavi@uthm.edu.my).

does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. It is possible for an element x to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [8]). For the sake of simplicity, we shall use the symbol $\varphi = \langle \varphi^-, \varphi^+ \rangle$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

III. Bipolar Fuzzy Finite State Machines

Definition 3.1: [6] A *bipolar fuzzy finite state machine* (bffsm, for short) is a triple $\mathcal{M} = (Q, X, \varphi)$, where Q and X are finite nonempty sets, called the set of states and the set of input symbols, respectively, and $\varphi = \langle \varphi^-, \varphi^+ \rangle$ is a bipolar fuzzy set in $Q \times X \times Q$.

Let X^* denote the set of all words of elements of X of finite length. Let λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

Definition 3.2: [6] Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Define a bipolar fuzzy set $\varphi_* = \langle \varphi_*^-, \varphi_*^+ \rangle$ in $Q \times X^* \times Q$ by

$$\varphi_*^-(q, \lambda, p) := \begin{cases} -1 & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

$$\varphi_*^+(q, \lambda, p) := \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

$$\varphi_*^-(q, xa, p) = \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi^-(r, a, p)]$$

$$\varphi_*^+(q, xa, p) = \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi^+(r, a, p)]$$

for all $p, q \in Q$, $x \in X^*$ and $a \in X$.

Lemma 3.3: [6] Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Then

$$\varphi_*^-(q, xy, p) = \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi_*^-(r, y, p)]$$

and

$$\varphi_*^+(q, xy, p) = \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi_*^+(r, y, p)]$$

for all $p, q \in Q$ and $x, y \in X^*$.

Definition 3.4: [6] Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm and let $p, q \in Q$. Then p is called a *bipolar immediate successor* of q if the following condition holds:

$$(\exists a \in X) \ (\varphi^-(q, a, p) < 0, \ \varphi^+(q, a, p) > 0).$$

We say that p is a *bipolar successor* of q if the following condition holds:

$$(\exists x \in X^*) \ (\varphi_*^-(q, x, p) < 0, \ \varphi_*^+(q, x, p) > 0).$$

We denote by $\mathcal{S}(q)$ the set of all bipolar successors of q. For any subset T of Q, the set of all bipolar successors of T, denoted by $\mathcal{S}(T)$, is defined to be the set

$$\mathcal{S}(T) := \cup \{\mathcal{S}(q) \mid q \in T\}.$$

Proposition 3.5: [6] For any bffsm $\mathcal{M} = (Q, X, \varphi)$, we have the following properties:

(1) $(\forall q \in Q) \ (q \in \mathcal{S}(q))$.
(2) $(\forall p, q, r \in Q) \ (p \in \mathcal{S}(q), \ r \in \mathcal{S}(p) \ \Rightarrow \ r \in \mathcal{S}(q))$.

A. Bipolar Submachine and Bipolar Connected

Definition 3.6: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Let $(\forall T \subseteq Q)$. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy set in $T \times X \times T$ and $\mathcal{N} = (T, X, \varphi_Q)$ be a bffsm. Then $\mathcal{N}$ is called a *bipolar submachine* of $\mathcal{M}$, if

(i) $\varphi \mid_{T \times X \times T} = \varphi_Q$,
i.e., $\varphi^- \mid_{T \times X \times T} = \varphi_Q^-$ and $\varphi^+ \mid_{T \times X \times T} = \varphi_Q^+$.
(ii) $S(T) \subseteq T$

We assume that $\emptyset = (\emptyset, X, \varphi)$ is a bffsm of $\mathcal{M}$. Obviously it $\mathcal{K}$ is a bipolar submachine of $\mathcal{N}$ and $\mathcal{M}$ is a bipolar submachine of $\mathcal{M}$, the $\mathcal{K}$ is a bipolar submachine of $\mathcal{M}$.

Definition 3.7: A bffsm $\mathcal{M} = (Q, X, \varphi)$is said to be *strongly bipolar connected* if $p \in S(q)$ for every $p, q \in Q$. A bipolar submachine $\mathcal{N} = (T, X, \varphi_Q)$ of a bffsm $\mathcal{M} = (Q, X, A)$ is said to be proper if $T \neq \emptyset$ and $T \neq Q$.

Theorem 3.8: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Let $\mathcal{N} = (T, X, \varphi_{Q_i}), i \in \Lambda$, be a family of bipolar submachines of $\mathcal{M} = (Q, X, \varphi)$. Then we have

(i) $\bigcap_{i \in \Lambda} \mathcal{N}_i = (\bigcap_{i \in \Lambda} T_i, X, \bigcap_{i \in \Lambda} \varphi_Q)$ is a bipolar submachine of $\mathcal{M}$.
(ii) $\bigcup_{i \in \Lambda} \mathcal{N}_i = (\bigcup_{i \in \Lambda} T_i, X, \varphi_*)$ is a bipolar submachine of $\mathcal{M}$ where $\varphi_* = (\varphi_*^-, \varphi_*^+)$ is given by $\varphi_*^- = \varphi^- \mid_{\bigcup_{i \in \Lambda} T_i \times X \times \bigcup_{i \in \Lambda} T_i}$ and $\varphi_*^+ = \varphi^+ \mid_{\bigcup_{i \in \Lambda} T_i \times X \times \bigcup_{i \in \Lambda} T_i}$.

