A Social Choice Function Proposal to Aggregate Preferences in a Group Decision Making Process

Tuncay GÜRBÜZ

Abstract— **In order to generate a single preference order for the society, a set of operations must be made on the preferences of the individuals forming that society. The aim of this paper is to propose a new Social Choice Function – SCF which will do this procedure. There are numerous functions that do this based on different intuitions or ideas. The function proposed in this paper has been developed with the intuitive approach that is based on the distances of the candidates to the ranks retrieved from the voters' ranking (as in the function of Cook & Seiford). The proposed function has a rather simple algorithm which, at each step, assigns a candidate to the closest possible rank while considering that distance has been covered for the next iteration. The function's process is designed in such way that the function will try to solve as much as possible the cases where there is indifference between candidates.**

***Index Terms*— Group Decision Making , Preference Voting, Social Choice Function, Voting,.**

I. Introduction

VOTING is a Multi-Criteria Decision Making – MCDM process made by the voter when he/she is about to choose a candidate among a set of candidates. It was observed that this method is quite unreliable when there are three or more candidates because the non-ranked voting system may result in the selection of the least popular candidate. That is why, a method of voting with which the voter would express his/her order of preference between the candidates is more suitable in case of three or more candidates. And once all the voters give their preference orderings for the candidates, it is SCFs' job to aggregate those preferences to come up with a final ranking that will represent the society's will in the most efficient possible way.

Each SCF has its own intuition/logic and therefore a mathematical base accordingly. Some consist of just counting the preferences whereas some rely on mathematical programming techniques such as Hungarian Algorithm. The importance here will not be how complicated the mathematical base, but the ability to give a possible ranking in order to represent the voters' will. Some of the well known SCFs that are used and still being used are:

--Condorcet's function, proposed by Marquis de Condorcet (1743 – 1794), measures the worst a candidate does against all other candidates and ranks them with the principle that the greater the function's value, the better the candidate.

--Borda's function, proposed by Borda (1733 – 1799), counts the number of individuals that preferred a candidate to each one of the other candidates. Again, the greater the function's value, the better the candidate.

--Copeland's function, proposed by Copeland in 1950, for a given candidate, measures the difference between the number of candidates that this given candidate has a strict simple majority over and the number of candidates that have strict simple majorities over that same candidate. Again, the greater the function's value, the better the candidate.

--Nanson's function, proposed by E.J. Nanson (1850 – 1936), is a Borda elimination procedure. Following the definition of Borda's function, it eliminates the candidate having the lowest Borda score, and then it calculates again the Borda scores for the remaining set of candidates until no more candidates can be deleted. The principle is that the later a candidate is eliminated, the better he/she is.

--Dodgson's function, proposed by C. L. Dodgson (1832 – 1898), measures the number of changes needed in voter's preference orders in order to create a simple majority winner for a given candidate. The principle is that the smaller the number, the better the candidate.

--Kemeny's function, proposed by J. G. Kemeny [1], finds the maximum amount of agreement between the consensus ranking and the voters' preference orderings.

--Cook & Seiford's function, proposed by W.D. Cook and L.M. Seiford in 1978 [2], with a definition of distance function as a measure of disagreement between rankings, uses the assignment algorithm in order to find the ranking that minimizes the total disagreement.

Some of the studies in this area in the last decades are: on superdictatorial domain characterization [3], on the equivalence of coalition strategy-proofness and Maskin monotonicity [4], on minimally manipulable anonymous SCFs [5], on superdictatorial domains for monotonic SCFs [6], on type two computability of SCFs [7], on fuzzy SCFs [8] and on a dictatorial domain for monotone SCFs [9].

In this study, the purpose is to propose a new SCF. Here,

Galatasaray University, Industrial Engineering Dept.
Çırağan cad. No.36 34357 Ortaköy, Istanbul, Turkey +902122274480
tuncaygurbuz09@gmail.com, tgurbuz@gsu.edu.tr
This research has been financially supported by Galatasaray University Research Fund.

it has to be noted that all SCFs are not obliged to give the same result. Each SCF has a way of computing the voters' preference orders at hand in order to give a final ranking of the candidates in order to aggregate and represent the voters' choice. The intuitive approach of the proposed function is to assign the candidate to a place/rank to which he/she has the minimum distance. As the iterations go, i.e. after each assignment, the function considers that this distance is covered and remaining candidates are that much closer to the remaining places/ranks.

