A New and Simple Method of Solving Large Linear Systems: Based on Cramer's Rule but Employing Dodgson's Condensation

Okoh Ufuoma

***Abstract*—The object of this paper is to introduce a new and fascinating method of solving large linear systems, based on Cramer's rule but employing Dodgson's condensation in its computations. This new method is very brief, straightforward, simple to understand, and unknown to teachers and students of mathematics, science, and engineering.**

***Index Terms*—matrix, determinant, linear systems, Cramer's rule, Dodgson's condensation**

I. INTRODUCTION

In the year 1866, Rev. Charles Lutwidge Dodgson (1832–1898), a British mathematician, most famous as Lewis Carroll for writing his nursery tale, *Alice's Adventures in Wonderland* (1865), and its sequel, *Through the Looking–glass* (1872) [3], [13], sent, after discovering a beautiful technique of evaluating large determinants by repeatedly reducing them to lower orders, a paper intriguingly entitled *On the Condensation of Determinants, being a new and brief Method for computing their arithmetical values* to the Royal Society of London, and the paper was published in the *Proceedings* of that erudite body [6], [1]. The condensation method is there employed to evaluate 4th and 5th order determinants, and in concluding his paper, Dodgson demonstrated how the method can be employed in finding the solutions of large linear systems of equations by giving examples for linear systems of 3 and 5 equations [6], for before his time evaluating large determinants and solving large linear systems were riddles to mathematicians.

The main aim of this paper is to introduce an alternative method, new and fascinating, to Dodgson's approach to solving large linear systems using his condensation method.

The remainder of this paper is in two sections. Section II gives, for the reader's convenience, a brief review of Dodgson's condensation. Section III deals with the alternative form of Dodgson's approach to solving linear systems of equations with large number of unknowns, based on Cramer's rule but employing Dodgson's condensation.

II. A BRIEF REVIEW OF DODGSON'S CONDENSATION

C.L. Dodgson deserves to be highly esteemed in the world of linear algebra for introducing the condensation method, an ingenious and remarkable method which at present many are revisiting. Dodgson's condensation of determinants consists of the following steps or rules [6], [13]:

1. Employ the elementary row and column operations to rearrange, if necessary, the given nth order determinant such that there are no zeros in its interior. The interior of a determinant is the minor formed after the first and last rows and columns of the determinant have been deleted.
2. Evaluate every 2nd order determinant formed by four adjacent elements. The values of the determinants form the $(n-1)$st order determinant.
3. Condense the $(n-1)$st order determinant in the same manner, dividing each entry by the corresponding element in the interior of the nth order determinant.
4. Repeat the condensation process until a single number is obtained. This number is the value of the nth order determinant.

To make the method clear, we consider an example. We want to condense the 4th order determinant

$$\begin{vmatrix} 2 & 1 & 3 & 5 \\ 4 & -2 & 7 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \end{vmatrix}$$

to a single number using Dodgson's condensation technique. We begin with

$$\begin{vmatrix} 2 & 1 & 3 & 5 \\ 4 & -2 & 7 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \end{vmatrix}.$$

By rule 2 this is condensed into

$$\begin{vmatrix} \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 4 & -2 \\ -8 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 7 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 7 & 6 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -8 & 3 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -6 \end{vmatrix} \end{vmatrix}$$

which, when evaluated, gives

$$\begin{vmatrix} -8 & 13 & -17 \\ -4 & -23 & -6 \\ -71 & -1 & -6 \end{vmatrix}.$$

Manuscript received July 09, 2013; revised July 31, 2013. Okoh Ufuoma is with the Department of Mathematics, Sonnicity Universal Schools, Ughelli, Delta State, Nigeria (tel: 08134813234; e-mail: okohufuoma@yahoo.com)