Proof: (i) Let $(q, x, p) \in (\bigcap_{i \in \Lambda} T_i, X, \bigcap_{i \in \Lambda} \varphi_Q)$. Then

$$\begin{aligned} (\inf_{i \in \Lambda} \varphi_Q^-)(q, x, p) &= \inf_{i \in \Lambda} \varphi_Q^-(q, x, p) \\ &= \inf_{i \in \Lambda} \varphi^-(q, x, p) = \varphi^-(q, x, p) \end{aligned}$$

and

$$\begin{aligned} (\sup_{i \in \Lambda} \varphi_Q^+)(q, x, p) &= \sup_{i \in \Lambda} \varphi_Q^+(q, x, p) \\ &= \sup_{i \in \Lambda} \varphi^+(q, x, p) = \varphi^+(q, x, p) \end{aligned}$$

Hence $\varphi \mid_{\bigcap_{i \in \Lambda} T_i \times X \times \bigcup_{i \in \Lambda} T_i} = \bigcap_{i \in \Lambda} \varphi_Q$. Now

$$S(\bigcap_{i \in \Lambda} T_i) \subseteq \bigcap_{i \in \Lambda} S(T_i) \subseteq \bigcap_{i \in \Lambda} T_i$$

Thus $\bigcap_{i \in \Lambda} \mathcal{N}_i$ is a bipolar submachine of $\mathcal{M}$.

(ii) Since $S(\bigcup_{i \in \Lambda} T_i) = \bigcup_{i \in \Lambda} S(T_i) \subseteq \bigcup_{i \in \Lambda} T_i$, $\bigcup_{i \in \Lambda} \mathcal{N}_i$ is a bipolar submachine of $\mathcal{M}$. ■

Theorem 3.9: A bffsm $\mathcal{M} = (Q, X, \varphi)$ is strongly bipolar connected if and only if $\mathcal{M} = (Q, X, \varphi)$ has a proper bipolar submachines.

Proof: Suppose that $\mathcal{M} = (Q, X, \varphi)$ is strongly bipolar connected. Let $\mathcal{N} = (T, X, \varphi_Q)$ be a bipolar submachine of $\mathcal{M}$ such that $T \neq \emptyset$. Then there exists $q \in T$. If $p \in Q$ then $p \in S(q)$ since $\mathcal{M}$ is strongly bipolar connected. It follows that $p \in S(q) \subseteq S(T) \subseteq T$ so that $T = Q$. Hence $\mathcal{M} = N$, i.e., $\mathcal{M}$ has no proper bipolar submachines. Let $p, q \in Q$ and let $\mathcal{N} = (S(q), X, \varphi_Q)$where $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ is given by

$$\varphi_Q^- = \varphi^- \mid_{S(q) \times X \times S(q)}$$

and

$$\varphi_Q^+ = \varphi^+ \mid_{S(q) \times X \times S(q)}$$

ISBN: 978-988-19251-6-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Then $\mathcal{N}$ is a bipolar submachine of $\mathcal{M}$ and $S(q) \neq \emptyset$, and so $S(q) = Q$. Thus $p \in S(q)$, and therefore $\mathcal{M}$ is strongly bipolar connected. ■

For an bffsm $\varphi = \langle \varphi^-, \varphi^+ \rangle$ in a set X, the *bipolar support* of φ is defined to be the set

$$Supp(\varphi) := \{x \in X \mid \varphi^-(x) < 0, \varphi^+(x) > 0\}.$$

For a bipolar fuzzy set $\varphi = \{(x, (\varphi^-(x), \varphi^+(x)) \mid x \in X\}$ and $(s,t) \in [-1,0] \times [0,1]$, we define

$$\varphi_t^- ;= \{x \in X \mid \varphi^-(x) \leq t\},$$

$$\varphi_t^+ ;= \{x \in X \mid \varphi^+(x) \geq t\},$$

which are called the *negative s-cut* of φ and the *positive t-cut* of φ, respectively.

The set $\varphi_{(s,t)} = \{x \in X \mid \varphi^-(x) \leq s, \varphi^+(x) \geq t\}$ is called an (s,t)-level subset of φ.