The construction of the study will be as follows: In Section 2 some preliminary information about voting, preference voting system and SCFs will be given. The proposed function will be explained in detail in Section 3. A numerical example including the comparison with existing methods will be given in Section 4. Finally Section 5 will give the concluding remarks with the properties of the proposed SCF.

II. Preliminaries

A. Voting

Voting is a group decision making method in a democratic society which expresses the will of the majority [10]. It is also a Multi-Criteria Decision Making – MCDM process whenever a voter is about to select a candidate, as the candidates are judged according to their capabilities, honesty, trustworthiness, political stance etc. The voter summarizes those criteria in his/her mind in order to form a utility function then decides according to that function. Hence it can be briefly said that a democratic voting process is a group decision making method under multi criteria.

When the voter has only one vote whereas there are many candidates, this is called a non-ranked voting system. This method is perfectly satisfactory when there are only two candidates and the winner is simply the one who has the majority of the votes, simple majority. However this method is quite unreliable when there are three or more candidates. The reason will be explained in detail in the following section.

B. Preferential Voting System - PVS

The most naive approach to the elections is to say that the candidate who gets the most votes wins. But with a more detailed look to the method cited above, it has to be asked whether or not those systems represent the people's will.

The cases, given by Dodgson [11], in the following example, demonstrate the injustices which may occur with those systems. Consider eleven voters who will vote for four candidates, namely a, b, c, and d, by representing their preference order – PO as it is given with two different cases in Tables I and II.

If those were the case of a non-ranked voting, in Case 1 presented in Table 1, although the candidate *a* is considered the best by three voters and 2nd by all the rest, the candidate b, who is selected the best by four voters however the worst by all other seven, is selected because he/she has

TABLE I
Case 1

PO	Voters										
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
1	a	a	a	b	b	b	b	c	c	c	d
2	c	c	c	a	a	a	a	a	a	a	a
3	d	d	d	c	c	c	c	d	d	d	c
4	b	b	b	d	d	d	d	b	b	b	b

TABLE II
Case 2

PO	Voters										
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
1	b	b	b	b	b	b	a	a	a	a	a
2	a	a	a	a	a	a	c	c	c	d	d
3	c	c	c	d	d	d	d	d	d	c	c
4	d	d	d	c	c	c	b	b	b	b	b

the maximum number of votes. On the other hand, in Case 2 presented in Table 2, no candidates other than *a* and *b* are defined as the best one and *b* would win this voting by the absolute majority although the candidate *a* is considered the best by five voters and second by the rest of them and *b* is considered best by six and the worst by all others.

As previously seen, since the non-ranked voting system may result in the selection of the least popular candidate, a method of voting that allows the voter to indicate his/her PO for the candidates is needed. By doing that, the voter will not only define the best candidate but will also define the ranking of the candidates according to him/her as presented in Table1 and Table 2. This is called the Preferential Voting which was first proposed by Chevalier de Borda in a paper he wrote in 1770 but not published until 1784 for unknown reasons [10].

The voting procedure is simple in practice nonetheless after polling is completed, the problem is to aggregate the individual preferences in order to form a social choice.

C. Social Choice Function - SCF

A SCF is a mapping which assigns a nonempty subset of the potential feasible subset to each ordered pair consisting of a potential feasible subset of alternatives and a schedule of profile of voter's preferences [12].

In order to generate a single preference order for the society, a set of operations must be made on the preferences of the individuals forming that society. In general terms, the problem is to define "fair" methods for amalgamating individual choices to yield a social decision.

III. Proposed Method

A. Retrieving the Distances from Voter's Ranking

Let r_{ij} represent the rank given to the candidate *j* (where $j = 1,\ldots,m$) by the voter i (where $i = 1,\ldots,n$). In this case, the candidate's distance from the consensus ranking *k* (where $k = 1,\ldots,m$) will be:

$$d_{jk} = \sum_{i=1}^{n} \left| r_{ij} - k \right| \qquad \forall j = 1,...,m \qquad (1)$$

For each candidate j, d_{j1} and d_{jm} values will be computed and collected in a square (mxm) matrix called D.