ISBN: 978-988-19252-3-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

This in turn, by rule 3, is condensed into

$$\begin{vmatrix} \begin{vmatrix} -8 & 13 \\ -4 & -23 \end{vmatrix} & \begin{vmatrix} 13 & -17 \\ -23 & -6 \end{vmatrix} \\ \begin{vmatrix} -4 & -23 \\ -71 & -1 \end{vmatrix} & \begin{vmatrix} -23 & -6 \\ -1 & -6 \end{vmatrix} \end{vmatrix}$$

which, being evaluated, furnishes

$$\begin{vmatrix} 236 & -469 \\ -1629 & 132 \end{vmatrix}.$$

We divide each element of the above 2×2 determinant by the corresponding element of the interior of the 4th order determinant,

$$\begin{bmatrix} -2 & 7 \\ 3 & 1 \end{bmatrix},$$

and have

$$\begin{vmatrix} \frac{236}{-2} & \frac{-469}{7} \\ \frac{-1629}{3} & \frac{132}{1} \end{vmatrix}$$

which gives

$$\begin{vmatrix} -118 & -67 \\ -543 & 132 \end{vmatrix}.$$

This, by rule 4, gives the value of -51957. Dividing this value by the interior $[-23]$ of the 3rd order determinant, we get 2259 which is the value of our original 4th order determinant.

Though Dodgson's condensation method is interesting and excellently suited to hand-computations [5] since it involves the evaluation of only 2nd order determinants, it has a great obstacle: the process cannot be continued when zeros (which Dodgson called *ciphers* in his paper [6]) occur in the interior of any one of the derived determinants, "since infinite values would be introduced by employing them as divisors"[6]. A solution to this problem, as Dodgson suggests, is to rearrange the original determinant and recommence the operation [6], [13].

Suppose now we want to find the value of the determinant

$$\begin{vmatrix} 2 & 1 & 3 & 5 \\ 4 & 6 & 2 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \end{vmatrix}.$$

We compute as follows:

$$\begin{vmatrix} 2 & 1 & 3 & 5 \\ 4 & 6 & 2 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \end{vmatrix}$$

$$\begin{vmatrix} \begin{vmatrix} 2 & 1 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 4 & 6 \\ -8 & 3 \end{vmatrix} & \begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -8 & 3 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -6 \end{vmatrix} \end{vmatrix}$$

$$\begin{vmatrix} 8 & -16 & 8 \\ 60 & 0 & -6 \\ -71 & -1 & -6 \end{vmatrix}.$$

We cannot continue the operation because of the zero which occurs in the interior of the derived 3th order determinant. Division by the zero will result in an infinite value. So we rearrange the original 4th order determinant by moving the top row to the bottom and moving all the other rows up once. Thus we have

$$\begin{vmatrix} 4 & 6 & 2 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \\ 2 & 1 & 3 & 5 \end{vmatrix}.$$

We recommence the operation:

$$\begin{vmatrix} 4 & 6 & 2 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \\ 2 & 1 & 3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 60 & 0 & -6 \\ -71 & -1 & -6 \\ -9 & 19 & 28 \end{vmatrix}$$

$$\begin{vmatrix} -60 & -6 \\ -1358 & 86 \end{vmatrix}.$$

We divide each element of the above 2×2 determinant by the corresponding element of the interior of the 4th order determinant,

$$\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix},$$

and have

$$\begin{vmatrix} \frac{-60}{3} & \frac{-6}{1} \\ \frac{-1358}{7} & \frac{86}{2} \end{vmatrix}$$

$$\begin{vmatrix} -20 & -6 \\ -194 & 43 \end{vmatrix}$$

which, when evaluated, gives the value of -2024. Dividing this value by the interior $[-1]$ of the 3rd order determinant, we get 2024 which is the value of the original 4th order determinant.

Dodgson, commenting on the problem of ciphers and comparing his method, despite the problem, with the famous Laplace method, writes [6]:

> The fact that, whenever ciphers occur in the interior of a derived block (matrix), it is necessary to recommence the operation, may be thought a great obstacle to the use of this method; but I believe it will be found in practice that, even though this should occur several times in the course of one operation, the whole amount of labour will still be much less than that involved in the old process of computation .

ISBN: 978-988-19252-3-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The process of recommencing the operation might be a reason why Dodgson's condensation has not obtained great popularity since it was invented.