Theorem 3.10: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm and $\mathcal{Q} = (Q, \varphi_Q, X, \varphi)$ be a bipolar subsystems of $\mathcal{M}$. Then

(i) $\mathcal{N} = (Supp(\mathcal{Q}), X, \varphi_{Q^{**}})$is a bipolar submachine of $\mathcal{M}$, where $\varphi_{Q^{**}} = (\varphi^-_{Q^{**}}, \varphi^+_{Q^{**}})$ is given by

$$\varphi^-_{Q^{**}} = \varphi^- \mid_{Supp(\mathcal{Q}) \times X \times Supp(\mathcal{Q})}$$

and

$$\varphi^+_{Q^{**}} = \varphi^+ \mid_{Supp(\mathcal{Q}) \times X \times Supp(\mathcal{Q})}$$

(ii) Let $\mathcal{N}_{(s,t)} = (\mathcal{Q}_{(s,t)}, X, \varphi_{Q^{**}_{(s,t)}})$ where

$$\mathcal{Q}_{(s,t)} := \{x \in Q \mid \varphi^-_Q(x) \leq s, \varphi^+_Q(x) \geq t\}$$

and $\varphi_{Q^{**}_{(s,t)}} = (\varphi^-_{Q^{**}_{(s,t)}}, \varphi^+_{Q^{**}_{(s,t)}})$ is given by

$$\varphi^-_{Q^{**}_{(s,t)}} = \varphi^- \mid_{\mathcal{Q}_{(s,t)}) \times X \times \mathcal{Q}_{(s,t)}}$$

and

$$\varphi^+_{Q^{**}_{(s,t)}} = \varphi^+ \mid_{\mathcal{Q}_{(s,t)}) \times X \times \mathcal{Q}_{(s,t)}},$$

$(s,t) \in [-1,0] \times [0,1]$. If $\mathcal{N}_{(s,t)}$ is a bipolar submachine of $\mathcal{M}$ for all $(s,t) \in [-1,0] \times [0,1]$, then $\mathcal{Q}$ is a bipolar subsystem of $\mathcal{M}$.

Proof: (i) Let $p \in S(Supp(\mathcal{Q}))$. Then $p \in S(q)$ for some $q \in Supp(\mathcal{Q}))$. Thus $\varphi^-_Q(q) < 0$ and $\varphi^+_Q(q) > 0$. Since $p \in S(q)$, there exists $x \in X^*$ such that $\varphi^-_{Q^*}(q,x,p) < 0$ and $\varphi^+_{Q^*}(q,x,p) > 0$. Since Q is a bipolar subsystem, it follows from Theorem 3.2 [6] that

$$\varphi^-_Q(q) \leq \varphi^-_Q(p) \vee \varphi^-_*(p,x,q) < 0$$

$$\varphi^+_Q(q) \geq \varphi^+_Q(p) \wedge \varphi^+_*(p,x,q) > 0$$

so that $p \in Supp(\mathcal{Q})$. Hence $S(Supp(\mathcal{Q})) \subseteq Supp(\mathcal{Q})$, and therefore $\mathcal{N}$ is a bipolar submachine of $\mathcal{M}$.

(ii) Let $p, q \in Q$ and $x \in X^*$. If $\varphi^-_Q(p) = s$ or $\varphi^-_{Q^*}(p,x,q) = s$, then

$$\varphi^-_Q(q) \leq s = \varphi^-_Q(p) \vee \varphi^-_{Q^*}(p,x,q).$$

If $\varphi^+_Q(p) = t$ or $\varphi^+_{Q^*}(p,x,q) = t$, then

$$\varphi^+_Q(q) \geq t = \varphi^+_Q(p) \wedge \varphi^+_{Q^*}(p,x,q).$$

Suppose $\varphi^-_Q(p) < 0$, $\varphi^-_{Q^*}(p,x,q) < 0$, $\varphi^+_Q(p) > 0$ and $\varphi^+_{Q^*}(p,x,q) > 0$. Let

$$\varphi^-_Q(p) \vee \varphi^-_{Q^*}(p,x,q) = s$$

$$\varphi^+_Q(p) \wedge \varphi^+_{Q^*}(p,x,q) = t.$$

Then $p \in \mathcal{Q}_{(s,t)}$. Since $\mathcal{N}_{(s,t)}$ is a bipolar submachine of $\mathcal{M}$, we have $S(\mathcal{Q}_{(s,t)}) \subseteq \mathcal{Q}_{(s,t)}$. Hence $p \in S(p) \subseteq S(\mathcal{Q}_{(s,t)}) \subseteq \mathcal{Q}_{(s,t)}$, and thus

$$\varphi^-_Q(q) \leq s \leq \varphi^-_Q(p) \vee \varphi^-_*(p,x,q)$$

and

$$\varphi^+_Q(q) \geq t \geq \varphi^+_Q(p) \wedge \varphi^+_*(p,x,q)$$

Using theorem 3.2 [6], we conclude that $\mathcal{Q}$ is a bipolar subsystem of $\mathcal{M}$. ■

The converse of Theorem 3.8 (ii) is not true in general. In fact, consider the bipolar subsystem $\mathcal{Q}$ in Example. Let $\frac{1}{2} < t \leq \frac{3}{4}$ and $-\frac{1}{2} < s \leq -\frac{1}{8}$. Let $\mathcal{N}_{(s,t)} = (\mathcal{Q}_{(s,t)}, X, \varphi_{Q^{**}_{(s,t)}})$ where $\varphi_{Q^{**}_{(s,t)}} = (\varphi^-_{Q^{**}_{(s,t)}}, \varphi^+_{Q^{**}_{(s,t)}})$ is given by

$$\varphi^-_{Q^{**}_{(s,t)}} = \varphi^- \mid_{\mathcal{Q}_{(s,t)}) \times X \times \mathcal{Q}_{(s,t)}}$$

and

$$\varphi^+_{Q^{**}_{(s,t)}} = \varphi^+ \mid_{\mathcal{Q}_{(s,t)}) \times X \times \mathcal{Q}_{(s,t)}} .$$