B. *Retrieving the Distances from Voter's Ranking*

Once the PV is made by the voters, we compute the distances for candidates using formula (1) in order to form D. By definition, for the first iteration, $D_1=D$. After this process, the step by step procedure of the proposed function is as follows:

--Initialization Step: Start. Set n=1,

--Step 1: Find $\min_{j,k=1,...,m} d_{jk} = d_{rt}$ in D_n.

--Step 2: If d_{rt} is unique, then assign rth candidate to tth rank. Go to Step 3. If d_{rt} is not unique, then one of the three situations can occur:

a. $\min_{j,k=1,...,m} d_{jk} = d_{rt} = d_{pl}$ *i.e.* two or more identical value on different columns for different candidates. Then, assign rth candidate to tth rank, pth candidate to lth rank and so on for all other candidates and ranks having the identical value. Go to Step 3a.

b. $\min_{j,k=1,...,m} d_{jk} = d_{rt} = d_{rl}$ *i.e.* two or more identical value on different columns for the same candidate. Or in other words a candidate is eligible for more than one ranking. In that case, assign the candidate to higher (resp. lower) ranking if the distance for the 1st place (resp. mth place) is smaller than the distance to the mth place (resp. 1st place) because this shows that the society wants to see this candidate closer to the 1st place (resp. mth place). Go to Step 3b.

c. $\min_{j,k=1,...,m} d_{jk} = d_{rt} = d_{pt}$ *i.e.* two or more identical value on same columns. Or in other words, two or more candidates are eligible for the same rank. In that case, assignment will be arbitrary. From this point forward, the algorithm can be run for each arbitrary choice and once the final results are obtained, their agreement level can be measured (for instance with Kemeny's function) in order to choose the best ranking to represent the society's will. Go to Step 3c.

--Step 3: Erase rth row and tth column of D_n. Substract d_{rt} from each remaining element. Set n=n+1. Form D_n with the new values.

a. In case of Step 2a, erase the related rows and columns of D_n. Substract d_{rt} from each remaining element. Set n=n+r where r is the number of related candidates (two or more). Form D_n with the new values.

b. In case of Step 2b, in D_n, erase rth row and the column related to the ranking the candidate has been assigned. Substract d_{rt} from each remaining element. Set n=n+1. Form D_n with the new values.

c. In case of Step 2c, erase tth column and the row of the assigned candidate in Dn. Substract d_{rt} from each remaining element. Set n=n+1. Form D_n with the new values.

--Step 4: If dim(D_n) = 2 then execute the evident last two assignments. Finish. Otherwise, set m=m-1 (or m=m-r in case of 3b). Go to Step 1.

IV. Numerical Example

A. *Data*

Consider the following preference orders made by 60 voters for five candidates *a*, *b*, *c*, *d* and *e*:

23 votes : *a P b P e P d P c*
17 votes : *e P b P c P a P d*
2 votes : *b P e P a P c P d*
10 votes : *c P a P e P d P b*
8 votes : *c P b P e P a P d*

From this preference voting, for each candidate, using formula (1), the following results are computed:

TABLE III
DISTANCES FROM PREFERENTIAL RANKINGS

Cand.	RANK					DISTANCE to				
	1st	*2nd*	*3rd*	*4th*	*5th*	*1st Pl.*	*2nd Pl.*	*3rd Pl.*	*4th Pl.*	*5th Pl.*
a	23	10	2	25	0	89	75	81	91	151
b	2	48	0	0	10	88	32	72	112	152
c	18	0	17	2	23	132	108	84	94	108
d	0	0	0	33	27	207	147	87	27	33
e	17	2	41	0	0	84	58	36	96	156

The distance matrix, D, is placed on the right hand side of the table. Left hand side of the table shows how many voters put a candidate to a specific ranking.

B. *Iterations*

Start, n=1

Iteration 1:

Step 1:

$$D_1 = D = \begin{matrix} & 1st & 2nd & 3rd & 4th & 5th \\ a \\ b \\ c \\ d \\ e \end{matrix} \begin{bmatrix} 89 & 75 & 81 & 91 & 151 \\ 88 & 32 & 72 & 112 & 152 \\ 132 & 108 & 84 & 94 & 108 \\ 207 & 147 & 87 & 27 & 33 \\ 84 & 58 & 36 & 96 & 156 \end{bmatrix}$$

$$\min_{j,k=1,...,m} d_{jk} = d_{44} = 27$$

Step 2: Candidate d is ranked 4^{th}.