In order to popularize Dodgson's condensation method some mathematicians recently revisited it in their papers, notably Adrian Rice and Eve Torrence [13], David Bressoud [5], and Francine Abeles [1]. David Bressoud says that condensation is "useful and deserves to be better known, especially since it is so well suited to parallel computation "[5]. Rice and Torrence, teachers of linear algebra, find Dodgson's method to be the most popular method among their students for evaluating large determinants. In [13] they write:

> But there is another method (Dodgson's condensation), first introduced in 1866 and widely ignored since, which can simplify the work involved in calculating determinants of large matrices considerably, and which, we believe, can still be of interest to today's students.

III. A NEW APPROACH TO SOLVING LINEAR SYSTEMS

The curiosity about Dodgson's condensation is not for nothing. In his paper of 1866, C.L. Dodgson showed how his condensation can be used to hand-solve large linear systems of equations and gave two examples to clarify his approach which, though effective and gives accurate solution set, is a little bit lengthy and not so easy to employ. (The reader who is interested in Dodgson's approach should see [6].) Because Dodgson's approach to solving linear systems may be quite lengthy, in this section we will introduce an alternative form of his approach which is based on Cramer's rule but employs Dodgson's condensation in its computations.

We begin with the general system of n simultaneous linear equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

where x_1, x_2, x_3,..., x_n are the unknowns, a_{ij} are the coefficients of the system, and b_1, b_2, b_3, ..., b_n are the constant terms. A solution of the system is a set of values of the unknowns that satisfies every equation of the system simultaneously.

A compact way of solving a linear system is by expressing it as a matrix, a rectangular array of numbers arranged in rows and columns and enclosed in brackets [4], [8]. The system (1) of linear equations can be written compactly in matrix form as $\mathbf{Dx} = \mathbf{b}$ where

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

is an $n \times n$ coefficient matrix of the coefficients a_{ij} , and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{3}$$

are $n \times 1$ column matrices of the unknowns x_i and the constants b_i respectively. Thus the system (1) can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{4}$$

The equation (4) is known as the nth order matrix equation.

A term closely related to matrix is determinant. The determinant of the $n \times n$ matrix $\mathbf{D}$ (2), called an nth order determinant, is denoted as $\left|\mathbf{D}\right|$ and written as

$$\left|\mathbf{D}\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{5}$$

where a_{ij} is any given number in row i and column j.

If we expand the determinant (5) by Laplace method, we obtain a single number which determines whether or not the matrix $\mathbf{D}$ has an inverse or the system (1) is solvable. If $\left|\mathbf{D}\right| = 0$, there is linear dependence among the equations of the system and no unique solution is possible. If $\left|\mathbf{D}\right| \neq 0$, there is no linear dependence among the equations of the system and a unique solution can be found or the system is solvable [7], [11], [13].

A classical solution formula which efficiently gives solutions of linear systems and is in common use, particularly among authors and students of science and engineering, is the famous Cramer's rule, named after its inventor, Gabriel Cramer (1704–1752), a Swiss mathematician, born in Geneva. Cramer described his rule for an arbitrary number of unknowns in an appendix in his very influential book *Introduction to the analysis of algebraic curves*, published in 1750 [2]. Cramer's rule states that the solution of the linear system (1) is

$$x_k = \frac{|\mathbf{D}_k|}{|\mathbf{D}|}$$

where x_k is the kth unknown, and $|\mathbf{D}_k|$ is the determinant of a matrix formed from the coefficient matrix by replacing the column of coefficients of x_k, i.e. the kth column of $\mathbf{D}$, with the column matrix $\mathbf{b}$ of constants b_1, b_2, b_3, ..., b_n. Thus, for the system (1), the solutions are

$$x_1 = \frac{|\mathbf{D}_1|}{|\mathbf{D}|},\ x_2 = \frac{|\mathbf{D}_2|}{|\mathbf{D}|},\ x_3 = \frac{|\mathbf{D}_3|}{|\mathbf{D}|},\ ,\ldots,\ x_n = \frac{|\mathbf{D}_n|}{|\mathbf{D}|}.$$