Now $\varphi^-_Q(q) = -\frac{1}{8}$, $\varphi^+_Q(q) = \frac{3}{4}$ and $\varphi^-_Q(q) + \varphi^+_Q(q) > 0$. Hence $q \in \mathcal{Q}_{(s,t)}$. Also $\varphi^-_Q(q,a,p) = -\frac{1}{2} < 0$ and $\varphi^+_Q(q,a,p) = \frac{1}{2} > 0$, and so $p \in S(q) \subset S(\mathcal{Q}_{(s,t)})$. But $\varphi^+_Q(p) = \frac{1}{2} < t$, and thus $p \notin \mathcal{Q}_{(s,t)}$. Hence $S(\mathcal{Q}_{(s,t)}) \not\subseteq \mathcal{Q}_{(s,t)}$, and therefore $\mathcal{N}_{(s,t)}$ is not a bipolar submachine of $\mathcal{M}$.

B. Bipolar Retrievable

Definition 3.11: A bffsm $\mathcal{M} = (Q, X, \varphi)$ is said to be *bipolar retrievable* if
$(\forall q \in Q)(\forall y \in X^*)(\exists t \in Q)$

$$(\varphi^-_{Q^*}(q,y,t) < 0, \varphi^+_{Q^*}(q,y,t) > 0)$$

$\Longrightarrow (\exists x \in X^*)(\varphi^-_{Q^*}(t,x,q) < 0, \varphi^+_{Q^*}(t,x,q) > 0)$

Definition 3.12: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm and let $q, r, s \in Q$. Then r and s are said to be *bipolar q-related* if there exists $y \in X^*$ such that $\varphi^-_{Q^*}(q,y,r) < 0, \varphi^-_{Q^*}(q,y,s) < 0, \varphi^+_{Q^*}(q,y,r) > 0$ and $\varphi^+_{Q^*}(q,y,s) > 0$.

We say that r and s are *bipolar q-twins* if

(i) r and s are bipolar q-related,

(ii) $S(r) = S(s)$.

Lemma 3.13: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Then the following assertions are equivalent.

(i) $\forall q, r, s \in Q$, if r and s are bipolar q-related, then r and s are bipolar q-twins.

(ii) $(\forall p, q, r \in Q)(\forall x, y \in X^*)$
$(\varphi^-_{Q^*}(q,y,r) < 0, \varphi^-_{Q^*}(q,yx,p) < 0,$
$\varphi^+_{Q^*}(q,y,r) > 0, \varphi^+_{Q^*}(q,yx,p) > 0 \Longrightarrow p \in S(r)).$

Proof: (i) ⇒ (ii) By Lemma 3.1[6], it is clear that r and s are bipolar q-related. It follows from (i) that r and s are bipolar q-twins so that $p \in S(s) = S(r)$.

(ii) ⇒ (i) Suppose that (ii) is valid. Let $q, r, s \in Q$ be such that r and s are bipolar q-related. Then there exists $y \in X^*$ such that $\varphi^-_{Q^*}(q,y,r) < 0, \varphi^-_{Q^*}(q,y,s) < 0, \varphi^+_{Q^*}(q,y,s) > 0$, and $\varphi^+_{Q^*}(q,y,s) > 0$. If $p \in S(s)$, then there exists $x \in X^*$ such that $\varphi^-_{Q^*}(s,x,p) < 0$, and $\varphi^+_{Q^*}(s,x,p) > 0$. Then

$$\varphi^-_{Q^*}(q,xy,p) = \inf_{t \in Q} [\varphi^-_{Q^*}(q,y,t) \vee \varphi^-_{Q^*}(t,x,p)] < 0$$

ISBN: 978-988-19251-6-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

and

$$\varphi^{+}_{Q^*}(q, xy, p) = \sup_{t \in Q}[\varphi^{+}_{Q^*}(q, y, t) \wedge \varphi^{+}_{Q^*}(t, x, p)] > 0.$$

Thus $p \in S(r)$ by hypothesis. Similarly if $p \in S(r)$ then $p \in S(s)$. Therefore r and s are bipolar q-twins. ■

Theorem 3.14: A bffsm $\mathcal{M} = (Q, X, \varphi)$ is a bipolar retrievable if and only if it satisfies

(i) $(\forall q \in Q)(\forall y \in X^*)(\exists t \in Q)(\varphi^{-}_{Q^*}(q, y, t) < 0, \varphi^{+}_{Q^*}(q, y, t) > 0)$
$\Longrightarrow (\exists x \in X^*)(\varphi^{-}_{Q^*}(q, yx, q) < 0, \varphi^{+}_{Q^*}(q, yx, q) > 0))$.