Step 3: n=2

$$D_2 = \begin{matrix} a \\ b \\ c \\ e \end{matrix} \overset{\begin{matrix} 1st & 2nd & 3rd & 5th \end{matrix}}{\begin{bmatrix} 62 & 48 & 54 & 124 \\ 61 & 5 & 45 & 125 \\ 105 & 81 & 57 & 81 \\ 57 & 31 & 9 & 129 \end{bmatrix}}$$

Step 4: $dim(D_n)$ = 4. m=4. Go to Step 1 for Iteration 2.

Iteration 2:

Step 1: $\min_{j,k=1,...,m} d_{jk} = d_{22} = 5$

Step 2: Candidate b is ranked 2^{nd}.

Step 3: n=3

$$D_3 = \begin{matrix} & 1st & 3rd & 5th \\ a & 57 & 49 & 119 \\ c & 100 & 52 & 76 \\ e & 52 & 4 & 124 \end{matrix}$$

Step 4: $dim(D_n) = 3$. $m=3$. Go to Step 1 for Iteration 3.

Iteration 3:

Step 1: $\min_{j,k=1,\ldots,m} d_{jk} = d_{32} = 4$

Step 2: Candidate e is ranked 3^{rd}.

Step 3: n=4

$$D_4 = \begin{matrix} & 1st & 5th \\ a & 53 & 115 \\ c & 96 & 72 \end{matrix}$$

Step 4: $dim(D_n) = 2$. Candidate a is ranked 1^{st} having the minimum distance value and finally candidate c is ranked 5^{th}. Finish.

Hence the final ranking will be as follows:

$$a \succ b \succ e \succ d \succ c$$

V. Concluding Remarks

In this study, a new SCF is proposed and its functioning is explained using a numerical case. To conclude the study, we can check the properties of "*decisiveness, neutrality, anonymity, monotonicity, unanimity, homogeneity*" for the proposed function. These terminology was introduced and discussed by May [13] and [14], Fishburn [15] and [16], Sen [17], Pattanaik [18] and [19], Smith [20] and others. It can be seen that the proposed function is:

--Decisive: The function is such that the voters' preferences lead to a defined and unique decision. With the given definitions, it is not possible to yield to two different consensus ranking.

--Neutral: The final decision will be reversed if every voter reverses his/her vote. If the rankings are reversed, then, values of the elements in matrix D will be reversed, *i.e.* the columns 1,…,m will be the columns m,…,1 respectively. Therefore, the final decision will be reversed.

--Anonymous: Every voter has the same importance. None of them has more power than the other when it comes to aggregate the individual preferences into a social choice.

--Monotone: If a voter changes his/her decision and moves a candidate x upward leaving all the other candidates same as before, in the final ranking, x will stand at least as well relative to each candidate as before. This modification will decrease the distance of that candidate to top ranks and also increase his/her distance to lower ranks. Hence the candidate will at least keep his/her previous place or be in a better place.

--Unanimous: A candidate x will win if every voter prefers x to y. It is obvious that in such case, distance of x for a better rank will be smaller than that of y. Hence the result $x \succ y$.

--Homogeneous: A voter indifferent among several voters can be replaced by several fractional voters holding symmetric views on them. The indifference between two candidates means that their distances to any rank are equal. On the other hand, two voters with symmetric views on those candidates will yield to two rankings with exchanged distance values from one ranking to another between these two candidates; which will yield to the same distances as before.

The consensus rankings found using other popular SCFs are as follows:

--Condorcet's	:	$a \succ b = e \succ c \succ d$
--Borda's	:	$e \succ b \succ a \succ c \succ d$
--Copeland's	:	$a = b \succ c = e \succ d$
--Nanson's	:	$a \succ b \succ e \succ c \succ d$
--Dodgson's	:	$e = b \succ a \succ c \succ d$
--Kemeny's	:	$a \succ b \succ e \succ c \succ d$
--Cook & Seiford's	:	$a \succ b \succ e \succ c \succ d$

For the example considered in this study, the consensus ranking found with Nanson's function gave the same result as the ranking found with Kemeny's function giving the maximum agreement and Cook&Seiford's functions giving the minimum disagreement. On the other hand, Condorcet's, Borda's, Copeland's and Dodgson's functions give different results including indifferences between some candidates.