Suppose we want to solve, by Cramer's rule, the following linear system of equations:

$$\begin{aligned} x_1 - 4x_2 - x_3 &= 11 \\ 2x_1 - 5x_2 - 2x_3 &= 39 \\ -3x_1 - 2x_2 + x_3 &= 1. \end{aligned}$$

We begin with the coefficient matrix

$$\mathbf{D} = \begin{bmatrix} 1 & -4 & -1 \\ 2 & -5 & -2 \\ -3 & -2 & 1 \end{bmatrix}$$

whose determinant is

$$|\mathbf{D}| = \begin{vmatrix} 1 & -4 & -1 \\ 2 & -5 & 2 \\ -3 & 2 & 1 \end{vmatrix} = 34.$$

We obtain $|\mathbf{D}_1|$, $|\mathbf{D}_2|$, and $|\mathbf{D}_3|$ by replacing respectively the first, second and third columns of $|\mathbf{D}|$ by the constant terms. So we get

$$|\mathbf{D}_1| = \begin{vmatrix} 11 & -4 & -1 \\ 39 & -5 & 2 \\ 1 & 2 & 1 \end{vmatrix} = -34,$$

$$|\mathbf{D}_2| = \begin{vmatrix} 1 & 11 & -1 \\ 2 & 39 & 2 \\ -3 & 1 & 1 \end{vmatrix} = -170,$$

$$|\mathbf{D}_3| = \begin{vmatrix} 1 & -4 & 11 \\ 2 & -5 & 39 \\ -3 & 2 & 1 \end{vmatrix} = 272.$$

Thus the solutions are

$$\begin{aligned} x_1 &= \frac{|\mathbf{D}_1|}{|\mathbf{D}|} = \frac{-34}{34} = -1, \\ x_2 &= \frac{|\mathbf{D}_2|}{|\mathbf{D}|} = \frac{-170}{34} = -5, \\ x_3 &= \frac{|\mathbf{D}_3|}{|\mathbf{D}|} = \frac{272}{34} = 8. \end{aligned}$$

It is often stated that Cramer's rule which gives solutions of linear systems as quotients of determinants is generally impractical [9], [11], quickly getting long and tedious as the number of the unknowns of the system increases. This is so because as the number of the unknowns increases, the number of determinants involved and their orders increase in equal proportion, causing one to give up hope of ever solving such a system. One way of curtailing the amount of labour, time and computation to a reasonable level when using Cramer's rule is to adopt a brief method of computing large determinants [9] such as Dodgson's condensation which we have already discussed in Section II. Here we shall never refer to this approach, but we shall discuss a new, simpler and better one which is derived from Cramer's rule but employs Dodgson's condensation in its calculations.

Now this new approach is exhibited in the following steps:

1. Form the $n \times 2n$ matrix:

$$\mathbf{S}_1 = \begin{bmatrix} \mathbf{D} & \mathbf{b} & \mathbf{D}' \end{bmatrix}$$

where $\mathbf{D}'$ is the array of numbers left when the last column of $\mathbf{D}$ is deleted.

2. Use Dodgson's condensation to condense $\mathbf{S}_1$ to $\mathbf{S}_2$, $\mathbf{S}_2$ to $\mathbf{S}_3$, and so on until the following row matrix is obtained:

$$\mathbf{S}_n = \begin{bmatrix} D & D_1 & D_2 & D_3 & \ldots & D_n \end{bmatrix}.$$

The values $D, D_1, D_2, D_3, \ldots, D_n$ are the elements of $\mathbf{S}_n$.

If n is even, the values of the unknowns or the solutions are

$$x_1 = -\frac{D_1}{D}, \quad x_2 = -\frac{D_2}{D}, \quad \ldots, \quad x_n = -\frac{D_n}{D}.$$

If n is odd, the values of the unknowns or the solutions are

$$x_1 = \frac{D_1}{D}, \quad x_2 = -\frac{D_2}{D}, \quad \ldots, \quad x_n = \frac{D_n}{D}.$$

To understand this new method it is wise to begin with the simplest case, the system of two equations:

$$\begin{aligned} 3x_1 - 4x_2 &= 2 \\ 2x_1 - 5x_2 &= -1. \end{aligned}$$

We begin with

$$\mathbf{S}_1 = \begin{bmatrix} 3 & -4 & 2 & 3 \\ 2 & -5 & -1 & 2 \end{bmatrix}.$$