(ii) $q, r, s \in Q$, if r and s are bipolar q-related then r and s are bipolar q-twins.

Proof: Obvious ■

IV. Bipolar fuzzy finite switchboard state machines

Definition 4.1: A bffsm $\mathcal{M} = (Q, X, \varphi)$ is said to be switching if it satisfies:

$$\varphi^{-}_{*}(q, a, p) = \varphi^{-}_{*}(p, a, q)$$

$$\varphi^{+}_{*}(q, a, p) = \varphi^{+}_{*}(p, a, q)$$

for all $p, q \in Q$, and $a \in X$.
A bffsm $\mathcal{M} = (Q, X, \varphi)$ is said to be commutative if it satisfies:

$$\varphi^{-}_{*}(q, ab, p) = \varphi^{-}_{*}(p, ba, q)$$

$$\varphi^{+}_{*}(q, ab, p) = \varphi^{+}_{*}(p, ba, q)$$

for all $p, q \in Q$, and $a, b \in X$.

If a bffsm $\mathcal{M} = (Q, X, \varphi)$ is both switching and commutative, we say that $\mathcal{M}$ is a bipolar fuzzy finite switchboard state machine (bffssm, for short).

Example 4.2: Let Let $\mathcal{M} = (Q, X, \varphi)$ is a bffsm, where $Q = \{p, q, r\}$, $X = \{a, b\}$ and let $\varphi_Q = (\varphi^{-}_{Q}, \varphi^{+}_{Q})$ is defined as follows.

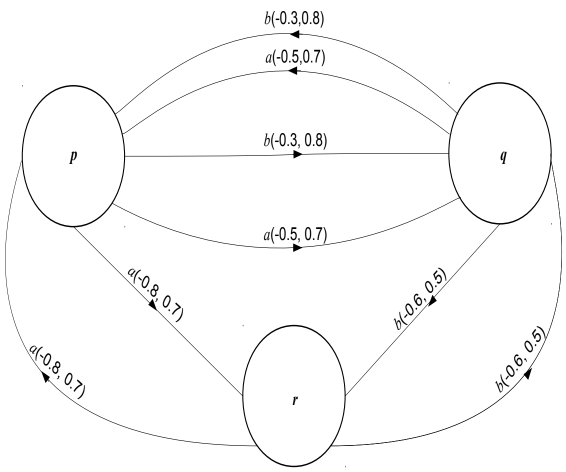

Fig. 1. Bipolar Fuzzy Finite Switchboard State Machine

Then, by routine computations and using Lemma 3.3 for all $p, q, r \in Q$ and $a, b \in X$, it is easy to see that $\mathcal{M} = (Q, X, \varphi)$ is a bffssm.

Proposition 4.3: If $\mathcal{M} = (Q, X, \varphi)$ is a commutative bffsm, then

$$\varphi^{-}_{*}(q, xa, p) = \varphi^{-}_{*}(q, ax, p)$$

$$\varphi^{+}_{*}(q, xa, p) = \varphi^{+}_{*}(q, ax, p)$$

for all $p, q \in Q$, $a \in X$ and $x \in X^*$.

Proof: Let $p, q \in Q$, $a \in X$ and $x \in X^*$. Suppose $|x| = n$. If $n = 0$, then $x = \lambda$. Thus

$$\begin{aligned}\varphi^{-}_{*}(q, xa, p) &= \varphi^{-}_{*}(q, \lambda a, p) = \varphi^{-}_{*}(q, a, p)\\ &= \varphi^{-}_{*}(q, a\lambda, p) = \varphi^{-}_{*}(q, xa, p)\end{aligned}$$

and

$$\begin{aligned}\varphi^{+}_{*}(q, xa, p) &= \varphi^{+}_{*}(q, \lambda a, p) = \varphi^{+}_{*}(q, a, p)\\ &= \varphi^{+}_{*}(q, a\lambda, p) = \varphi^{+}_{*}(q, ax, p).\end{aligned}$$

Suppose the result is true for all $u \in X^*$ with $|u| = n - 1$, $n > 0$. Let $b \in X$ be such that $x = ub$. Then

$$\begin{aligned}\varphi^{-}_{*}(q, xa, p) &= \varphi^{-}_{*}(q, uba, p)\\ &= \inf_{r \in Q}[\varphi^{-}_{*}(q, u, r) \vee \varphi^{-}_{*}(r, ba, p)]\\ &= \inf_{r \in Q}[\varphi^{-}_{*}(q, u, r) \vee \varphi^{-}_{*}(r, ab, p)] = \varphi^{-}_{*}(q, uab, p)\\ &= \inf_{r \in Q}[\varphi^{-}_{*}(q, ua, r) \vee \varphi^{-}(r, b, p)]\\ &= \inf_{r \in Q}[\varphi^{-}_{*}(q, au, r) \vee \varphi^{-}(r, b, p)]\\ &= \varphi^{-}_{*}(q, aub, p) = \varphi^{-}_{*}(q, ax, p)\end{aligned}$$