The function proposed in this study gave the same result with Nanson's, Kemeny's and Cook&Seiford's with only one difference: rankings of two candidates ranked as the last ones have interchanged with the proposed function. Here it has to be noted again that there is no such necessity like every SCF has to give the same result. The importance is the ability to give a possible ranking which is appropriate to the voters' rankings. Therefore, the proposed function is useful in order to get a final ranking in order to represent a group of voters' will. On the other hand, the function has a relatively simple algorithm to run, it gives a usable solution and its process is designed in a way that the function will try to solve the cases of indifference between two candidates. Hence, the functionality is also remarkable.

For further research, simulation runs for different number of voters and different number of candidates can be executed in order to find the similarities between the results found with the proposed SCF and existing SCFs.

References

[1] Kemeney, J.G. (1959). Mathematics Without Numbers, Dadalus, Vol. 88, 577 – 591.

[2] Cook, W.D., Seiford, L.M., "Priority Ranking and Consensus Formation", Management Science, Vol.24, No. 16, pp. 1721 – 1732, 1978.

[3] Sanver, M.R., "A Characterization of Superdictatorial Domains for Strategy-proof Social Choice Funtions", Mathematical Social Sciences, 54, 257 – 260, 2007.

[4] Takamiya, K., "Domains of Social Choice Functions on Which Coalition Strategy-proofness and Maskin Monotonicity Are Euivalent", Economics Letters, 95, 348 – 354,2007.

[5] Maus, S., Peters, H., Storcken, T., "Minimally Manipulable Anonymous Social Choice Functions", Mathematical Social Sciences, 53, 239 – 254, 2007.

[6] Kutlu, L., "Superdictatorial Domains for Monotonic Social Choice Functions", Economics Letters, 97, 151 – 154, 2008.

[7] Tanaka, Y., "Type Two Computability of Social Choice Functions and the Gibbard-Satterthwaite Theorem in an Infinite Society", Applied Mathematics and Computation, 192, 168 – 174, 2007.

[8] Abdelaziz, F.B., Figueira, J.R., Meddeb, O., "On the Manipulability of the Fuzzy Social Choice Functions", Fuzzy Sets and Systems, Vol.159, 177 – 184, 2008.

[9] Kutlu, L., "A Dictatorial Domain for Monotone Social Choice Functions", Economics Letters, Vol. 105, 14 – 16, 2009.

[10] Hwang, C.L., Lin, M.J., "Group Decision Making Under Multiple Criteria – Methods and Applications", Springer-Verlag, Berlin Heidelberg, 1987.
[11] Dodgson, C. L., "A Discussion of the Various Methods of Procedure in Conducting Elections (1873), Suggestions as to the Best Method of Taking Votes, Where More Than Two Issues are to be Voted on (1874), A Method of Taking Votes on More Than Two Issues (1876), (See Black, D., (1958), The Theory of Committees and Elections", pp. 189 – 238.
[12] Fishburn, P. C., "Social Choice Functions", SIAM Review, 16(1), 63 – 90, 1974.
[13] May, K.O., "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision", Econometrica, Vol.20, No.4, pp.680 – 684, 1952.
[14] May, K.O., "A Note on the Complete Independence of the Conditions for Simple Majority Decision, Econometrica, Vol.21, pp.172 – 173, 1953.
[15] Fishburn, P. C., "The Theory of Social Choice", Princeton University Press, Princeton, New Jersey, 1973.
[16] Fishburn, P. C., "Condorcet Social Choice Functions", SIAM, Journal of Applied Mathematics, Vol.33, No.3, pp.469 – 489, 1977.
[17] Sen, A.K., "Collective Choice and Social Welfare", Holden-Day Inc., San Fransisco, 1970.
[18] Pattanaik, P.K., "On Social Choice with Quasitransitive Individual Preferences", Journal of Economic Theory, Vol.2, pp. 267 – 275, 1970.
[19] Pattanaik, P.K., "Strategy and Group Choice", North-Holland Publishing Company, New York, 1978.
[20] Smith, J.H., "Aggregation of preferences with variable electorate", Econometrica, Vol.41, No.6, pp. 1027 – 1041, 1973.

ISBN: 978-988-19253-1-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)