We apply rule 2 of Dodgson's condensation and get the following:

$$\mathbf{S}_2 = \begin{bmatrix} -7 & 14 & 7 \end{bmatrix}.$$

The values of the unknowns are thus

$$\begin{aligned} x_1 &= -\frac{14}{-7} = 2, \\ x_2 &= -\frac{7}{-7} = 1. \end{aligned}$$

We now solve by the new approach the system of three equations which we gave as an instance of Cramer's rule :

$$\begin{aligned} x_1 - 4x_2 - x_3 &= 11 \\ 2x_1 - 5x_2 + 2x_3 &= 39 \\ -3x_1 + 2x_2 + 2x_3 &= 1. \end{aligned}$$

We start with

$$\mathbf{S}_1 = \begin{bmatrix} 1 & -4 & -1 & 11 & 1 & -4 \\ 2 & -5 & 2 & 39 & 2 & -5 \\ -3 & 2 & 1 & 1 & -3 & 2 \end{bmatrix},$$

apply rule 2 of Dodgson's condensation and get

$$\mathbf{S}_2 = \begin{bmatrix} 3 & -13 & -61 & -17 & 3 \\ -11 & -9 & -37 & -119 & -11 \end{bmatrix}.$$

ISBN: 978-988-19252-3-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Next, we condense $\mathbf{S}_2$ in a similar fashion, but this time, we famously divide each element (number) of the resulting matrix by the corresponding element of the interior matrix of $\mathbf{S}_1$ to get $\mathbf{S}_3$. Thus we have

$$\begin{aligned}\mathbf{S}_3 &= \begin{bmatrix} \frac{-170}{-5} & \frac{-68}{2} & \frac{6630}{39} & \frac{544}{2} \end{bmatrix} \\ &= \begin{bmatrix} 34 & -34 & 170 & 272 \end{bmatrix}.\end{aligned}$$

The values of the unknowns are thus

$$\begin{aligned}x_1 &= \frac{-34}{34} = -1, \\ x_2 &= -\frac{170}{34} = -5, \\ x_3 &= \frac{272}{34} = 8.\end{aligned}$$

In the above two instances of the new approach, we see that, among the popular methods of solving linear systems of 2 and 3 equations, such as Cramer's rule and Gaussian elimination, none is simpler or more fascinating than the alternative form of Dodgson's condensation. For this reason, this new and superior approach deserves serious consideration and merits the special attention of a wider audience as linear systems of 2 and 3 equations are the most commonly discussed in texts and encountered by teachers and students of mathematics, science, and engineering; for the reader of this paper will need little convincing that linear systems of 2 and 3 equations appear with great frequency in these disciplines and their applications.

As another instance of the approach, let us solve the system of four linear equations:

$$\begin{aligned}2x_1 + x_2 + 2x_3 + x_4 &= 6, \\ x_1 - x_2 + x_3 + 2x_4 &= 6, \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1, \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10.\end{aligned}$$

We begin with

$$\mathbf{S}_1 = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 & 2 & 1 & 2 \\ 1 & -1 & 1 & 2 & 6 & 1 & -1 & 1 \\ 4 & 3 & 3 & -3 & -1 & 4 & 3 & 3 \\ 2 & 2 & -1 & 1 & 10 & 2 & 2 & -1 \end{bmatrix},$$

apply rule 2 of Dodgson's condensation and get

$$\mathbf{S}_2 = \begin{bmatrix} -3 & 3 & 3 & -6 & -6 & -3 & 3 \\ 7 & -6 & -9 & 16 & 25 & 7 & -6 \\ 2 & -9 & 0 & -29 & -42 & 2 & -9 \end{bmatrix}.$$