and

$$\begin{aligned}\varphi^{+}_{*}(q, xa, p) &= \varphi^{+}_{*}(q, uba, p)\\ &= \sup_{r \in Q}[\varphi^{+}_{*}(q, u, r) \wedge \varphi^{+}_{*}(r, ba, p)]\\ &= \sup_{r \in Q}[\varphi^{+}_{*}(q, u, r) \wedge \varphi^{+}_{*}(r, ab, p)] = \varphi^{+}_{*}(q, uab, p)\\ &= \sup_{r \in Q}[\varphi^{+}_{*}(q, ua, r) \wedge \varphi^{+}(r, b, p)]\\ &= \sup_{r \in Q}[\varphi^{+}_{*}(q, au, r) \wedge \varphi^{+}(r, b, p)]\\ &= \varphi^{+}_{*}(q, aub, p) = \varphi^{+}_{*}(q, ax, p)\end{aligned}$$

This completes the proof. ■

Example 4.4: Let $\mathcal{M} = (Q, X, \varphi)$ is a bffsm, where $Q = \{p, q, r\}$, $X = \{a, b\}$ and let $\varphi_Q = (\varphi^{-}_{Q}, \varphi^{+}_{Q})$ is defined as follows.

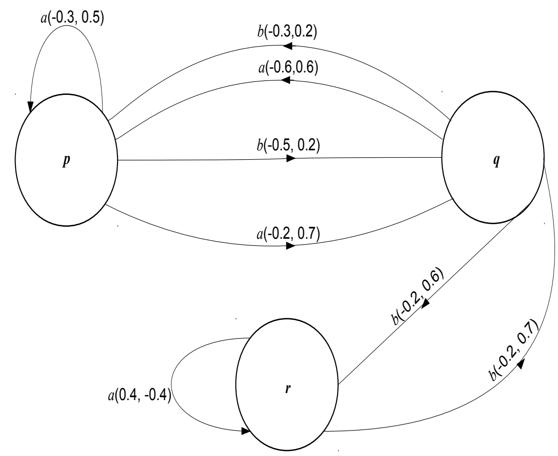

Fig. 2. $\mathcal{M}$

Routine computations show that $\mathcal{M}$ is a commutative, but $\mathcal{M}$ is not switching since $\varphi^{-}_{*}(q, a, p) \neq \varphi^{-}_{*}(p, a, q)$ and $\varphi^{+}_{*}(q, a, p) \neq \varphi^{+}_{*}(p, a, q)$.

Proposition 4.5: If $\mathcal{M} = (Q, X, \varphi)$ is an bffssm, then

$$\varphi_*^-(q, x, p) = \varphi_*^-(p, x, q)$$

$$\varphi_*^+(q, x, p) = \varphi_*^+(p, x, q)$$

for all $p, q \in Q$ and $x \in X^*$.

Proof: Let $p, q \in Q$ and $x \in X^*$. We prove the result by induction on $|x| = n$. Since $x = \lambda$ whenever $n = 0$, we have

$$\varphi_*^-(q, x, p) = \varphi_*^-(q, \lambda, p) = \varphi_*^-(p, \lambda, q) = \varphi_*^-(p, x, q)$$

and

$$\varphi_*^+(q, x, p) = \varphi_*^+(q, \lambda, p) = \varphi_*^+(p, \lambda, q) = \varphi_*^+(p, x, q).$$

Hence the result is true for $n = 0$. Assume that the result is valid for all $u \in X^*$ with $|u| = n - 1$; $n > 0$, that is,

$$\varphi_*^-(q, u, p) = \varphi_*^-(p, u, q)$$

$$\varphi_*^+(q, u, p) = \varphi_*^+(p, u, q).$$

Let $a \in X$ and $x \in X^*$ be such that $x = ua$. Then

$$\begin{aligned}\varphi_*^-(q, x, p) &= \varphi_*^-(q, ua, p) = \inf_{r \in Q}[\varphi_*^-(q, u, r) \vee \varphi^-(r, a, p)] \\ &= \inf_{r \in Q}[\varphi_*^-(r, u, q) \vee \varphi^-(p, a, r)] \\ &= \inf_{r \in Q}[\varphi_*^-(r, u, q) \vee \varphi_*^-(p, a, r)] \\ &= \inf_{r \in Q}[\varphi_*^-(p, a, r) \vee \varphi_*^-(r, u, q)] = \varphi_*^-(p, au, q) \\ &= \varphi_*^-(q, ua, p) = \varphi_*^-(p, x, q)\end{aligned}$$

and

$$\begin{aligned}\varphi_*^+(q, x, p) &= \varphi_*^+(q, ua, p) = \sup_{r \in Q}[\varphi_*^+(q, u, r) \wedge \varphi^+(r, a, p)] \\ &= \sup_{r \in Q}[\varphi_*^+(r, u, q) \wedge \varphi^+(p, a, r)] \\ &= \sup_{r \in Q}[\varphi_*^+(r, u, q) \wedge \varphi_*^+(p, a, r)] \\ &= \sup_{r \in Q}[\varphi_*^+(p, a, r) \wedge \varphi_*^+(r, u, q)] = \varphi_*^+(p, au, q)] \\ &= \varphi_*^+(q, ua, p) = \varphi_*^+(p, x, q)\end{aligned}$$

This shows that the result is true for $|u| = n$. This completes the proof. ■

Proposition 4.6: If $\mathcal{M} = (Q, X, \varphi)$ is a bffssm, then

$$\varphi_*^-(q, xy, p) = \varphi_*^-(q, yx, p)$$

$$\varphi_*^+(q, xy, p) = \varphi_*^+(q, yx, p)$$

for all $p, q \in Q$ and $x, y \in X^*$.

Proof: Let $p, q \in Q$ and $x, y \in X^*$. Assume that $|y| = n$. If $n = 0$, then $y = \lambda$ and so

$$\varphi_*^-(q, xy, p) = \varphi_*^-(q, x\lambda, p) = \varphi_*^-(q, \lambda x, p) = \varphi_*^-(q, yx, p)$$

and

$$\varphi_*^+(q, xy, p) = \varphi_*^+(q, x\lambda, p) = \varphi_*^+(q, \lambda x, p) = \varphi_*^+(q, yx, p).$$

Suppose that $\varphi_*^-(q, xu, p) = \varphi_*^-(q, ux, p)$ and $\varphi_*^+(q, xu, p) = \varphi_*^+(q, ux, p)$ for every $u \in X^*$ with $|u| = n$, $n > 0$. Let $y = ua$ where $a \in X$ and $u \in X^*$ with $|u| = n - 1$, $n > 0$. Then

$$\begin{aligned}\varphi_*^-(q, xy, p) &= \varphi_*^-(q, xua, p) \\ &= \inf_{r \in Q}[\varphi_*^-(q, xu, r) \vee \varphi^-(r, a, p)] \\ &= \inf_{r \in Q}[\varphi_*^-(q, ux, r) \vee \varphi^-(r, a, p)] \\ &= \inf_{r \in Q}[\varphi_*^-(r, ux, q) \vee \varphi^-(p, a, r)] \\ &= \inf_{r \in Q}[\varphi^-(p, a, r) \vee \varphi_*^-(r, ux, q)] = \varphi_*^-(p, aux, q) \\ &= \inf_{r \in Q}[\varphi_*^-(p, au, r) \vee \varphi^-(r, x, q)] \\ &= \inf_{r \in Q}[\varphi_*^-(p, ua, r) \vee \varphi^-(r, x, q)] \\ &= \varphi_*^-(p, uax, q) = \varphi_*^-(q, uax, p) = \varphi_*^-(q, yx, p)\end{aligned}$$

and

$$\begin{aligned}\varphi_*^+(q, xy, p) &= \varphi_*^+(q, xua, p) \\ &= \sup_{r \in Q}[\varphi_*^+(q, xu, r) \wedge \varphi^+(r, a, p)] \\ &= \sup_{r \in Q}[\varphi_*^+(q, ux, r) \wedge \varphi^+(r, a, p)] \\ &= \sup_{r \in Q}[\varphi_*^+(r, ux, q) \wedge \varphi^+(p, a, r)] \\ &= \sup_{r \in Q}[\varphi^+(p, a, r) \wedge \varphi_*^+(r, ux, q)] = \varphi_*^+(p, aux, q) \\ &= \sup_{r \in Q}[\varphi_*^+(p, au, r) \wedge \varphi^+(r, x, q)] \\ &= \sup_{r \in Q}[\varphi_*^+(p, ua, r) \wedge \varphi^+(r, x, q)] \\ &= \varphi_*^+(p, uax, q) = \varphi_*^+(q, uax, p) = \varphi_*^+(q, yx, p)\end{aligned}$$

This completes the proof. ■

Note that X^* is a semigroup with identity λ with respect to the binary operation concatenation of two words. Let $x, y \in X^*$. Define a relation $\sim$ on X^* by $x \sim y$ if and only if $\varphi_*^-(q, x, p) = \varphi_*^-(q, y, p)$ and $\varphi_*^+(q, x, p) = \varphi_*^+(q, y, p)$ for all $p, q \in Q$. Obviously $\sim$ is an equivalence relation on X^*. Let $x, y, z \in X^*$ be such that $x \sim y$, and let $p, q \in Q$. Then

$$\begin{aligned}\varphi_*^-(q, xz, p) &= \inf_{r \in Q}[\varphi_*^-(q, x, r) \vee \varphi_*^-(r, z, p)] \\ &= \inf_{r \in Q}[\varphi_*^-(q, y, r) \vee \varphi_*^-(r, z, p)] = \varphi_*^-(q, yx, p)\end{aligned}$$

$$\begin{aligned}\varphi_*^+(q, xz, p) &= \sup_{r \in Q}[\varphi_*^+(q, x, r) \wedge \varphi_*^+(r, z, p)] \\ &= \sup_{r \in Q}[\varphi_*^+(q, y, r) \wedge \varphi_*^+(r, z, p)] = \varphi_*^-(q, yx, p)\end{aligned}$$

Hence $xz \sim yz$. Similarly, $zx \sim zy$. Therefore $\sim$ is a congruence relation on the semigroup X^*. For any $x \in X^*$., we denote $[x] = \{y \in X^* \mid x \sim y\}$ and $S(\mathcal{M}) = \{[x] \mid x \in X^*\}$.