Again we condense $\mathbf{S}_2$ and divide each element (number) of the resulting matrix by the corresponding element of the interior matrix of $\mathbf{S}_1$ to obtain $\mathbf{S}_3$:

$$\mathbf{S}_3 = \begin{bmatrix} \frac{-3}{-1} & \frac{-9}{1} & \frac{-6}{2} & \frac{-54}{6} & \frac{33}{1} & \frac{-3}{-1} \\ \frac{-5}{3} & \frac{-81}{3} & \frac{261}{-3} & \frac{53}{-1} & \frac{344}{4} & \frac{-51}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 & -3 & -9 & 33 & 3 \\ -17 & -27 & -87 & -53 & 86 & -17 \end{bmatrix}.$$

Finally, we condense $\mathbf{S}_3$ and divide each element (number) of the resulting matrix by the corresponding element of the interior matrix of $\mathbf{S}_2$ to obtain $\mathbf{S}_4$:

$$\begin{aligned}\mathbf{S}_4 &= \begin{bmatrix} \frac{-234}{-6} & \frac{702}{-9} & \frac{-624}{16} & \frac{975}{25} & \frac{-819}{7} \end{bmatrix} \\ &= \begin{bmatrix} 39 & -78 & -39 & 39 & -117 \end{bmatrix}.\end{aligned}$$

Thus the values of the unknowns are

$$\begin{aligned}x_1 &= -\frac{-78}{39} = 2, \\ x_2 &= -\frac{-39}{39} = 1, \\ x_3 &= -\frac{39}{39} = -1, \\ x_4 &= -\frac{-117}{39} = 3.\end{aligned}$$

I now proceed to give a proof of the validity of this new approach. We begin with the $n \times 2n$ matrix:

$$\begin{aligned}\mathbf{S}_1 &= \begin{bmatrix} \mathbf{D} & \mathbf{b} & \mathbf{D}' \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 & a_{11} & \dots & a_{1(n-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & b_n & a_{n1} & \dots & a_{n(n-1)} \end{bmatrix}\end{aligned}$$

where $\mathbf{D}'$ is the matrix formed by deleting the last column of the coefficient matrix $\mathbf{D}$ of system (1) and $\mathbf{b}$ is the column matrix containing the constants of the system. We employ Dodgson's condensation to condense $\mathbf{S}_1$ until we finally arrive at a row matrix $\mathbf{S}_n$ consisting of $n+1$ elements. Let us denote these elements by D, D_1, D_2, $D_3, \dots, D_n$, so that

$$\begin{aligned}D &= \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \\ &= |\mathbf{D}| \\ D_1 &= \begin{vmatrix} a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n2} & \dots & a_{nn} & b_n \end{vmatrix} \\ &= (-1)^{n-1} \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix} \\ &= (-1)^{n-1} |\mathbf{D}_1| \\ D_2 &= \begin{vmatrix} a_{13} & \dots & a_{1n} & b_1 & a_{11} \\ \vdots & & \vdots & \vdots & \vdots \\ a_{n3} & \dots & a_{nn} & b_n & a_{n1} \end{vmatrix} \\ &= - \begin{vmatrix} a_{11} & b_1 & a_{13} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & b_n & a_{n3} & \dots & a_{nn} \end{vmatrix} \\ &= -|\mathbf{D}_2|\end{aligned} \tag{6}$$

ISBN: 978-988-19252-3-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\begin{aligned} D_3 &= \begin{vmatrix} a_{14} & \dots & a_{1n} & b_1 & a_{11} & a_{12} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{n4} & \dots & a_{nn} & b_n & a_{n1} & a_{n2} \end{vmatrix} \\ &= (-1)^{n-1} \begin{vmatrix} a_{11} & a_{12} & b_1 & a_{14} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & b_n & a_{n4} & \dots & a_{nn} \end{vmatrix} \\ &= (-1)^{n-1} |\mathbf{D}_3| \\ &\vdots \\ D_n &= \begin{vmatrix} b_1 & a_{11} & \dots & a_{1(n-1)} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n1} & \dots & a_{n(n-1)} \end{vmatrix} \\ &= (-1)^{n-1} \begin{vmatrix} a_{11} & \dots & a_{1(n-1)} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{n(n-1)} & b_n \end{vmatrix} \\ &= (-1)^{n-1} |\mathbf{D}_n| . \end{aligned}$$