Define a binary operation $\odot$ on $S(\mathcal{M})$ by $[x] \odot [y] = [xy]$ for all $[x], [y] \in S(\mathcal{M})$. Obviously $\odot$ is well-defined and associative. For every $[x] \in S(\mathcal{M})$, we have

$$[x] \odot [y] = [x\lambda] = [x] = [\lambda x] = [\lambda] \odot [x].$$

This means that $[\lambda]$ is the identity of $(S(\mathcal{M}), \odot)$. Now let $x \in X^*$ and $x = x_1 x_2 \cdots x_n$ where $x_1, x_2, \cdots, x_n \in X$. For every $p, q \in Q$ we obtain

$$\varphi_*^-(q, x, p) = \inf_{r_1, r_2, \cdots, r_{n-1} \in Q}[\varphi^-(q, x_1, r_1) \vee \varphi^-(r_1, x_2, r_2) \vee \cdots \vee \varphi^-(r_{n-1}, x_n, p)]$$

$$\varphi_*^+(q, x, p) = \sup_{r_1, r_2, \cdots, r_{n-1} \in Q}[\varphi^+(q, x_1, r_1) \vee \varphi^+(r_1, x_2, r_2) \vee \cdots \vee \varphi^-(r_{n-1}, x_n, p)]$$

ISBN: 978-988-19251-6-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Since the image φ is finite, the image of φ_* is also finite. Hence we have the following theorem.

Theorem 4.7: Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Define a binary operation$\odot$ on $S(\mathcal{M})$ by $[x] \odot [y] = [xy]$ for all $[x], [y] \in S(\mathcal{M})$. Then $(S(\mathcal{M}), \odot)$ is a finite semigroup with identity.

V. Conclusion

In a nutshell, the bipolar fuzzy sets constitute a generalization of Zadeh's fuzzy set theory. Relatively, bipolar fuzzy sets have potential impacts on many fields including artificial intelligence, computer science, information science, cognitive science, decision science, management science, economics, neural science, quantum computing, medical science, and social science. The bipolar fuzzy models for fuzzy finite state machines give more precision, flexibility, informative, and compatibility to the system as compared to the classical and intuitionistic fuzzy models for finite state machines [14]. Subsequently, this paper has introduced the bipolar fuzzy finite switchboard state machines concept and investigated some of its related properties. Based on the results, more studies in bipolar fuzzy transformation semigroups and bipolar fuzzy topology associated with a bipolar fuzzy finite state machine are proposed as future direction. In addition, attempts would be made to locate an example of real life problem in respect to philosophical study.

Acknowledgment

This research work is supported by under the FRGS (Vot No: 0821), Ministry of Higher Education, Malaysia. The authors are highly grateful to Dr Y. B. Jun, Korea for his valuable comments and suggestions for improving the paper.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems **20** (1) (1989), 87–96.
[2] D. Dubois, and H. Prade, *Fuzzy sets and systems: Theory and Applications,* Academic Press, 1980.
[3] D. Dubois, S. Kaci, and H. Prade, *Bipolarity in reasoning and decision, an introduction, in: Int. Conf. on Inf. Pro. Man. Unc.* IPMU'04, (2004), 959–966.
[4] Y. B. Jun, *Intuitionistic fuzzy finite state machines,* J. Appl. Math. Comput. **17**(1–2) (2005), 109–120.
[5] Y. B. Jun, *Intuitionistic fuzzy finite switchboard state machines,* J. Appl. Math Comput.**20** (1–2) (2006), 315–325.
[6] Y. B. Jun and J. Kavikumar, *Bipolar fuzzy finite state machines,* Bull. Malays. Math. Sci. Soc. (2) **34** (1) (2011), 181–188.
[7] H. V. Kumbhojkar, and S. R. Chaudhari, *On covering of products of fuzzy finite state machines*, Fuzzy Sets and Systems **125** (2002), 215–222.
[8] K. M. Lee, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets,* J. Fuzzy Logic Intelligent Systems **14** (2004), 125–129.
[9] Y. Li and W. Pedrycz, *Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids,* Fuzzy Sets and Systems **156** (1), 68–92.
[10] D. S. Malik, J. N. Mordeson, and M. K. Sen, *Submachines of fuzzy finite state machines*, J. Fuzzy Math. **2** (1994), 781–792.
[11] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.
[12] W. R. Zhang, *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis*, Proceedings of IEEE Conf., (1994), 305–309.
[13] W. R. Zhang, *Bipolar fuzzy sets*, Proceedings of FUZZ-IEEE, (1998), 835–840.
[14] X.W. Zhang, and Y.M. Li, *Intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata*, Soft Computing, (13) **6** (2009), 611-616.
[15] H. J. Zimmermann, *Fuzzy set theory and its applications,* Kluwer-Nijhoff Publishing, 1985.