From Cramer's rule, it is known that $|\mathbf{D}|$ is the determinant of the coefficient matrix $\mathbf{D}$ and $|\mathbf{D_k}|$ is the determinant of the matrix formed from the original coefficient matrix by replacing the kth column of $\mathbf{D}$ by the column with the elements $b_1,\dots,b_n$ [10],[11], [12], [14]. Thus, the solution of the system (1) is given by the formulas

$$x_1 = \frac{|\mathbf{D}_1|}{|\mathbf{D}|},\ x_2 = \frac{|\mathbf{D}_2|}{|\mathbf{D}|},\ x_3 = \frac{|\mathbf{D}_3|}{|\mathbf{D}|},\ \dots,\ x_n = \frac{|\mathbf{D}_n|}{|\mathbf{D}|}. \tag{7}$$

But, from (6) we have the following:

$$\begin{aligned} D &= |\mathbf{D}| \\ D_1 &= (-1)^{n-1} |\mathbf{D}_1| \\ D_2 &= -|\mathbf{D}_2| \\ D_3 &= (-1)^{n-1} |\mathbf{D}_3| \\ &\vdots \\ D_n &= (-1)^{n-1} |\mathbf{D}_n| . \end{aligned} \tag{8}$$

Hence, the solution (7) of the system (1) becomes, after a stylish manipulation by equating (7) and (8), the noteworthy formulas:

$$\begin{aligned} x_1 &= (-1)^{n-1} \frac{D_1}{D} \\ x_2 &= -\frac{D_2}{D} \\ x_3 &= (-1)^{n-1} \frac{D_3}{D} \\ &\vdots \\ x_n &= (-1)^{n-1} \frac{D_n}{D}. \end{aligned}$$

ACKNOWLEDGEMENT

The author thanks the reviewers of this paper for their immense suggestions which were considered, thereby improving and expanding the paper.

REFERENCES

[1] F.F. Abeles, "Determinants and Linear Systems: Charles L. Dodgson's View", *British Journal for the History of Science* 19(1986) pp 331-335.

[2] G. Cramer, *Intr. l'analyse de lignes courbes algbriques*, Geneva, 1750 pp 657-659.

[3] D. Crystal, *The Cambridge Encyclopedia*, Cambridge University Press, 1995, pp 219.

[4] H.K. Dass, *Advanced Engineering Mathematics*, S. Chand, New Delhi, 19th Edition, pp 220-371.

[5] M.B. David, "Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture", *MAA Spectrum Series*, Cambridge University Press (1999)

[6] C.L.Dodgson, "Condensation of Determinants, Being a New and Brief Method for Computing their Arithmetic Values", *Proc. Roy. Soc. Ser. A*, 15(1866), pp 150-155.

[7] E. T. Dowling, *Introduction to Mathematical Economics*, McGraw – Hill International Edition, 2001, pp 199-253.

[8] E. Egbe, G. A. Odili, O.O. Ugbebor, *Further Mathematics*, Africana-First Publishers Limited, Onitsha, pp 113- 128.

[9] K. Habgood, I. Arel, "A Condensation–based application of Cramer's rule for solving large–scale linear systems", *Journal of Discrete Algorithms* (2011), doi: 10.1016/j.jda.2011.06.007.

[10] J. Hornsby, L. Lial, K. Rockswold, *Precalculus*, Addison Wesley, 3rd Edition, 2003 pp 480–557.

[11] Kreyszig, *Advanced Engineering Mathematics*, Wiley, Singapore, 8th Edition, pp 304-368.

[12] C.D. Miller, V.E. Heeren, J. Hornsby, *Mathematical ideas*, Pearson Addison Wesley, New York, 10th Edition, 2004, pp 456-480.

[13] A. Rice, E. Torrence, "Shutting up like a telescope : Lewis Carroll's "Curious"Condensation Method for Evaluating Determinants", *College Mathematics Journal* 38(March):pp 85-95.

[14] J. Stewart, L. Redlin, S. Watson, *Precalculus*, Thomson Brooks/cole, 5th Edition, pp 635-715.